Published in final edited form as:

Ann Biomed Eng. 2011 March ; 39(3): 1032–1040. doi:10.1007/s10439-010-0212-2.

Determination of the Critical Buckling Pressure of Blood Vessels Using the Energy Approach

Hai-Chao Han^{1,2}

¹Department of Mechanical Engineering, University of Texas at San Antonio, Biomedical Engineering Program, UTSA-UTHSCSA, San Antonio, TX 78249, USA

²Institute of Mechanobiology & Medical Engineering, Shanghai Jiaotong University, Shanghai, China

Abstract

The stability of blood vessels under lumen blood pressure is essential to the maintenance of normal vascular function. Differential buckling equations have been established recently for linear and nonlinear elastic artery models. However, the strain energy in bent buckling and the corresponding energy method have not been investigated for blood vessels under lumen pressure. The purpose of this study was to establish the energy equation for blood vessel buckling under internal pressure. A buckling equation was established to determine the critical pressure based on the potential energy. The critical pressures of blood vessels with small tapering along their axis were estimated using the energy approach. It was demonstrated that the energy approach yields both the same differential equation and critical pressure for cylindrical blood vessel buckling as obtained previously using the adjacent equilibrium approach. Tapering reduced the critical pressure of blood vessels compared to the cylindrical ones. This energy approach provides a useful tool for studying blood vessel buckling and will be useful in dealing with various imperfections of the vessel wall.

Keywords

Artery; Vein; Blood vessel; Bent buckling; Stability; Instability; Tortuosity; Energy equation; Strain energy; Buckling equation; Critical pressure; Tapering; Nonlinear elastic

Introduction

The structural stability of blood vessels under the lumen blood pressure load is essential to the maintenance of normal vascular function. It has been well documented that arteries and veins may collapse when the internal lumen pressure is too low and/or the external pressure exceeds the internal pressure.^{1,7,9,29} Recently, we have demonstrated that long arteries and veins may buckle and become tortuous along their axis due to hypertensive lumen pressure or reduced axial stretch ratio.^{12,24} While buckling equations have been established for linear and nonlinear elastic artery and vein models using the adjacent equilibrium approach^{4,12–14,16,23,24} and dynamic stability approach,^{31,32} the energy approach needs investigation.

Address correspondence to Hai-Chao Han, Department of Mechanical Engineering, University of Texas at San Antonio, Biomedical Engineering Program, UTSA-UTHSCSA, San Antonio, TX 78249, USA. hchan@utsa.edu.

Associate Editor Cheng Dong oversaw the review of this article.

Energy methods have been used as a powerful tool for studying complex mechanical systems and have been used in dealing with buckling problems of engineering structures.^{2,34} Energy methods demonstrate great advantages in estimating the critical load for complex engineering systems when exact solutions are difficult to obtain from differential equations. However, the energy methods for blood vessel bent buckling under internal pressure have not been investigated. Real arteries and veins demonstrate various geometric and material variations, i.e., “imperfections,” such as initial curvature and tapering.^{10,17} These variations often need to be considered in order to better understand artery behavior but they pose tremendous challenges in solving the differential buckling equations. We have previously demonstrated that cylindrical blood vessels buckle under internal pressure through a different mechanism than columns under axial compression.^{12–14,16} Thus, the energy equation for Euler column buckling cannot be applied directly to the blood vessels. Therefore, it is necessary to develop an energy method as an alternative approach to estimate the critical pressure for realistic arteries with various imperfections.

Accordingly, the objective of this study was to establish the energy equations for blood vessel buckling under internal pressure. These equations will provide a new approach to study the buckling of arteries and veins and other pressurized vessels such as ureters.

Methods

The Strain Energy in Buckled Blood Vessels

We modeled blood vessels as cylindrical tubes of nonlinear elastic materials. Taking into account the residual stress in the arteries,^{3,18} we modeled arteries as open sectors at their initial “zero-stress state” with an opening angle of $2\Theta_0$ (Fig. 1). The initial inner radius, outer radius, and length of the arteries were designated as R_i , R_e , and L , respectively. Similarly, the inner radius, outer radius, and length at the loaded state were designated as r_i , r_e , and l , respectively, under an internal pressure p and axial tension N . The initial cylindrical coordinates of a point at the zero-stress configuration were denoted by (R, Θ, Z) , while the corresponding cylindrical coordinates of the point in the vessel wall under the internal pressure and axial tension (without buckling) were denoted by (r, ϕ, z) (see Fig. 1). Accordingly, we have

$$\begin{aligned} r &= r(R, p, N) \\ \phi &= \frac{\pi}{\Theta_0} \Theta \\ z &= \lambda_z^0 Z \end{aligned} \quad (1)$$

where λ_z^0 is the axial stretch ratio.

When the vessel starts to buckle, the small deflection of the vessel in the lateral direction is designated by $u_c(z)$, with z being the axial coordinate of the central axis (Fig. 2). The buckling deflection leads to an uneven deformation at the concave and convex sides of the vessel and the internal pressure p will generate a lateral distributed load q due to the difference in the surface areas at the concave and convex sides.^{12,16}

$$q(z) = -p\pi r_i^2 \left(\frac{d^2 u_c}{dz^2} \right) = -p\pi r_i^2 u_c'' \quad (2)$$

Accordingly, the incremental (virtual) work done by q at δu_c is^{22,34}

$$\delta W_q = - \int_0^l \left[p \pi r_i^2 u_c'' \delta u_c \right] dz. \quad (3)$$

Integrating this equation yields

$$W_q = \int_0^{u_c} \delta W_q = - p \pi r_i^2 \left[u_c' u_c \right]_0^l + \frac{1}{2} p \pi r_i^2 \int_0^l (u_c')^2 dz. \quad (4a)$$

For fixed or pinned ends, $\left[u_c' u_c \right]_0^l = 0$, we have

$$W_q = \frac{1}{2} p \pi r_i^2 \int_0^l (u_c')^2 dz. \quad (4b)$$

Further, the length change (axial displacement) of the vessel due to the buckling is⁹

$$\Delta l = \frac{1}{2} \int_0^l (u_c')^2 dz. \quad (5)$$

Thus, the work done by the axial tension N is

$$W_N = - \frac{1}{2} \int_0^l N (u_c')^2 dz. \quad (6)$$

On the other hand, the strain energy in the deformed vessel can be expressed as

$$U = \frac{1}{2} \int_0^l \left[EI (u_c'')^2 \right] dz, \quad (7)$$

where $EI = EI(p, \lambda_z^0, z)$ is the cross-sectional bending modulus of the vessel, which is a function of the pressure for nonlinear elastic arterial walls.^{9,11,14,31,34} Detailed equations to determine EI for nonlinear anisotropic arterial walls with the Fung strain energy function were given in our previous papers^{13,14} and are listed in the Appendix.

In addition, the support from contiguous tissues can be modeled as an elastic matrix (substrate). Since the deflection at buckling is small, we assumed that the reaction force is proportional to the deflection with an elastic constant k_s for the matrix.^{14,34} Thus, the strain energy stored in the matrix is

$$U_s = \frac{1}{2} \int_0^l k_s u_c^2 dz. \quad (8)$$

Therefore, using the potential energy expression for beam-columns,^{2,22,34} we can express the potential energy of the blood vessel as

$$\Pi = U + U_s - W_N - W_q, \quad (9)$$

i.e.,

$$\Pi = \frac{1}{2} \int_0^l \left[EI(u_c'')^2 + k_s u_c^2 \right] dz - \frac{1}{2} \int_0^l \left[(p\pi r_i^2 - N) (u_c')^2 \right] dz + p\pi r_i^2 \left[u_c' u_c \right]_0^l. \quad (10)$$

For fixed or pinned ends, $\left[u_c' u_c \right]_0^l = 0$, it becomes

$$\Pi = \frac{1}{2} \int_0^l \left[EI(u_c'')^2 + k_s u_c^2 - (p\pi r_i^2 - N) (u_c')^2 \right] dz. \quad (11)$$

The Critical Pressure

At buckling, the strain energy equals the work done by the loads, and the potential energy given in Eq. (11) equals zero.^{2,22,34} This yields the following buckling equation

$$p_{cr} = \frac{\int_0^l \left[EI(u_c'')^2 + N(u_c')^2 + k_s u_c^2 \right] dz}{\pi r_i^2 \int_0^l (u_c')^2 dz}. \quad (12)$$

This equation determines the critical pressure when the mode shape u_c is known. It gives an approximation of the critical pressure when u_c is close to but not the exact mode shape. In general, the mode shape can be expressed in terms of a Fourier series

$$u_c = \sum_n \left[C_n \cdot \sin\left(\frac{n\pi z}{l}\right) + D_n \cdot \cos\left(\frac{n\pi z}{l}\right) \right] \quad (13)$$

that satisfies the boundary conditions. The critical pressure and mode shape (i.e., the coefficients C_n and D_n) can be determined by minimizing the p_{cr} given in Eq. (12).

Results

Results obtained using the buckling energy equation are presented here to demonstrate the validation of the theory and its applications.

The Differential Equation of Blood Vessel Buckling

As a validation of the energy equations, we derived the differential equation for blood vessel buckling using the potential energy equations. Using Eq. (11), the virtual work done for virtual deflection δu_c is

$$\delta \Pi = \int_0^l \left[(EI u_c'') \delta (u_c'') + k_s u_c \delta (u_c) \right] dz - \int_0^l \left[(p\pi r_i^2 - N) u_c' \right] \delta (u_c') dz \quad (14)$$

Integrating by parts, we have

$$\delta \Pi = \int_0^l \left\{ (EI u_c'')'' + [(p\pi r_i^2 - N) u_c']' + k_s u_c \right\} \delta u_c dz + [EI u_c'' \delta u_c]_0^l - \left\{ [(EI u_c'')' - (p\pi r_i^2 - N) u_c'] \delta u_c \right\}_0^l. \quad (15)$$

Therefore, using $\delta \Pi = 0$ at equilibrium for any arbitrary δu_c leads to

$$(EI u_c'')'' + [(p\pi r_i^2 - N) u_c']' + k_s u_c = 0. \quad (16)$$

And the boundary conditions at both ends

$$u_c = 0 \quad \text{or} \quad (EI u_c'')' + (p\pi r_i^2 - N) u_c' = 0 \quad (17a)$$

and

$$u_c' = 0 \quad \text{or} \quad EI u_c'' = M = 0. \quad (17b)$$

When EI and N are constant along the axial direction of the vessel, Eq. (16) becomes

$$EI u_c'' + (p\pi r_i^2 - N) u_c'' + k_s u_c = 0. \quad (18)$$

It is seen that the energy approach yields the same differential equation as we obtained previously using the adjacent equilibrium approach.^{14,16} The solutions for this equation also can be found in our previous work.^{14,16} This agreement validated the potential energy equations.

Buckling Pressure of Arteries with Pinned or Fixed Ends

For cylindrical blood vessels, either thin-walled or thick-walled, either linear or nonlinear elastic materials, EI is constant along the axial location. The equations for critical pressure can be derived from buckling equation (12) as follows.

For vessels with pinned ends, by using mode shape

$$u_c = C_n \cdot \sin\left(\frac{n\pi z}{l}\right), \quad n=1, 2, 3, \dots, \quad (19)$$

we have

$$p_{cr} = \frac{EI}{\pi r_i^2} \left(\frac{n\pi}{l}\right)^2 + \frac{k_s}{\pi r_i^2} \left(\frac{n\pi}{l}\right)^{-2} + \frac{N}{\pi r_i^2}. \quad (20)$$

For vessels with fixed ends, by using mode shape

$$u_c^+ = \frac{C_n}{2} \left[1 - \cos\left(\frac{2n\pi z}{l}\right) \right] \quad (21a)$$

or

$$u_c^- = C_n \left[\sin\left(\frac{2\beta_n \pi z}{l} - \beta_n \pi\right) - \left(\frac{2z}{l} - 1\right) \sin(\beta_n \pi) \right], \quad (21b)$$

where $\beta_n = 1.40, 2.46, 3.47, 4.48, \dots$ are the roots of $\beta\pi = \tan \beta\pi$,¹⁶ we have

$$p_{cr} = \frac{EI}{\pi r_i^2} \left(\frac{2n\pi}{l}\right)^2 + \frac{3k_s}{\pi r_i^2} \left(\frac{2n\pi}{l}\right)^{-2} + \frac{N}{\pi r_i^2} \quad (22a)$$

or

$$p_{cr} = \frac{EI}{\pi r_i^2} \left(\frac{2\beta_n \pi}{l}\right)^2 + \frac{5k_s}{3\pi r_i^2} \left(\frac{2\beta_n \pi}{l}\right)^{-2} + \frac{N}{\pi r_i^2}. \quad (22b)$$

These equations determine the critical pressure of a vessel buckling within an elastic matrix and are similar to those obtained previously using the adjacent equilibrium approach.^{14,16} These results further validated buckling equation (12). Note that the minimum pressure can be reached at a higher order buckling mode ($n > 1$) within an elastic matrix ($k_s > 0$) thus arteries buckle into multiple waves, i.e., tortuous shapes.

The Critical Pressure of a Thin-Walled Artery with Tapered Wall Thickness

For blood vessels with irregular shapes and imperfections, it is convenient to use the strain energy approach to approximate the critical pressure. As an example, we examined the buckling of a thin-walled artery with tapered wall thickness. The artery was assumed to have a constant lumen radius r_i , but a tapered outer radius r_e , i.e., a decreasing wall thickness t along the axis given by

$$t(z)=t_0 \left(1-\alpha \frac{z}{l}\right), \quad \alpha=\frac{t_0-t^{\text{distal}}}{t_0}, \quad (23)$$

where α is the wall thickness tapering ratio, with t_0 and t^{distal} being the wall thickness at the proximal and distal ends, respectively. The vessel wall was assumed to be thin and homogenous with an incremental modulus E_{inc} . Accordingly,

$$EI \approx \pi E_{\text{inc}} r_i^3 t = \pi E_{\text{inc}} r_i^3 t_0 \left(1-\alpha \frac{z}{l}\right). \quad (24)$$

For small tapering ratios, the buckling mode shapes were expected to be close to the sinusoidal mode shapes of cylindrical tubes. Thus, we used the first two sine terms of Eq. (13) to approximate the mode shape for a vessel with pinned ends, i.e.,

$$u_c = C_1 \cdot \sin\left(\frac{\pi z}{l}\right) + C_2 \cdot \sin\left(\frac{2\pi z}{l}\right). \quad (25)$$

By taking Eqs. (24) and (25) into buckling equation (12) and ignoring the matrix support ($k_s = 0$), we obtained

$$p_{\text{cr}} = \frac{E_{\text{inc}} r_i t_0 \left(\frac{\pi}{l}\right)^2 \left[\left(1 - \frac{\alpha}{2}\right) \left(1 + 16 \left(\frac{C_2}{C_1}\right)^2\right) + \frac{128}{9} \frac{\alpha C_2}{\pi^2 C_1} \right]}{1 + 4 \left(\frac{C_2}{C_1}\right)^2} + \frac{N}{\pi r_i^2}. \quad (26)$$

The mode shape (C_2/C_1) that makes the p_{cr} minimal was determined by letting

$$\frac{\partial p_{\text{cr}}}{\partial (C_2/C_1)} = 0 \quad (27)$$

which yielded

$$\frac{C_2}{C_1} = \frac{27 \left(1 - \frac{\alpha}{2}\right) - \sqrt{27^2 \left(1 - \frac{\alpha}{2}\right)^2 + 64^2 (\alpha/\pi^2)^2}}{128\alpha/\pi^2}. \quad (28)$$

The critical pressure was then determined by taking Eq. (28) back into Eq. (26).

In addition, letting $C_2 = 0$ in Eq. (26) yielded

$$p_{\text{cr}} = E_{\text{inc}} r_i t_0 \left(\frac{\pi}{l}\right)^2 \left(1 - \frac{\alpha}{2}\right) + \frac{N}{\pi r_i^2}. \quad (29)$$

It is seen that gradual wall thinning along vessel length reduced the critical pressure of the vessel slightly.

Numerical simulations were performed for arteries with a lumen radius $r_i = 2.4$ mm, length $L = 25$ mm ($l = \lambda_z^0 L$), and wall thickness $t_0 = 0.268$ mm at the proximal end based on the rabbit aorta data and our previous simulations.^{3,13,14} The Young's modulus was estimated to be 100 kPa.¹³ Our results showed that tapering reduced the critical pressure slightly when using a single sine function ($C_2 = 0$) to approximate the mode shape (Fig. 3). When using a modified mode shape with two sine terms ($C_2 \neq 0$), we found that the C_2 term was actually very small compared to C_1 (Fig. 4, top) and the mode shapes changed very little from the base mode shape (Fig. 4, bottom). Accordingly, the critical pressure obtained from the buckling equation reduced slightly with the wall thickness tapering ratio, and the difference generated by including the C_2 terms was negligible (Fig. 5). These results demonstrated that tapered wall thickness had little effect on the mode shape and the critical pressure.

Critical Pressure of a Tapered Thick-Walled Nonlinear Artery

Arteries are commonly modeled as thick-walled nonlinear elastic tubes. Thus, we further considered an artery with tapered lumen radius and outer radius at its zero-stress state.

$$R_i(z) = R_{i0} \left(1 - \beta_i \frac{Z}{L} \right), \quad R_e(z) = R_{e0} \left(1 - \beta_e \frac{Z}{L} \right), \quad (30)$$

where

$$\beta_i = \frac{R_{i0} - R_{id}}{R_{i0}}, \quad \beta_e = \frac{R_{e0} - R_{ed}}{R_{e0}} \quad (31)$$

are the lumen and outer tapering ratios with R_{i0} , R_{e0} , and R_{id} , R_{ed} being the lumen and outer radius at the proximal and distal ends, respectively. For vessels with pinned ends without surrounding tissue support ($k_s = 0$), we used the first sine term of Eq. (13) to approximate the mode shape, i.e.,

$$u_c = C_1 \cdot \sin \left(\frac{\pi z}{l} \right). \quad (32)$$

Taking it into buckling energy equation (12), we have

$$p_{cr} = \frac{\int_0^l \left[\left(\frac{\pi}{l} \right)^2 EI \sin^2 \left(\frac{\pi z}{l} \right) + N \cos^2 \left(\frac{\pi z}{l} \right) \right] dz}{\pi r_i^2 \left(\frac{l}{2} \right)}. \quad (33)$$

Simulations were performed for rabbit arteries with $R_{i0} = 3.92$ mm, $R_{e0} = 4.52$ mm, opening angle $\Theta_0 = 71.4^\circ$, initial length $L = 25$ mm, and an axial stretch ratio of $\lambda_z^0 = 1.5$.^{3,13} The arterial wall was modeled as a nonlinear, incompressible, homogenous, orthotropic material with Fung strain energy function.^{8,20}

$$w_0 = \frac{1}{2} b_0 e^Q + K[(1+2E_r)(1+2E_\theta)(1+2E_z) - 1] \quad (34a)$$

with

$$Q = b_1 E_\theta^2 + b_2 E_z^2 + b_3 E_r^2 + 2b_4 E_\theta E_z + 2b_5 E_z E_r + 2b_6 E_r E_\theta \quad (34b)$$

Wherein K is a Lagrangian multiplier; E_θ , E_z , and E_r are the circumferential, axial, and radial Green strains, respectively; b_0 to b_6 are material property constants, with $b_0 = 22.40$ kPa, $b_1 = 1.067$, $b_2 = 0.4775$, $b_3 = 0.0499$, $b_4 = 0.0903$, $b_5 = 0.0585$, $b_6 = 0.0042$.^{3,13,14} For given R_i and R_e , the deformed dimensions (radius) were determined by solving the pressure equation (A1) in the Appendix and the corresponding N and EI were determined using Eqs. (A2) and (A3) given in the Appendix.

The critical pressure of the cylindrical vessel ($\beta_i = 0$, $\beta_e = 0$) was determined using Eq. (33) in conjunction with Eq. (A1) in the Appendix.¹⁴ Since Eq. (33) was derived for vessels with cylindrical lumens (r_i independent of z), we selected vessels with specific outer wall tapering ratios to achieve a constant lumen radius r_i at a given pressure. To this end, we used the trial and error approach to determine the distal lumen radius R_{id} (thus lumen tapering ratio β_i) and distal outer radius R_{ed} (thus outer tapering ratio β_e) to achieve a constant lumen radius for each given pressure using Eq. (A1) since the tapering was very small. We then used Eq. (33) to estimate the corresponding hypothetical “critical pressures” of these vessels. The vessel with the estimated critical pressure equal to the given pressure was selected as the tapered vessel since it has a constant lumen radius at the actual critical pressure. Using this inverse approach, we found the tapering ratios for a series of reduced critical pressure values. Simulation results are shown in Fig. 6. It is seen that tapering reduced the critical pressure of the vessels.

Discussion

This study established the energy equations for blood vessel buckling under internal pressure and axial stretch. We demonstrated that the differential equation of blood vessel buckling can be established using the energy approach, and the critical pressure can be estimated based on buckling mode shapes.

Blood vessels are generally considered stable under lumen pressure. However, we have recently demonstrated that blood vessels buckle when the lumen pressure reaches a critical value.^{12–16} Here, we further validated the buckling theory using the energy approach and demonstrated that the energy equation provides a very useful tool for determining the critical buckling load for cylindrical or tapered arteries. While the model results were not directly validated with experimental data in this study, the agreement between current results and previous model results^{12,14} that have been experimentally validated supports the conclusions from the current study.

Note that this study was aimed at determining the critical load that initiates artery buckling, not the post-buckling deformation. Similar to engineering columns, the critical load for the initiation of artery buckling can be determined within the scope of small deformation. Like engineering structures, the deflection can become large when the load increases beyond the critical pressure, and a buckled artery may develop into the large tortuosity observed

clinically. The development of large deflection “post-buckling” and tortuous vessels, however, is beyond the scope of this article.

While arteries were modeled as a nonlinear elastic anisotropic material, the surrounding matrix was modeled as a linear elastic material with constant stiffness k_s . Since the deflection is small at critical buckling, it is reasonable to model the matrix using the linear elastic approximation. In general, the stiffness should be interpreted as an equivalent linear elastic modulus for a nonlinear elastic matrix. In any regards, the use of constant or variable stiffness k_s would not change the conclusion that the critical pressure can be estimated through the energy approach.

Though the pressure load is generally considered to be a “follower force,” our results demonstrated that the work done by load q at a small deflection can be determined based on the deflection, independent of the path (see Eq. 4). Thus, the potential energy can be determined and used to determine the critical pressure of buckling.

One limitation of the study was that the pressure was assumed to be static. Arteries *in vivo* are under pulsatile pressure and flow. A dynamic analysis of artery buckling was given by Rachev and Gleason^{31,32}, and further work needs to be done to study artery buckling behavior under pulsatile pressure. Static models provide a good tool for estimating buckling pressures. It is similar to using the mean pressure to estimate the mean stress and strain in arteries, while the pulsatile pressure leads to cyclic stress and strain. Furthermore, the proposed energy equation can directly apply to small arteries and arterioles as well as veins, in which the pulse pressure is dampened down and the pressure can be approximated as a steady pressure.

For vessels with a small taper along its axis, our results demonstrated that the tapering reduced the buckling pressure slightly. For the vessels with similar dimensions in the two examples presented in this article, the linear elastic wall model predicted a higher critical pressure than the nonlinear elastic wall model, consistent with our previous analysis.¹³ The effect of tapering on the critical pressure predicted by the thick-walled nonlinear model was relatively larger than the effect predicted by the thin-walled linear model. One possible reason for this difference is that the first example, which used the thin-walled model, compared vessels of different wall thickness tapering at a constant lumen radius, while the second example, which used the thick-walled model, compared vessels of different initial lumen radius tapering. In addition to the tapered wall thickness, the lumen radius in the second example also decreased with increasing lumen tapering ratio due to reduced critical pressure. The decrease in lumen radius amplified the reduction in critical pressure.

While our models predicted a lower than physiologic critical pressure for rabbit arteries, arteries *in vivo* normally have a higher critical pressure due to higher axial stretch ratios and surrounding tissue support.^{13,14} Thus, arteries *in vivo* do not buckle under normal conditions. However, arteries do buckle and become tortuous under hypertensive pressure or even under normal physiological pressure due to reduction in axial tension or wall weakening with elastin degradation.^{6,21,35} The primary goal of this study was to establish the basic energy equation of blood vessel buckling. The buckling of tapered blood vessels with a significantly tapered lumen radius along its axis or with surrounding tissue support is beyond the scope of this article and needs further investigation. Some interesting numerical simulation results on the buckling of tapered arteries and the effect of surrounding tissue support can be found in our previous reports.^{5,14,26}

The current results broaden our understanding of vascular biomechanics and shed light on the stability of arteries and veins. A high prevalence of tortuosity has been observed in vessels with hypertensive pressure, diminishing axial stretch and other vascular diseases.

6,19,25,28 We demonstrated that arteries buckle under hypertensive pressure or reduced axial stretch ratios. The buckling would change lumen blood flow and wall stress, and result in remodeling of the arterial wall.^{27,30,33,36} Thus, mechanical buckling could be one of the possible mechanisms that leads to tortuous vessels. Understanding the biomechanics of blood vessel buckling may have wide implications in vascular physiology and pathology, as well as vascular surgery.

Acknowledgments

This work was supported by CAREER award 0644646 from the National Science Foundation, grant R01HL095852 from the National Institute of Health, and Grant 10928206 from NSF of China. The author thanks Professor Zhenbang Kuang and Dr. Fangsen Cui for their helpful comments on the manuscript.

Appendix

For cylindrical arteries with an orthotropic nonlinear elastic wall described by the Fung-type strain energy function given in Eq. (34a), integrating the equation of equilibrium and applying the boundary conditions yield the lumen pressure as:

$$p = \int_{r_i}^{r_e} [(1+2E_\theta)(b_1E_\theta + b_4E_z + b_6E_r) - (1+2E_r)(b_6E_\theta + b_5E_z + b_3E_r)] b_0 e^Q \frac{d\xi}{\xi} \quad (\text{A1})$$

and the axial tension N as

$$N = \pi r_i^2 p + \pi \int_{r_i}^{r_e} [2(1+2E_z)(b_4E_\theta + b_2E_z + b_5E_r) - (1+2E_r)(b_6E_\theta + b_5E_z + b_3E_r) - (1+2E_\theta)(b_1E_\theta + b_4E_z + b_6E_r)] b_0 e^Q r dr \quad (\text{A2})$$

When arteries buckle (bend), the flexural rigidity EI is given by^{13,14}

$$EI = H = \pi \int_{r_i}^{r_e} \left[J_z - \frac{1}{3} J_\theta \right] b_0 e^Q r^3 dr + \frac{\pi r_i^3}{3} \int_{r_i}^{r_e} [J_\theta] b_0 e^Q dr \quad (\text{A3})$$

wherein

$$\begin{aligned} J_z &= 2g_z[(1+2E_z)g_z - (1+2E_r)g_r] + 2g_z + (1+2E_z)b_2 - (1+2E_r)b_5 \\ J_\theta &= 2g_z[(1+2E_\theta)g_\theta - (1+2E_r)g_r] + (1+2E_\theta)b_4 - (1+2E_r)b_5 \end{aligned}$$

with

$$\begin{aligned} g_\theta &= b_1E_\theta + b_4E_z + b_6E_r \\ g_z &= b_4E_\theta + b_2E_z + b_5E_r \\ g_r &= b_6E_\theta + b_5E_z + b_3E_r \end{aligned} \quad (\text{A4})$$

Note that there was an error in one of the previous publications¹³ and the error has been corrected in the current equations (A3) and (A4).

References

1. Aoki T, Ku DN. Collapse of diseased arteries with eccentric cross section. *J Biomech* 1993;26(2): 133–142. [PubMed: 8429056]
2. Bazant, ZP.; Cedolin, L. *Stability of Structures*. Mineola, NY: Dove Publication, Inc; 1991.
3. Chuong CJ, Fung YC. On residual stresses in arteries. *J Biomech Eng* 1986;108(2):189–192. [PubMed: 3079517]
4. Cui F, Zhang Y. Comment on “A biomechanical model of artery buckling” published on *Journal of Biomechanics* (volume 40, issue 16, pages 3672–3678). *J Biomech* 2010;43(4):801–802. author reply 802–803. [PubMed: 19879584]
5. Datir P, Lee AY, Lamm SD, Han HC. Effects of geometric variations on the buckling of arteries. *Int J Appl Mech*. 2010
6. Del Corso L, Moruzzo D, Conte B, Agelli M, Romanelli AM, Pastine F, Protti M, Pentimone F, Baggiani G. Tortuosity, kinking, and coiling of the carotid artery: expression of atherosclerosis or aging? *Angiology* 1998;49(5):361–371. [PubMed: 9591528]
7. Drzewiecki G, Field S, Moubarak I, Li JK. Vessel growth and collapsible pressure–area relationship. *Am J Physiol* 1997;273(4 Pt 2):H2030–H2043. [PubMed: 9362275]
8. Fung, YC. *Biomechanics: Mechanical Properties of Living Tissues*. 2nd. New York: Springer Verlag; 1993.
9. Fung, YC. *Biomechanics: Circulation*. 2nd. Vol. Chap. 4. New York: Springer; 1997.
10. Fung YC, Liu SQ. Change of residual strains in arteries due to hypertrophy caused by aortic constriction. *Circ Res* 1989;65(5):1340–1349. [PubMed: 2805247]
11. Gere, JM. *Mechanics of Materials*. 6th. Vol. Chap. 11. Belmont, CA: Brooks/Cole; 2004.
12. Han HC. A biomechanical model of artery buckling. *J Biomech* 2007;40(16):3672–3678. [PubMed: 17689541]
13. Han HC. Nonlinear buckling of blood vessels: a theoretical study. *J Biomech* 2008;41(12):2708–2713. [PubMed: 18653191]
14. Han HC. Blood vessel buckling within soft surrounding tissue generates tortuosity. *J Biomech* 2009;42(16):2797–2801. [PubMed: 19758591]
15. Han HC. The mechanical buckling of curved arteries. *Mol Cell Biomech* 2009;6(2):93–100. [PubMed: 19496257]
16. Han HC. The theoretical foundation for artery buckling under internal pressure. *J Biomech Eng* 2009;131(12):124501. [PubMed: 20524735]
17. Han HC, Fung YC. Species dependence of the zero-stress state of aorta: pig versus rat. *J Biomech Eng* 1991;113(4):446–451. [PubMed: 1762442]
18. Han HC, Fung YC. Direct measurement of transverse residual strains in aorta. *Am J Physiol* 1996;270(2 Pt 2):H750–H759. [PubMed: 8779853]
19. Hiroki M, Miyashita K, Oda M. Tortuosity of the white matter medullary arterioles is related to the severity of hypertension. *Cerebrovasc Dis* 2002;13(4):242–250. [PubMed: 12011548]
20. Humphrey, JD. *Cardiovascular Solid Mechanics: Cells, Tissues and Organs*. New York: Springer-Verlag; 2009.
21. Jackson ZS, Dajnowiec D, Gotlieb AI, Langille BL. Partial off-loading of longitudinal tension induces arterial tortuosity. *Arterioscler Thromb Vasc Biol* 2005;25(5):957–962. [PubMed: 15746437]
22. Kuang, ZB. *Fundamental of Nonlinear Continuum Mechanics*. Xi'an: Xi'an Jiaotong University Press; 1989.
23. Lee A, Han HC. A thin-walled nonlinear model for vein buckling. *Cardiovasc Eng Technol* 2010;1(4):282–289.
24. Martinez R, Fierro CA, Shireman PK, Han HC. Mechanical buckling of veins under internal pressure. *Ann Biomed Eng* 2010;38(4):1345–1353. [PubMed: 20094913]
25. Nichols, WW.; O'Rourke, MF. *McDonald's Blood Flow in Arteries: Theoretical, Experimental, and Clinical Principles*. 4th. Vol. Chap. 16. London: Arnold Publisher; 1998.

26. Northcutt, A.; Datir, P.; Han, HC. Computational simulations of buckling of oval and tapered arteries. In: Chien, S., editor. *Tributes to Yuan-Cheng Fung on His 90th Birthday: From Cell to Human*. Singapore: World Scientific Publishing Co.; 2009. p. 53-64.
27. Northcutt, A.; Gaddy, A.; Feng, Y.; Han, HC. A numerical study of blood flow in sinusoidal-shaped arteries. *Biomedical Engineering Society (BMES) Annual Fall Meeting*; St Louis, MO. 2008.
28. Pancera P, Ribul M, Presciuttini B, Lechi A. Prevalence of carotid artery kinking in 590 consecutive subjects evaluated by Echocolor Doppler. Is there a correlation with arterial hypertension? *J Intern Med* 2000;248(1):7–12. [PubMed: 10947875]
29. Pedley TJ, Luo XY. Modelling flow and oscillations in collapsible tubes. *Theor Comput Fluid Dyn* 1998;10:277–294.
30. Qiao AK, Guo XL, Wu SG, Zeng YJ, Xu XH. Numerical study of nonlinear pulsatile flow in S-shaped curved arteries. *Med Eng Phys* 2004;26(7):545–552. [PubMed: 15271282]
31. Rachev A. A theoretical study of mechanical stability of arteries. *J Biomech Eng* 2009;131(5):051006. [PubMed: 19388776]
32. Rachev, A.; Gleason, R. Dynamic instability of arteries. Effects of perivascular tissue. *ASME Summer Bioengineering Conference*; Lake Ta Hoe, CA. 2009.
33. Smedby O, Bergstrand L. Tortuosity and atherosclerosis in the femoral artery: what is cause and what is effect? *Ann Biomed Eng* 1996;24(4):474–480. [PubMed: 8841722]
34. Timoshenko, SP.; Gere, JM. *Theory of Elastic Stability*. 2nd. New York: McGraw-Hill; 1963.
35. Wagenseil JE, Nerurkar NL, Knutsen RH, Okamoto RJ, Li DY, Mecham RP. Effects of elastin haploinsufficiency on the mechanical behavior of mouse arteries. *Am J Physiol Heart Circ Physiol* 2005;289(3):H1209–H1217. [PubMed: 15863465]
36. Wood NB, Zhao SZ, Zambanini A, Jackson M, Gedroyc W, Thom SA, Hughes AD, Xu XY. Curvature and tortuosity of the superficial femoral artery: a possible risk factor for peripheral arterial disease. *J Appl Physiol* 2006;101(5):1412–1418. [PubMed: 16825527]

FIGURE 1.

Top Schematic illustration of a cylindrical vessel under internal pressure p and axial tension N . *Bottom* Schematic illustration of the definitions of the initial coordinates (R, Θ, Z) at the zero-stress state and deformed coordinates (r, ϕ, z) at the loaded state. The deformed lumen radius, r_i , outer radius, r_e , and vessel length, l , are also illustrated.

FIGURE 2. Schematics illustrating the deflection $u_c(z)$ and the pressure-generated distributed lateral load $q(z)$ in a buckled blood vessel.

FIGURE 3.

Critical pressure plotted as a function of wall thickness tapering ratio α in blood vessels with a constant lumen radius but tapered wall thickness (wall thinning). The *dashed line* and *solid line* are for an axial stretch ratio of 1.3 and 1.5, respectively. The vessel wall thickness to lumen radius ratio was 0.11 and Young's modulus was 100 kPa.

FIGURE 4.

Buckling mode shape of arteries with tapered wall thickness. The C_2/C_1 ratio, which indicates the change in the mode shape plotted as a function of the wall thickness tapering ratio (*top*) and the corresponding mode shapes at different levels of wall thickness tapering ratio (*bottom*). The mode shape shifted slightly towards the distal end with increasing wall thickness tapering ratio.

FIGURE 5.

Comparison of the critical pressure plotted as functions of the wall thickness tapering ratio with and without considering the changes in mode shape ($C_2 \neq 0$ and $C_2 = 0$, respectively). The axial stretch ratio was 1.5 and the vessel dimensions are the same as given in Fig. 3.

FIGURE 6.

The effect of tapering on the critical pressure of thick-walled arteries with a nonlinear elastic Fung strain energy function. The outer radius tapering ratio (*Top*), the corresponding wall thickness tapering ratio (*middle*), and critical pressure (*bottom*) plotted as functions of the lumen radius tapering ratio.