Congruences for the Andrews spt function

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Ramanujan-type congruences for the Andrews $spt(n)$ partition function have been found for prime moduli $5 \leq \ell \leq 37$ in the work of Andrews [Andrews GE, (2008) *J Reine Angew Math* 624:133–142] and Garvan [Garvan F, (2010) *Int J Number Theory* 6:1–29]. We exhibit unexpectedly simple congruences for all $\ell \geq 5$. Confirming a conjecture of Garvan, we show that if $\ell \geq 5$ is prime and $(\frac{-\ell}{\ell}) = 1$, then $spt[\frac{\ell^2(\ell n + \delta) + 1}{24}] \equiv 0 \pmod{\ell}$. This congruence gives $(\ell - 1)/2$ arithmetic progressions modulo ℓ^3 which support a mod ℓ congruence. This result follows from the surprising fact that the reduction of a certain mock theta function modulo ℓ , for every $\ell \geq 5$, is an eigenform of the Hecke operator $T(\ell^2)$.

harmonic Maass form | partitions

1. Introduction and Statement of Results

Andrews (1) recently introduced the function $spt(n)$, which counts the number of smallest parts among the integer partitions of n . For $n = 4$ we have

$$4, \quad 3 + \underline{1}, \quad \underline{2} + \underline{2}, \quad 2 + \underline{1} + \underline{1}, \quad \underline{1} + \underline{1} + \underline{1} + \underline{1}.$$

The smallest parts are underlined, and so we have that $spt(4) = 10$. Andrews (1) proved the following elegant Ramanujan-type congruences:

$$\begin{aligned} spt(5n + 4) &\equiv 0 \pmod{5}, & spt(7n + 5) &\equiv 0 \pmod{7}, \\ spt(13n + 6) &\equiv 0 \pmod{13}. \end{aligned}$$

Inspired by these congruences, Garvan (2) has obtained several new congruences. Recently, Folsom and Ono (3) (see also ref. 4) confirmed conjectures of Garvan and Sellers, and these results provide simple congruences modulo 2 and 3.

The situation is more complicated for primes $\ell \geq 5$. It is known that there are infinitely many congruences of the form

$$spt(an + b) \equiv 0 \pmod{\ell}.$$

This fact follows from work of Bringmann (5) (see also refs. and 7) on $N_2(n)$, the second rank moment, combined with earlier work of Ahlgren and Ono on $p(n)$ (8–10). However, explicit examples are only known for $\ell \leq 37$. For example, Garvan (2) has obtained

$$\begin{aligned} spt(19^4 \cdot 11 \cdot n + 22,006) &\equiv 0 \pmod{11}, \\ spt(7^4 \cdot 17 \cdot n + 243) &\equiv 0 \pmod{17}, \\ spt(5^4 \cdot 19 \cdot n + 99) &\equiv 0 \pmod{19}, \\ spt(13^4 \cdot 29 \cdot n + 18,583) &\equiv 0 \pmod{29}. \end{aligned}$$

The moduli of the arithmetic progressions above involve (fourth) powers of special auxiliary primes, a feature shared by the congruences which arise from this theory. The congruences are constructed using these special primes, and these primes are guaranteed to exist by the theory of odd modular ℓ -adic Galois representations and the Chebotarev density theorem. To find a congruence, one is then required to search, prime by prime, for an auxiliary prime. This task is analogous to the simpler problem of finding the smallest prime $p \equiv 1 \pmod{\ell}$.

We establish unique universal congruences for $spt(n)$ without relying on the existence of such primes. For aesthetics, we define $\hat{s}(n)$ and $\hat{p}(n)$ by

$$\mathcal{S}(q) = \sum_{n=0}^{\infty} \hat{s}(n)q^n := \sum_{n=1}^{\infty} spt(n)q^{24n-1}, \quad [1]$$

$$\mathcal{P}(q) = \sum_{n=-1}^{\infty} \hat{p}(n)q^n := \sum_{n=0}^{\infty} p(n)q^{24n-1}. \quad [2]$$

We obtain the following congruences relating $\hat{s}(n)$, $\hat{p}(n)$, and the Legendre symbol $(\frac{\cdot}{\ell})$.

Theorem 1.1. If $\ell \geq 5$ is prime, then

$$\hat{s}(\ell^2 n) \equiv \left(\frac{3}{\ell}\right) \left[1 - \left(\frac{-n}{\ell}\right)\right] \cdot \left[\hat{s}(n) + \frac{n}{12} \cdot \hat{p}(n)\right] \pmod{\ell}.$$

Remark: Theorem 1.1 may be reformulated in terms of the “mock theta function”

$$\begin{aligned} M(q) := \mathcal{S}(q) + \frac{1}{12} \cdot q \frac{d}{dq} \mathcal{P}(q) &= -\frac{1}{12} \cdot q^{-1} + \frac{35}{12} \cdot q^{23} \\ &\quad + \frac{65}{6} \cdot q^{47} + \dots \end{aligned} \quad [3]$$

We refer to $M(q)$ as a mock theta function because it is the holomorphic part of a harmonic Maass form. Although $M(q)$ is not an eigenform of any Hecke operators, Theorem 1.1 is equivalent to the assertion, for every prime $\ell \geq 5$ that

$$M(q)|T(\ell^2) \equiv \left(\frac{3}{\ell}\right) \cdot M(q) \pmod{\ell}.$$

Theorem 1.1 immediately gives the following corollary.

Corollary 1.2. Suppose that $\ell \geq 5$ is prime. Then the following are true

1. If $(\frac{-n}{\ell}) = 1$, then

$$\hat{s}(\ell^2 n) \equiv 0 \pmod{\ell}.$$

2. We have that

$$\hat{s}(\ell^3 n) \equiv \left(\frac{3}{\ell}\right) \hat{s}(\ell n) \pmod{\ell}.$$

Remark: Corollary 1.2 (1) gives distinct $0 < b_{\ell}(1) < \dots < b_{\ell}\left(\frac{\ell-1}{2}\right) < \ell^3$ for which

$$spt[\ell^3 n + b_{\ell}(m)] \equiv 0 \pmod{\ell}.$$

Indeed, if $(\frac{-\delta}{\ell}) = 1$, then Corollary 1.2 (1) implies that

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*The spt -congruence in your paper was originally conjectured by me. This conjecture was inspired by work of Tina Garrett's students.

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$$spt\left[\frac{\ell^2(\ell n + \delta) + 1}{24}\right] \equiv 0 \pmod{\ell}.$$

These congruences were conjectured by Garvan in July 2008*. Garvan's conjecture was inspired by work done by Garrett and her students in October 2007. For $\ell = 11$ the general result gives the five congruences:

$$\begin{aligned} spt(11^3n + 479) &\equiv spt(11^3n + 842) \equiv spt(11^3n + 1,084) \\ &\equiv spt(11^3n + 1,205) \equiv spt(11^3n + 1,326) \equiv 0 \pmod{11}. \end{aligned}$$

In Section 2 we prove Theorem 1.1 and Corollary 1.2 using work of Bringmann (5), and of Bruinier and Ono (11). In Section 3 we conclude with several illuminating examples.

2. Proofs

We assume that the reader is familiar with basic facts about modular forms and harmonic Maass forms [for background, see refs. (12–14)]. In ref. 1, Andrews obtained the following generating function for $spt(n)$:

$$\begin{aligned} q^{\frac{1}{24}} \mathcal{S}(q^{\frac{1}{24}}) &= \sum_{n=1}^{\infty} spt(n) q^n = \frac{1}{(q)_{\infty}} \cdot \sum_{n=1}^{\infty} \frac{q^n \cdot \prod_{m=1}^{n-1} (1 - q^m)}{1 - q^n} \\ &= q + 3q^2 + 5q^3 + \dots, \end{aligned} \quad [4]$$

where q is a formal parameter and $(q)_{\infty} = \prod_{n=1}^{\infty} (1 - q^n)$. If we let $q := e^{2\pi i z}$, where z is in the upper half of the complex plane, then we have the following important theorem† of Bringmann (5) which relates this generating function to a certain harmonic Maass form.

Theorem 2.1. Define the function $\mathcal{M}(z)$ by

$$\mathcal{M}(z) := \mathcal{S}(q) - \frac{\mathcal{D}(24z)}{12} - \frac{i}{4\pi\sqrt{2}} \cdot \int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{[-i(\tau + z)]^{\frac{3}{2}}} \cdot d\tau,$$

where $\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is Dedekind's eta-function, and where

$$\mathcal{D}(24z) := \frac{1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^{24n}}{\eta(24z)}.$$

Then $\mathcal{M}(z)$ is a weight $3/2$ harmonic Maass form on $\Gamma_0(576)$ with Nebentypus $\chi_{12}(\bullet) := (\frac{\bullet}{12})$.

2.1 Producing Modular Forms. We use Theorem 2.1 to obtain modular forms from the harmonic Maass form $\mathcal{M}(z)$. By the q -series manipulations in ref. 1 and [2], it is known that

$$\begin{aligned} q \frac{d}{dq} \mathcal{P}(q) &= -\mathcal{D}(24z) = -\frac{1}{q} + 23q^{23} + 94q^{47} + 213q^{71} \\ &\quad + 475q^{95} + 833q^{119} + \dots \end{aligned}$$

Therefore, [1] and [3] imply that $M(z) = M(q)$ is the holomorphic part of $\mathcal{M}(z)$, and so it is a mock theta function.

For each prime $\ell \geq 5$, we let $T(\ell^2)$ be the index ℓ^2 Hecke operator for weight $3/2$ harmonic Maass forms with Nebentypus χ_{12} . On q series, these operators are defined by

$$\begin{aligned} \left[\sum a(n) q^n \right] | T(\ell^2) &:= \sum \left[a(\ell^2 n) + \left(\frac{3}{\ell} \right) \left(\frac{-n}{\ell} \right) a(n) \right. \\ &\quad \left. + \ell a(n/\ell^2) \right] q^n. \end{aligned} \quad [5]$$

We define $M_{\ell}(z)$ by

$$\begin{aligned} M_{\ell}(z) &= M_{\ell}(q) = \sum a_{\ell}(n) q^n \\ &:= M(q) | T(\ell^2) - \left(\frac{3}{\ell} \right) (1 + \ell) M(q). \end{aligned} \quad [6]$$

The following theorem is crucial to the proof of Theorem 1.1.

Theorem 2.2. Suppose that $\ell \geq 5$ is prime, and that

$$F_{\ell}(z) := \eta(z)^{\ell^2} \cdot M_{\ell}(z/24).$$

Then $F_{\ell}(z)$ is a weight $(\ell^2 + 3)/2$ holomorphic modular form on $\text{SL}_2(\mathbb{Z})$.

Proof: The operator $\xi := 2iy^{\frac{3}{2}} \cdot \frac{\partial}{\partial z^2}$, where $y = \text{Im}(z)$, has the property that $\xi(\mathcal{M}) = -\frac{1}{8\pi} \cdot \eta(24z)$. Because $\eta(24z)$ is an eigenform of the weight $1/2$ Hecke operators, Lemma 7.4 of ref. 11 implies that $M_{\ell}(z)$ is a weight $3/2$ weakly holomorphic modular form on $\Gamma_0(576)$ with Nebentypus χ_{12} . Here we used the fact that the eigenvalue of $\eta(24z)$ for the index ℓ^2 weight $1/2$ Hecke operator is $\chi_{12}(\ell)(1 + \ell^{-1})$.

It is straightforward to check that $M_{\ell}(z)$ has coefficients in $\frac{1}{12}\mathbb{Z}$, and has the property that

$$M_{\ell}(z) = -\frac{\ell}{12} \cdot q^{-\ell^2} + \left(\frac{3}{\ell} \right) \cdot \frac{\ell}{12} \cdot q^{-1} + \sum_{\substack{n \geq 23 \\ n \equiv 23 \pmod{24}}} a_{\ell}(n) q^n. \quad [7]$$

Here we have used the fact that $\ell^2 \equiv 1 \pmod{24}$. Therefore, it follows that

$$F_{\ell}(24z) = \eta(24z)^{\ell^2} M_{\ell}(z) = -\frac{\ell}{12} + \dots$$

is a weight $(\ell^2 + 3)/2$ weakly holomorphic modular form on $\Gamma_0(576)$ with trivial Nebentypus whose nonzero coefficients are supported on exponents which are multiples of 24. In particular, we have that $F_{\ell}(z) = F_{\ell}(z + 1)$. To prove that $F_{\ell}(z)$ is a weakly holomorphic modular form on $\text{SL}_2(\mathbb{Z})$, it suffices to prove that

$$F_{\ell}(-1/z) = z^{\frac{\ell^2+3}{2}} F_{\ell}(z).$$

To this end, let W be the Fricke involution (see ref. 13, section 3.2) which acts on weight $3/2$ modular forms on $\Gamma_0(576)$ by

$$(f|W)(z) := (\sqrt{24} \cdot \sqrt{-iz})^{-3} \cdot f\left(-\frac{1}{576z}\right).$$

If f has Nebentypus χ , and if $\ell \nmid 576$ is prime, then it is well known that

$$f|W|T(\ell^2) = \overline{\chi(\ell^2)} \cdot f|T(\ell^2)|W.$$

If we let $A_{\ell}(z) := F_{\ell}(24z)$, then this commutation relation implies that

*Theorem 2.1 corrects a sign error in ref. 5.

$$\begin{aligned}
T_{11}^{(3)}(q) = & 3,421,567,149,001,730,876,538,911,832q^{13} \\
& + 721,427,557,133,531,761,496,593,371,848,380, \\
& \quad 785,660,101,905,536q^{37} \\
& + 120,494,776,849,783,345,014,198,876,429,157,577, \\
& \quad 016,120,072,623,960,718,684,904,344q^{61} \\
& + \dots \equiv 4q^{13} + 3q^{37} + 7q^{61} + \dots \pmod{11}.
\end{aligned}$$

1. Andrews GE (2008) The number of smallest parts in the partitions of n . *J Reine Angew Math* 624:133–142.
2. Garvan F (2010) Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank. *Int J Number Theory* 6:1–29.
3. Folsom A, Ono K (2008) The spt -function of Andrews. *Proc Natl Acad Sci USA* 105:20152–20156.
4. Bringmann K, Folsom A, Ono K (2009) q -series and weight $3/2$ Maass forms. *Compos Math* 145:541–552.
5. Bringmann K (2008) On the explicit construction of higher deformations of partition statistics. *Duke Math J* 144:195–233.
6. Bringmann K, Garvan F, Mahlburg K (2009) Partition statistics and quasiweak Maass forms. *Int Math Res Notices* 1:63–97.
7. Bringmann K, Ono K (2010) Dyson's ranks and Maass forms. *Ann Math* 171:419–499.
8. Ahlgren S (2000) Distribution of the partition function modulo composite integers. *M. Math Ann* 318:795–803.
9. Ahlgren S, Ono K (2001) Congruence properties for the partition function. *Proc Natl Acad Sci USA* 98:12882–12889.
10. Ono K (2000) Distribution of the partition function modulo m . *Ann Math* 151:293–307.
11. Bruinier JH, Ono K Heegner divisors, L -functions, and harmonic weak Maass forms. *Ann Math*, 172 pp:2135–2181.
12. Bruinier JH, Funke J (2004) On two geometric theta lifts. *Duke Math J* 125:45–90.
13. Ono K (2004) *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and Q -series*, Conference Board of the Mathematical Sciences (American Mathematical Society, Providence, RI), 102.
14. Ono K (2008) Unearthing the visions of a master: Harmonic Maass forms and number theory. *Proceedings of the 2008 Harvard-MIT Current Developments in Mathematics Conference* (International Press, Somerville, MA), pp 347–454.

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