MEROMORPHIC FUNCTIONS, COVERING PROPERTIES, AND CLUSTER SETS

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1. *Introduction.*—Let $w = f(z)$ be a single-valued meromorphic function defined in a domain D in the extended z -plane, and let W denote the extended w -plane. Consider a fixed finite point z_0 of the boundary Γ of D , and let E be a bounded closed set of capacity¹ zero such that $z_0 \in E \subset \Gamma$. We assume that if z_0 is not an accumulation point of $\Gamma - E$, then z_0 is an essential singularity of $f(z)$. The cluster set $C_D(f, z)$ ($z \in \Gamma$) is defined as follows: $w \in C_D(f, z)$ if and only if $w \in W$ and there exist points $z_n \in D$ ($n = 1, 2, \dots$) such that $z_n \rightarrow z$ and $f(z_n) \rightarrow w$. Set $C_D = C_D(f, z_0)$. We now define the boundary cluster set $C_{\Gamma - E} = C_{\Gamma - E}(f, z_0)$: $w \in C_{\Gamma - E}$ if and only if $w \in W$ and there exist points $z_n \in \Gamma - E$ and $w_n \in C_D(f, z_n)$ ($n = 1, 2, \dots$) such that $z_n \rightarrow z_0$ and $w_n \rightarrow w$. Clearly C_D and $C_{\Gamma - E}$ are closed ($C_{\Gamma - E} = \phi$ if and only if z_0 is not an accumulation point of $\Gamma - E$). A known result, which contains an extension of the maximum principle and also the theorem of Weierstrass on essential isolated singularities, is that any component of $W - C_{\Gamma - E}$ that is not contained in C_D does not intersect C_D , that is, the set $\Omega = C_D - C_{\Gamma - E}$ is open.² Let $\Delta_1, \dots, \Delta_q$ ($q \geq 3$) be domains in W , and let $\bar{\Delta}_\nu$ and $\partial\Delta_\nu$ denote the closure and boundary, respectively, of Δ_ν ($\nu = 1, \dots, q$). Suppose that each $\partial\Delta_\nu$ is a single Jordan curve, no two $\bar{\Delta}_\nu$ intersect, and that all $\partial\Delta_\nu$ are contained in the same component Ω_0 of Ω . We prove the inequalities (1) and (2) below, which have analogues in Ahlfors' theory³ and are derived using basically Ahlfors' method. These inequalities apply in particular to the following two cases. *Case I:* D is simply connected. *Case II:* $E = \{z_0\}$ and $\Delta_\nu \subset \Omega_0$ ($\nu = 1, \dots, q$). As corollaries of (1) and (2) we find that in either Case I or Case II there exist $q - 2$ of the domains Δ_ν such that every neighborhood of z_0 contains a relatively compact component (whose image under $f(z)$ is consequently all of Δ_ν) of the preimage $f^{-1}(\Delta_\nu)$; and if $q \geq 5$, then in either Case I or Case II there exist $q - 4$ of the domains Δ_ν such that every neighborhood of z_0 contains a relatively compact component of $f^{-1}(\Delta_\nu)$ that $f(z)$ maps one-to-one onto Δ_ν . These conclusions in Case II (and under the assumption that z_0 is not an isolated point of Γ) follow from Kunugui's results,⁴ and in this case the inequalities (1) and (2) are closely related to those derived by Kunugui. Related theorems on Riemann surfaces are given by Sario and Noshiro.⁵ We also mention the earlier theorem of Noshiro,⁶ which is contained in (1). We conclude this note with a discussion of functions in the unit disk.

2. *Statements of Basic Results.*—We assume that E is contained in a single component Γ_0 of Γ , an assumption which holds in either Case I or Case II. Let Δ_0 be any domain such that $\partial\Delta_0$ is a Jordan curve, $\bar{\Delta}_0 \cap \bar{\Delta}_\nu = \phi$ ($\nu = 1, \dots, q$), and $\bar{\Delta} \subset \Omega_0$, where $\bar{\Delta} = W - \bigcup_{\nu=0}^q \bar{\Delta}_\nu$. Clearly Δ can be an arbitrary Jordan domain (bounded by finitely many Jordan curves) satisfying $\bar{\Delta} \subset \Omega$. In

proving the results stated above, we can clearly suppose without loss of generality that each $\partial\Delta_\nu$ ($\nu = 0, \dots, q$) is an analytic Jordan curve that does not contain the projection of any branch point of the Riemann surface $\mathfrak{s}$ onto which $f(z)$ maps D . Choose $\delta > 0$ such that $C_D(f, z) \cap \bar{\Delta} = \phi$ if $|z - z_0| \leq \delta$ and $z \in \Gamma - E$, and such that $E \cap \mathfrak{C} = \phi$ and $D \cap \mathfrak{C} \neq \phi$, where $\mathfrak{C} = \{|z - z_0| = \delta\}$. The requirement $E \cap \mathfrak{C} = \phi$ is possible because a compact set of capacity zero has linear measure zero.⁷ Set $\mathfrak{D} = \{|z - z_0| < \delta\}$ and $E' = E \cap \mathfrak{D}$. Then E' is a compact set of capacity zero. By the Evans-Selberg theorem,⁸ there exists a single-valued harmonic function $u(z)$ defined in the complement of E' (relative to the finite z -plane) such that $u(z) + \log |z|$ has a removable singularity at ∞ and $u(z) \rightarrow +\infty$ as z tends to any point of E' . It is easy to see that for any real number a , $\{z: u(z) < a\}$ is connected and is bounded by finitely many analytic curves, together with the point ∞ . Let a_0 be any real number sufficiently large so that $\{z: u(z) \geq a_0\} \subset \mathfrak{D}$. Set

$$V = V(a) = \{z: a_0 < u(z) < a\} (a > a_0), \quad B = B(a) = \{z: u(z) = a\} (a \geq a_0),$$

and $U = U(a) = f^{-1}(\Delta) \cap V$. Note that there are at most finitely many components of $f^{-1}(\partial\Delta) \cap V$, and that any such component that is relatively compact in V is compact. Also, U has at most finitely many components, each of which is relatively compact in D . Consider any component I_ν of $f^{-1}(\Delta_\nu) \cap V$ that is relatively compact in V . It is bounded by finitely many compact components of $f^{-1}(\partial\Delta_\nu)$ and possibly also a subset of Γ . These components of $f^{-1}(\partial\Delta_\nu)$ form the boundary of a domain I_ν^* that contains I_ν and is relatively compact in V .⁹ Note that if D is simply connected $I_\nu^* = I_\nu$. This also holds if $\Delta_\nu \subset \Omega_0$, provided we in this case require of δ that $C_D(f, z) \cap \bar{\Delta}_\nu = \phi$ if $|z - z_0| \leq \delta$ and $z \in \Gamma - E$. Note that there are at most finitely many I_ν . An arbitrary I_ν ($\nu = 0, \dots, q$) is denoted by I , and the corresponding I_ν^* by I^* . Let $e(I^*)$ denote the Euler characteristic of I^* , that is, $e(I^*)$ is the number of components of ∂I^* minus 2. Let $|\Delta|$ denote the spherical area of Δ , and set

$$M = M(a) = \frac{1}{|\Delta|} A(a), \quad \text{where} \quad A(a) = \iint_U \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} dx dy (z = x + iy).$$

We prove that

$$\liminf_{a \rightarrow +\infty} \sum_{\nu=0}^q \left(1 + \frac{1}{M} \sum e(I_\nu^*) \right) \leq 2, \quad \lim_{a \rightarrow +\infty} M = \infty, \quad (1)$$

where it is understood that for a particular ν , $\sum e(I_\nu^*) = 0$ if there is no I_ν . For each I_ν^* the subset of $\mathfrak{s}$ that corresponds under $f(z)$ to ∂I_ν^* covers each point of $\partial\Delta_\nu$ the same finite number of times. For a particular ν the minimum over all simply connected I_ν^* of these numbers is denoted by m_ν . If for a particular ν there is no simply connected I_ν^* , we set $m_\nu = \infty$. We prove that

$$\liminf_{a \rightarrow +\infty} \sum_{\nu=0}^q \left(1 - \frac{1}{m_\nu} \right) \leq 2. \quad (2)$$

3. *Proof of (1) and (2).*—Throughout the proof $a \geq a_0$. There exists a set α of real numbers that is bounded below and has no finite accumulation point such that $u_x(z) - iu_y(z) \neq 0$ if $z \in B$ and $a \notin \alpha$. Set

$$\gamma = \gamma(a) = f^{-1}(\Delta) \cap B, \quad L = L(a) = \int_{\gamma} \frac{|f'(z)|}{1 + |f(z)|^2} |dz|, \quad L_0 = L(a_0).$$

Require now that $a > a_0$. Let U^* be the interior of $U \cup (\cup \bar{I}^*)$, and let U_0 and U_0^* denote arbitrary components of U and U^* , respectively. For a particular U_0 , the relative distance between two points of U_0 is defined to be the infimum of the lengths of all polygonal Jordan arcs lying in U_0 and joining those two points. The completion $\hat{U}_0$ of U_0 with this relative metric is a compact bordered surface, whose set of interior points is U_0 ; and in an obvious way $f(z)$ defines $\hat{U}_0$ as a covering surface of $\bar{\Delta}$ on which we consider a different metric, obtained by lifting the spherical metric from the w -plane. We consider $\hat{U}_0$ instead of $\bar{U}_0$ because ∂U_0 might have multiple points. The Euler characteristic of $\hat{U}_0$ is denoted simply by $e(U_0)$. Similarly we consider $e(U_0^*)$. It is readily seen that

$$\sum e(U_0^*) = \sum e(U_0) + \sum e(I^*).$$

Since $E' \subset \Gamma_0$, and since each component of B encloses (in its interior) at least one point of E' , the following holds: for each U_0 , all except possibly two components of ∂U_0 are compact components of $f^{-1}(\partial \Delta) \cap V$ that do not enclose any component of B . It follows that each U_0^* is simply or doubly connected, that is, $e(U_0^*)$ is -1 or 0 . By Hurwitz' formula no simply connected U_0 is relatively compact in V , and we see that each simply connected U_0 is a U_0^* . Thus it follows from the displayed equation that $-\sum e(I^*) \geq \sum e^+(U_0)$, where $e^+(U_0)$ is the maximum of 0 and $e(U_0)$. Hence by Ahlfors' fundamental theorem¹⁰

$$-\sum e(I^*) \geq \sum e^+(U_0) \geq (q-1)M - k(L_0 + L), \quad (3)$$

where k depends only on Δ .

The set $\Delta - R_D(f, z_0)$ has inner capacity zero,¹¹ where $R_D(f, z_0)$ denotes the set of values assumed by $f(z)$ in every neighborhood of z_0 , and in particular the former set has two-dimensional measure zero. Thus there exists a sufficiently large $a_1' > a_0$ such that $A(a_1') > |\Delta|/2$. Similarly, there exists a sufficiently large $a_2' > a_1'$ such that the spherical area of the Riemannian image under $f(z)$ of $f^{-1}(\Delta) \cap \{z: a_1' < u(z) < a_2'\}$ is greater than $|\Delta|/2$. Continuing in this way, we see that $A(a) \rightarrow \infty$ as $a \rightarrow +\infty$.

We now use a classical argument and a method of G. af Hällström and M. Tsuji to prove that

$$L(a)^2 \leq 2\pi A'(a) \quad (a \notin \alpha). \quad (4)$$

Let a' and a'' be any numbers such that $a_0 < a' < a''$ and such that the closed interval $[a', a'']$ does not intersect α . We require for the proof of (4) that $a' < a < a''$. Note that for any sufficiently large $h > 0$, $B \cup \{|z| = h\}$ is the boundary of a domain. Thus it follows from Green's formula and the fact that

$u(z) + \log|z|$ is harmonic at ∞ that

$$\int_B - \frac{\partial u}{\partial n} ds = 2\pi, \quad (5)$$

where B has the positive orientation and n is the inner normal, both with respect to $\{z: u(z) < a\}$, and s denotes arc length. For a given $\epsilon > 0$, subtract from $f^{-1}(\Delta) \cap \{z: a' < u(z) < a''\}$ the vertical and horizontal straight lines through $\pm j\epsilon$ and $\pm ij\epsilon$, respectively ($j = 0, 1, \dots$), and let U_n ($n = 1, \dots, n_0$) be the finitely many components of the resulting open set. Since $[a', a''] \cap \alpha = \phi$, we can choose ϵ sufficiently small so that each $\bar{U}_n$ is contained in an open disk on which the function $\zeta_n(z) = u(z) + iv_n(z)$ is one-to-one, where $v_n(z)$ is a single-valued conjugate of $u(z)$ in this open disk. Let U'_n be the image of U_n under $\zeta = \zeta_n(z)$, and let $s_n(a)$ be the intersection of U'_n and the vertical line $\{\zeta: u = a\}$ ($\zeta = u + iv$). Let $z = \varphi_n(\zeta)$ ($\zeta \in U'_n$) be the inverse function of $\zeta = \zeta_n(z)$, and set $h_n(\zeta) = f(\varphi_n(\zeta))$ ($\zeta \in U'_n$). Then

$$A(a) - A(a') = \sum_n \int_{a'}^a \int_{s_n(u)} \frac{|h'_n(\zeta)|^2}{(1 + |h_n(\zeta)|^2)^2} dv du, \\ L(a) = \sum_n \int_{s_n(a)} \frac{|h'_n(\zeta)|}{1 + |h_n(\zeta)|^2} dv. \quad (6)$$

Let all of the following integration be over $s_n(a)$. By Schwarz' inequality

$$\int \frac{|h'_n(\zeta)|}{1 + |h_n(\zeta)|^2} dv \leq \left(\int dv \right)^{1/2} \left(\int \frac{|h'_n(\zeta)|^2}{(1 + |h_n(\zeta)|^2)^2} dv \right)^{1/2}. \quad (7)$$

Since the integrand in (5) is nonnegative, it follows from (5) that $\sum_n \int dv \leq 2\pi$. Thus by (6), (7), and Cauchy's inequality,

$$L(a)^2 \leq \left(\sum_n \int dv \right) \left(\sum_n \int \frac{|h'_n(\zeta)|^2}{(1 + |h_n(\zeta)|^2)^2} dv \right) \leq 2\pi A'(a),$$

which proves (4).

We now prove that $\liminf_{a \rightarrow +\infty} L(a)/A(a) = 0$. Otherwise there would exist constants $K > 0$ and $\bar{a} > a_0$ such that $L(a) \geq KA(a)$ if $a \geq \bar{a}$. Choose a sequence $\{a_n\}$ such that $\bar{a} < a_n < a_{n+1}$ ($n = 1, 2, \dots$), $a_n \rightarrow +\infty$, and for each n the open interval (a_n, a_{n+1}) does not intersect α . We have

$$\frac{1}{2\pi} (a_{n+1} - a_n) \leq \int_{a_n}^{a_{n+1}} \frac{A'(a)}{L(a)^2} da \leq \frac{1}{K^2} \left(\frac{1}{A(a_n)} - \frac{1}{A(a_{n+1})} \right),$$

because the first inequality follows from (4), and the second follows from the inequality $L(a) \geq KA(a)$ ($a \geq \bar{a}$). By summing for $n = 1, \dots, n' - 1$, we obtain

$$\frac{1}{2\pi} (a_{n'} - a_1) \leq \frac{1}{K^2} \left(\frac{1}{A(a_1)} - \frac{1}{A(a_{n'})} \right),$$

which is a contradiction since $A(a_{n'}) \rightarrow \infty$ as $n' \rightarrow \infty$.

Since $A(a) \rightarrow +\infty$ as $a \rightarrow +\infty$, (1) follows from (3) and the fact that $\liminf$

$L/A = 0$ ($A = A(a)$). We now prove (2). We noted that $f(z)$ defines $\hat{U}_0$ as a covering surface of $\bar{\Delta}$. Consider for a particular Δ_ν the spherical length (lifted from the w -plane) of the part over $\partial\Delta_\nu$ of the boundary of this covering surface, and divide this length by the spherical length of $\partial\Delta_\nu$. For a particular Δ_ν the sum over all U_0 of these quotients is denoted by $M(\partial\Delta_\nu)$. If $m_\nu \neq \infty$, we have for each ν

$$-m_\nu \sum e(I_\nu, *) \leq M(\partial\Delta_\nu) \leq M + k'(L_0 + L),$$

where k' depends only on Δ , the second inequality following from Ahlfors' second covering theorem.¹² If $m_\nu = \infty$, these inequalities hold with m_ν replaced by an arbitrary positive integer. Thus by (3)

$$(q-1)M \leq M \sum_{\nu=0}^q \frac{1}{m_\nu} + k''(L_0 + L),$$

where $k'' = k + (q+1)k'$. From this inequality and the fact that $\liminf L/A = 0$, (2) readily follows.

4. *Functions in the Unit Disk.*—In this section D and Γ denote the open unit disk and unit circle, respectively, and $w = f(z)$ is a function meromorphic in D . Let A be an open arc of Γ containing the fixed point $z_0 \in \Gamma$. Let E be a Borel set of inner capacity zero such that $z_0 \in E \subset A$, and associate with each $e^{i\theta} \in A - E$ an arc Λ_θ at $e^{i\theta}$ ($\Lambda_\theta \subset D$ and $\Lambda_\theta \cup \{e^{i\theta}\}$ is a Jordan arc). Set $\Omega^* = C_D - C_{\Gamma-E}^*$, where $C_D = C_D(f, z_0)$ and where $C_{\Gamma-E}^* = C_{\Gamma-E}^*(f, z_0)$ is Noshiro's boundary cluster set defined relative to the family $\{\Lambda_\theta\}$.¹³ By Noshiro's theorem,¹⁴ Ω^* is open. Also known¹⁵ is that if $W = R_D(f, z_0)$ contains at least three points, then for any component Ω_0^* of Ω^* , $\Omega_0^* - R_D(f, z_0)$ contains at most one point. We can now prove considerably more. Let $\Delta_1, \dots, \Delta_q$ ($q \geq 3$) be domains such that each $\partial\Delta_\nu$ is a single Jordan curve, no two $\bar{\Delta}_\nu$ intersect, and all $\partial\Delta_\nu$ are contained in the same component Ω_0^* of Ω^* . We prove that *there exist $q-2$ of the domains Δ_ν such that every neighborhood of z_0 contains a relatively compact component of $f^{-1}(\Delta_\nu)$, and that if $q \geq 5$, there exist $q-4$ of the domains Δ_ν such that every neighborhood of z_0 contains a relatively compact component of $f^{-1}(\Delta_\nu)$ that $f(z)$ maps one-to-one onto Δ_ν .*

Proof: Let Δ_ν ($\nu = 0, \dots, q$) be domains such that each $\partial\Delta_\nu$ is an analytic Jordan curve, no two $\bar{\Delta}_\nu$ intersect, and such that if we set $\Delta' = W - \bigcup_{\nu=0}^q \bar{\Delta}_\nu$, then $\bar{\Delta}' \subset \Omega_0^*$, $\partial\Delta_\nu \subset \Delta'$ ($\nu = 1, \dots, q$), and $\bar{\Delta}_\nu' \subset \Delta_\nu$ ($\nu = 1, \dots, q$). Let Δ_ν'' ($\nu = 0, \dots, q$) be domains such that each $\partial\Delta_\nu''$ is an analytic Jordan curve, no two $\bar{\Delta}_\nu''$ intersect, $\bar{\Delta}_0' \subset \Delta_0''$, and $\bar{\Delta}_\nu \subset \Delta_\nu''$ ($\nu = 1, \dots, q$). Require also that no $\partial\Delta_\nu'$ or $\partial\Delta_\nu''$ contains the projection of any branch point of the Riemann surface $\mathfrak{S}$ onto which $f(z)$ maps D . Set $\Delta'' = W - \bigcup \bar{\Delta}_\nu''$. Then $\bar{\Delta}'' \subset \Delta'$. Choose $\delta > 0$ such that if $\mathfrak{D} = \{|z - z_0| < \delta\}$ and $\gamma = \Gamma \cap \mathfrak{D}$, then $\gamma \subset A$ and $C_{\Lambda_\theta}(f, e^{i\theta}) \cap \bar{\Delta}' = \phi$ for $e^{i\theta} \in \gamma - E$. Let γ_0 be an open subarc of γ such that $z_0 \in \gamma_0$ and the end points $e^{i\theta_1}$ and $e^{i\theta_2}$ of γ_0 are in $\gamma - E$. Let Λ_1 and Λ_2 be arcs at $e^{i\theta_1}$ and $e^{i\theta_2}$, respectively, such that $\Lambda_j \subset \Lambda_{\theta_j} \cap \mathfrak{D}$, $f(\Lambda_j) \cap \bar{\Delta}' = \phi$ ($j = 1, 2$), $\Lambda_1 \cap \Lambda_2 = \phi$, and the initial points (i.e., end points in D) of Λ_1 and Λ_2 are the end points of an open rectilinear segment S that does not intersect $\Lambda_1 \cup \Lambda_2$. Let V

be the bounded domain whose boundary is $S \cup \Delta_1 \cup \Delta_2 \cup \gamma_0$. At most finitely many components of $V \cap f^{-1}(\Delta'')$ have boundary points on S . Thus, since $V \cap f^{-1}(\Delta'')$ contains points arbitrarily near z_0 , either (i) some component U^* of $V \cap f^{-1}(\Delta'')$ is relatively compact in V , or (ii) some component of $V \cap f^{-1}(\Delta'')$ is not relatively compact in D . We first consider the case where (i) holds. Let U_0 be the smallest simply connected domain containing U^* . Let S_0 be the part of S that corresponds under $f(z)$ to U_0 . Then the relative boundary of S_0 is a single curve lying over some $\partial\Delta_{\nu''}$ and covering each point of $\partial\Delta_{\nu''}$ the same finite number of times μ . We suppose without loss of generality that $0 \in \Delta_{\nu''}$ and $\infty \notin \bar{\Delta}_{\nu''}$. Consider the part over $\Delta_{\nu''} - \{0\}$ of the Riemann surface of $w^{1/\mu}$, and attach this part to S_0 across the relative boundary of S_0 . The resulting Riemann surface is conformally equivalent to the finite plane, and by applying Ahlfors' theorem,¹⁶ we see that there exist $q + 1 - 2$ of the domains $\Delta_{\nu''}$ ($\nu = 0, \dots, q$) such that V contains a relatively compact component of $f^{-1}(\Delta_{\nu''})$. Thus since $\Delta_{\nu} \subset \Delta_{\nu''}$ ($\nu = 1, \dots, q$), we have the conclusion (a): there are $q - 2$ of the domains Δ_{ν} ($\nu = 1, \dots, q$) such that V contains a relatively compact component of $f^{-1}(\Delta_{\nu})$. Similarly,¹⁷ (b) if $q \geq 5$, there are $q - 4$ of the domains Δ_{ν} ($\nu = 1, \dots, q$) such that V contains a relatively compact component of $f^{-1}(\Delta_{\nu})$ that $f(z)$ maps one-to-one onto Δ_{ν} . We now consider the case where (ii) holds. Let U' be a component of $V \cap f^{-1}(\Delta')$ that contains a component of $V \cap f^{-1}(\Delta'')$ that is not relatively compact in D . Let U be the smallest simply connected domain containing U' . Then $\partial U \subset \partial U'$, and for some $z_1 \in \gamma_0$, $C_U(f, z_1) \cap \bar{\Delta}'' \neq \phi$. Since $C_{\Delta_0}(f, e^{i\theta}) \cap \bar{\Delta}' = \phi$ for each $e^{i\theta} \in \gamma - E$, it is easy to see that each point of $\Gamma \cap \partial U$ is accessible through U , that is, each such point is an end point of an open Jordan arc lying in U . Thus each point of $(\gamma - E) \cap \partial U$ is accessible through both U and $D - \bar{U}$, and consequently each such point is one end point of a component of $f^{-1}(\partial\Delta')$. It follows that the set $(\gamma - E) \cap \partial U$ is at most countable. Since the union of two Borel sets of inner capacity zero has inner capacity zero,¹⁸ the compact set $E_1 = \Gamma \cap \partial U$ has capacity zero. Clearly $C_{\Gamma_1 - E_1}(f, z_1) \subset \partial\Delta'$, where $\Gamma_1 = \partial U$ and the boundary cluster set is defined relative to U . Thus, since $C_U(f, z_1) \cap \Delta' \neq \phi$, $\Delta' \subset C_U(f, z_1)$.¹⁹ Since $\partial\Delta_{\nu} \subset \Delta'$ ($\nu = 1, \dots, q$), it follows from the results stated in this note's introduction that the conclusions (a) and (b) hold also in the present case. Since δ can be arbitrarily small, the proof is complete.

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¹ In this note "capacity" means "logarithmic capacity."

² See Noshiro, K., *Cluster Sets* (Berlin, 1960), p. 17; and for the case where z_0 is not an accumulation point of $\Gamma - E$, see also p. 10, Proof of Theorem 1.

³ Ahlfors, L. V., "Zur Theorie der Überlagerungsflächen," *Acta Math.*, **65**, 157-194 (1935).

⁴ Kunugui, K., "Sur la théorie des fonctions méromorphes et uniformes," *Japan J. Math.*, **18**, 583-614 (1943), p. 606.

⁵ Sario, L., and K. Noshiro, *Value Distribution Theory* (Princeton, 1966), chap. 6.

⁶ Noshiro, K., *op. cit.*, p. 20.

⁷ See Nevanlinna, R., *Eindeutige Analytische Funktionen* (Berlin, 1953), pp. 126 and 151.

⁸ See, for example, Noshiro, K., *op. cit.*, p. 7.

⁹ We verify this as follows. We only need to consider the case where $\Gamma \cap \partial I_{\nu} \neq \phi$. Let V_0 be the component of V that contains I_{ν} . Since $D \cap \bar{V}_0 \neq \phi$, the unbounded component of the complement of $\bar{V}_0$ intersects D , and since each bounded component of the complement of $\bar{V}_0$

intersects E' , each such bounded component intersects D . Thus D intersects every component of ∂V_0 . It follows that ∂V_0 and I_ν are contained in one component u of the complement of $\Gamma \cap \partial I_\nu$. Clearly $f^{-1}(\partial \Delta_\nu) \cap \partial I_\nu$ is also contained in u , and consequently there exists a continuum $\mathfrak{z}$ that contains ∂V_0 and $f^{-1}(\partial \Delta_\nu) \cap \partial I_\nu$, and does not intersect $\Gamma \cap \partial I_\nu$. Therefore there exists a finite collection of bounded domains $\mathfrak{z}_j$ such that each $\partial \mathfrak{z}_j$ is a Jordan curve, no two $\mathfrak{z}_j$ intersect, $\Gamma \cap \partial I_\nu \subset \bigcup \mathfrak{z}_j$, each $\mathfrak{z}_j$ intersects $\Gamma \cap \partial I_\nu$, and no $\mathfrak{z}_j$ intersects $\mathfrak{z}$. It follows that for each j , $\partial \mathfrak{z}_j \subset I_\nu$ and $\mathfrak{z}_j \subset V_0$. Thus we can set $I_\nu^* = I_\nu \cup (\bigcup \mathfrak{z}_j)$.

¹⁰ Ahlfors, L. V., *op. cit.*, p. 168.

¹¹ See Noshiro, K., *op. cit.*, p. 19; and for the case where z_0 is not an accumulation point of $\Gamma - E$, see also p. 10, Proof of Theorem 2.

¹² Ahlfors, L. V., *op. cit.*, p. 165.

¹³ Noshiro, K., *op. cit.*, p. 40.

¹⁴ *Ibid.*

¹⁵ See McMillan, J. E., "On cluster sets of meromorphic functions," these PROCEEDINGS, **56**, 787-788 (1966).

¹⁶ Ahlfors, L. V., *op. cit.*, p. 191.

¹⁷ *Ibid.*

¹⁸ See Tsuji, M., *Potential Theory in Modern Function Theory* (Tokyo, 1959), p. 57.

¹⁹ See Noshiro, K., *op. cit.*, p. 17.