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Probabilities for separating sets of order statistics

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Abstract

Consider a set of order statistics that arise from sorting samples from two different populations, each with their own, possibly different distribution functions. The probability that these order statistics fall in disjoint, ordered intervals and that of the smallest statistics, a certain number come from the first populations is given in terms of the two distribution functions. The result is applied to computing the joint probability of the number of rejections and the number of false rejections for the Benjamini–Hochberg false discovery rate procedure.

Keywords

Benjamini and Hochberg procedure; block matrix; permanent; multiple comparison

1. Introduction

Glueck *et al.* [1] gave explicit expressions for the probability that arbitrary subsets of order statistics fall in disjoint, ordered intervals on the set of real numbers. In this paper, we extend this work and consider two sets of real-valued, independent but not necessarily identically distributed random variables. We give expressions in terms of cumulative distribution functions for the probability that arbitrary subsets of order statistics fall in disjoint, ordered intervals and that of the smallest statistics, a certain number come from one set. We have been unable to find any previous papers on this topic. This problem is of interest in calculating probabilities for the Benjamini and Hochberg [2] multiple comparisons procedure.

2. A simple example

Consider the following simple example. Let $X_1, X_2 \in [0, 1]$ be independent random variables. Denote by $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ the marginal cumulative distribution functions and by $F_{X_1, X_2}(x_1, x_2)$ the joint cumulative distribution function of X_1 and X_2 . Assume that the cumulative distribution functions are continuous. Let $Y_1 = \min\{X_1, X_2\}$ and let $Y_2 = \max\{X_1, X_2\}$ be the order statistics. For $i = 1, 2$, write the marginal cumulative distribution function of Y_i as $F_{Y_i}(y_i)$ and the joint cumulative distribution function as $F_{Y_1, Y_2}(y_1, y_2)$, for $y_1 \leq y_2$. This joint cumulative distribution function is also continuous [3, p. 10].

Choose numbers $b_1 < b_2$, $b_1, b_2 \in (0, 1)$. We wish to find the probabilities

$$\alpha = \Pr\{(y_1 < b_1) \wedge (y_2 > b_2)\}, \quad (1)$$

$$\beta = \Pr\{(y_1 < b_1) \wedge (y_2 > b_2) \wedge (x_1 < b_1)\} \quad (2)$$

and

$$\gamma = \Pr\{(y_1 < b_1) \wedge (y_2 > b_2) \wedge \neg(x_1 < b_1)\}. \quad (3)$$

and express them in terms of the distribution functions F_{X_1} and F_{X_2} . First, we will find the probabilities directly. So,

$$\begin{aligned} \beta &= \Pr\{(x_1 < b_1) \wedge (x_2 > b_2)\} \\ &= F_{X_1}(b_1)[1 - F_{X_2}(b_2)] \end{aligned} \quad (4)$$

and

$$\begin{aligned} \gamma &= \Pr\{(x_1 > b_2) \wedge (x_2 < b_1)\} \\ &= [1 - F_{X_1}(b_2)]F_{X_2}(b_1). \end{aligned} \quad (5)$$

Equations (4) and (5) follow directly from the independence of the random variables and the definition of the cumulative distribution functions. Since

$$\begin{aligned} \{(y_1 < b_1) \wedge (y_2 > b_2)\} &= \{(y_1 < b_1) \wedge (y_2 > b_2) \wedge (x_1 < b_1)\} \\ &\quad \cup \{(y_1 < b_1) \wedge (y_2 > b_2) \wedge \neg(x_1 < b_1)\} \end{aligned} \quad (6)$$

and the union is disjoint, it follows that

$$\alpha = \beta + \gamma. \quad (7)$$

For a problem with more than two order statistics, the number of cases one needs to consider and the number of possible combinations of statistics, subsets, and bounds makes a direct approach impractical. An algorithmic approach to obtaining γ and β will allow the generalization to an arbitrary number of order statistics.

Using the assumption that the distribution functions are continuous, simple set operations, and the definition of distribution function, we obtain that the probability of the union (6) is

$$\mathcal{A} = \Pr\{(y_1 < b_1) \wedge \neg(y_2 < b_2)\} \quad (8)$$

$$= \Pr\{y_1 < b_1\} - \Pr\{(y_1 < b_1) \wedge (y_2 < b_2)\} \quad (9)$$

$$= F_{Y_1}(b_1) - F_{Y_1, Y_2}(b_1, b_2). \quad (10)$$

The cumulative distributions of the order statistics can be written [4],

$$F_{Y_1}(b_1) = F_{X_1}(b_1)F_{X_2}(b_1) + F_{X_1}(b_1)[1 - F_{X_2}(b_1)] + [1 - F_{X_1}(b_1)]F_{X_2}(b_1) \quad (11)$$

$$F_{Y_1, Y_2}(b_1, b_2) = F_{X_1}(b_1)[F_{X_2}(b_2) - F_{X_2}(b_1)] + [F_{X_1}(b_2) - F_{X_1}(b_1)]F_{X_2}(b_1) + F_{X_1}(b_1)F_{X_2}(b_1). \quad (12)$$

Then, substituting Equations (11) and (12) into Equation (10), we can write $\mathcal{A}$ in terms of the distribution functions of X_1 and X_2 ,

$$\begin{aligned} \mathcal{A} = & F_{X_1}(b_1)[1 - F_{X_2}(b_1)] + [1 - F_{X_1}(b_1)]F_{X_2}(b_1) \\ & - F_{X_1}(b_1)[F_{X_2}(b_2) - F_{X_2}(b_1)] - [F_{X_1}(b_2) - F_{X_1}(b_1)]F_{X_2}(b_1) \end{aligned} \quad (13)$$

$$= F_{X_1}(b_1)[1 - F_{X_2}(b_2)] + [1 - F_{X_1}(b_2)]F_{X_2}(b_1). \quad (14)$$

We now interpret the terms in the sum in Equation (14). The term that includes $F_{X_1}(b_1)$ as a factor is the probability of an event in which $x_1 < b_1$ occurs, and the term that includes $1 - F_{X_1}(b_2)$ as a factor is the probability of an event in which $x_1 > b_2$. Since $b_1 < b_2$, the two events are disjoint, and, consequently, Equation (7) follows again.

To summarize, we have expressed the probability in terms of the joint distribution of the order statistics, which was in turn written in terms of the distribution functions of the random variables. Finally, by recognizing terms that corresponded to a partition, we decomposed $\mathcal{A}$ into a sum of β and γ , the two probabilities of interest.

3. General case

The logic used in this simple, two random variables example can be generalized to an arbitrary number of random variables. Consider a set of order statistics that arise from sorting samples from two different populations, each with their own, possibly different distribution function. We wish to find the probability that these order statistics fall in a given union of intervals and that of the smallest statistics, a certain number come from one population.

For this general case, we need to introduce some notation and definitions. Let X_i , $i = 1, \dots, m$, be independent but not necessarily identically distributed real-valued random variables with

values in the interval $[0, 1]$ and continuous cumulative distribution functions $F_{X_i}(x_i)$. Partition the set $\{X_1, X_2, \dots, X_m\}$ into two subsets,

$$S_1 = \{X_1, X_2, \dots, X_n\}, \quad S_2 = \{X_{n+1}, X_{n+2}, \dots, X_m\}. \quad (15)$$

For example, one can consider measurements for males or females, or for two different populations of breast cancer, slow or fast growing. The order statistics $Y_1, Y_2, \dots, Y_m$ are random variables defined by sorting the values of X_i . Thus $Y_1 \leq Y_2 \leq \dots \leq Y_m$. Denote the realizations of the order statistics by $y_1 \leq y_2 \leq \dots \leq y_m$.

The arguments of the joint cumulative distribution function of order statistics are customarily written omitting redundant arguments; thus for $1 \leq e \leq m$ let $1 \leq n_1 < n_2 < \dots < n_e \leq m$, denote the indices of the order statistics of interest. The joint cumulative distribution function of the set $\{Y_{n_1}, Y_{n_2}, \dots, Y_{n_e}\}$, which is a subset of the complete set of order statistics, is defined as

$$F_{Y_{n_1}, \dots, Y_{n_e}}(y_1, \dots, y_e) = \Pr(\{Y_{n_1} \leq y_1\} \cap \{Y_{n_2} \leq y_2\} \cap \dots \cap \{Y_{n_e} \leq y_e\}). \quad (16)$$

Suppose we are given $s \leq m$ disjoint intervals

$$(c_q, d_q), \quad 0 = c_1 < d_1 < c_2 < \dots < c_s < d_s = 1, \quad (17)$$

and integers

$$k_q \geq 0, \quad \sum_{q=1}^s k_q = m, \quad (18)$$

where $k_0 = 0$ and k_q is the number of order statistics that fall in the q th interval. Define

$w_{q,1} = 1 + \sum_{i=1}^{q-1} k_i$, and $w_{q,k_q} = \sum_{i=1}^q k_i$ to be the subscripts of the largest and smallest order statistics, respectively, that fall in the q th interval. In the case when $k_q = 1$, we have $w_{q,1} = w_{q,k_q}$. Using this notation, the event that exactly k_q of the order statistics fall in the q th interval is

$$\{c_1 < Y_{w_{1,1}} < \dots < Y_{w_{1,k_1}} < d_1 \wedge \dots \wedge c_s < Y_{w_{s,1}} < \dots < Y_{w_{s,k_s}} < d_s\}, \quad (19)$$

or in a more compact notation (21) given below. Now let B be another random event. The following theorem gives the probability of this event intersected with the event (19), in terms of the cumulative distribution functions of the order statistics relative to the event B . This distribution function is defined by

$$F_{Y_{n_1}, \dots, Y_{n_e}; B}(y_1, \dots, y_e) = \Pr(\{Y_{n_1} \leq y_1\} \cap \{Y_{n_2} \leq y_2\} \cap \dots \cap \{Y_{n_e} \leq y_e\} \cap B). \quad (20)$$

Contrary to the usual convention, we do not require that the indices of the order statistics in the cumulative distribution function (20) are sorted, because that would result in a complication of the notation in the next theorem (additional renumbering of the arguments).

THEOREM 1 *Denote the event*

$$E = \bigcap_{q=1}^s (\{c_q < Y_{w_{q,1}}\} \cap \{Y_{w_{q,k_q}} < d_q\}). \quad (21)$$

Then

$$\begin{aligned} \Pr(E \cap B) = & F_{Y_{w_{1,k_1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,k_s}}; B}(d_1, d_2, \dots, d_q) \\ & - \sum_{q=1}^s F_{Y_{w_{1,k_1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,k_s}}, Y_{w_{q,k_q}}; B}(d_1, d_2, \dots, d_q, c_q) \\ & + \sum_{\substack{r,t=1 \\ r < t}}^s F_{Y_{w_{1,k_1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,k_s}}, Y_{w_{r,1}}, Y_{w_{t,1}}; B}(d_1, d_2, \dots, d_q, c_r, c_t) \\ & \vdots \\ & + (-1)^s F_{Y_{w_{1,1}}, Y_{w_{1,k_1}}, Y_{w_{2,1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,1}}, Y_{w_{s,k_s}}; B}(c_1, d_1, c_2, d_2, \dots, c_s, d_s). \end{aligned} \quad (22)$$

Proof By standard set operations,

$$E = \bigcap_{q=1}^s \{c_q < Y_{w_{q,1}}\} \cap \bigcap_{q=1}^s \{Y_{w_{q,k_q}} < d_q\} \quad (23)$$

and

$$\bigcap_{q=1}^s \{c_q < Y_{w_{q,1}}\} = \bigcap_{q=1}^s \{Y_{w_{q,1}} \leq c_q\}^C = \left(\bigcup_{q=1}^s \{Y_{w_{q,1}} \leq c_q\} \right)^C, \quad (24)$$

where C denotes the complement. Therefore,

$$E \cap B = \left(\bigcup_{q=1}^s \{Y_{w_{q,1}} \leq c_q\} \right)^C \cap F, \quad (25)$$

where the event F is defined by

$$F = \bigcap_{q=1}^s \{Y_{w_{q,k_q}} < d_q\} \cap B. \quad (26)$$

By the additivity of probability, it follows Equation from (25) that

$$\Pr(E \cap B) = \Pr(F) - \Pr\left(\bigcup_{q=1}^s \{Y_{w_{q,1}} \leq c_q\} \cap F\right) = \Pr(F) - \Pr\left(\bigcup_{q=1}^s A_q\right), \quad (27)$$

where $A_q = \{Y_{w_{q,1}} \leq c_q\} \cap F$. Using the additivity of probability again, we have

$$\Pr\left(\bigcup_{q=1}^s A_q\right) = \sum_{q=1}^s \Pr(A_q) - \sum_{\substack{r,t=1 \\ r < t}}^s \Pr(A_r \cap A_t) + \dots \quad (28)$$

$$+(-1)^s \Pr \left(\bigcap_{q=1}^s A_q \right) \quad (29)$$

Now putting Equations (26)–(29) together and using the continuity of the cumulative distribution functions, we obtain

$$\begin{aligned} \Pr(E \cap B) = & \underbrace{\Pr \left(\bigcap_{q=1}^s \{Y_{w_{q,k_q}} \leq d_q\} \cap B \right)}_{F_{Y_{w_{1,k_1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,k_s}}; B}(d_1, d_2, \dots, d_s)} \\ & - \sum_{r=1}^s \underbrace{\Pr \left(\{Y_{w_{r,1}} \leq c_r\} \cap \bigcap_{q=1}^s \{Y_{w_{q,k_q}} \leq d_q\} \cap B \right)}_{F_{Y_{w_{1,k_1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,k_s}}, Y_{w_{r,1}}; B}(d_1, d_2, \dots, d_s, c_r)} \\ & + \sum_{\substack{r,t=1 \\ r < t}}^s \underbrace{\Pr \left(\{Y_{w_{r,1}} \leq c_r\} \cap \{Y_{w_{t,1}} \leq c_t\} \cap \bigcap_{q=1}^s \{Y_{w_{q,k_q}} \leq d_q\} \cap B \right)}_{F_{Y_{w_{1,k_1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,k_s}}, Y_{w_{r,1}}, Y_{w_{t,1}}; B}(c_1, d_1, d_2, \dots, d_s, c_r, c_t)} \\ & \vdots \\ & + (-1)^s \underbrace{\Pr \left(\bigcap_{q=1}^s \{Y_{w_{q,1}} \leq c_q\} \cap \bigcap_{q=1}^s \{Y_{w_{q,k_q}} \leq d_q\} \cap B \right)}_{F_{Y_{w_{1,1}}, Y_{w_{1,k_1}}, Y_{w_{2,1}}, Y_{w_{2,k_2}}, \dots, Y_{w_{s,1}}, Y_{w_{s,k_s}}; B}(c_1, d_1, c_2, d_2, \dots, c_s, d_s)} \end{aligned}$$

which concludes the proof.

From now on assume that B is the event that exactly j elements of S_1 fall in the interval $(0, y_1)$, for a given $j \leq n$. This event is shown in Table 1. Thus, to compute the probability of interest, it is enough evaluate the cumulative distribution functions relative to the event B of the order statistic, given by Equation (20). An efficient method for the computation of cumulative distribution functions of order statistics from two populations was proposed by Glueck *et al.* [5]. Here we need a slight generalization, involving the event B , which requires a different proof.

THEOREM 2 Denote the index vector $\mathbf{i} = (i_0, i_1, \dots, i_{e+1})$ and the summation index set

$$\mathcal{J} = \left\{ \mathbf{i}: \begin{array}{l} 0 = i_0 \leq i_1 \leq \dots \leq i_e \leq i_{e+1} = m, \\ \text{and } i_a \geq n_a \text{ for all } 1 \leq a \leq k \end{array} \right\}. \quad (30)$$

Suppose that $F_{X_i}(x) = F(x)$, for all $1 \leq i \leq n$, and $F_{X_i}(x) = G(x)$, for all $n+1 \leq i \leq m$. Then the cumulative distribution function relative to the event B (20) is given by

$$F_{Y_{n_1}, \dots, Y_{n_e}; B}(y_1, \dots, y_e) = \sum_{\mathbf{i} \in \mathcal{J}} \sum_{\boldsymbol{\lambda}} \prod_{a=1}^{e+1} \frac{n!(m-n)!}{\lambda_a!(i_a - i_{a-1} - \lambda_a)!} \times [F(y_a) - F(y_{a-1})]^{\lambda_a} [G(y_a) - G(y_{a-1})]^{i_a - i_{a-1} - \lambda_a}, \quad (31)$$

where $y_0 = 0$, $y_{e+1} = 1$, and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{e+1})$ ranges over all integer vectors such that $\lambda_1 = j$ and

$$\lambda_1 + \lambda_2 + \dots + \lambda_{e+1} = n, \quad 0 \leq \lambda_a \leq i_a - i_{a-1}. \quad (32)$$

Proof Denote by $A_{\mathbf{i}, \boldsymbol{\lambda}}$ the event that exactly $i_a - i_{a-1}$ of the random variables X_i fall in the interval (y_{a-1}, y_a) , and exactly λ_a of those are elements of S_1 . When $a = 1$, $(y_{a-1}, y_a) = (y_0, y_1) = (0, y_1)$. If B occurs, $\lambda_1 = j$. Then from the binomial theorem,

$$\Pr(A_{\mathbf{i}, \boldsymbol{\lambda}}) = \prod_{a=1}^{e+1} \frac{n!(m-n)!}{\lambda_a!(i_a - i_{a-1} - \lambda_a)!} [F(y_a) - F(y_{a-1})]^{\lambda_a} [G(y_a) - G(y_{a-1})]^{i_a - i_{a-1} - \lambda_a}. \quad (33)$$

Since the events $A_{\mathbf{i}, \boldsymbol{\lambda}}$ for different $(\mathbf{i}, \boldsymbol{\lambda})$ are disjoint, the result follows.

The only difference between Theorem 2 and the result by Glueck *et al.* [1] is the added condition $\lambda_1 = j$.

In the case of two random variables, we recover the same results as the direct method in Section 2. With $m = 2$, $n = 1$, $s = 2$, $c_1 = 0$, $d_1 = b_1$, $c_2 = b_2$, $d_2 = 1$, $S_1 = \{X_1\}$, $S_2 = \{X_1\}$, $k_1 = 1$, $k_2 = 1$, $Y_{w1,1} = Y_{w1,k1} = Y_1$, $Y_{w2,1} = Y_{w2,k1} = Y_2$, using Theorems 1 and 2 yields

$$(E \cap B) = \gamma, \quad (34)$$

when $j = 0$, and

$$(E \cap B) = \beta, \quad (35)$$

when $j = 1$.

In conclusion, for two sets of real-valued, independent but not necessarily identically distributed random variables, we have now given an expression for the probability that arbitrary subsets of order statistics fall in disjoint, ordered intervals and that of the smallest statistics, a certain number come from one set.

4. Concluding example

The methods of this paper can be used to calculating the joint probability of the number of rejections and the number of false rejection for the Benjamini–Hochberg [2] procedure. A rejection of a hypothesis for which the null holds is a false rejection. Given an false discovery rate $\alpha \in (0, 1)$, hypotheses H_i , $i = 1, \dots, m$, p -values X_i , and the corresponding order statistics for the p -values $Y_i = X_{(e)}$ (the random variables X_i sorted in nondecreasing order $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(m)}$), the procedure produces a nondecreasing sequence of numbers $b_i = i\alpha/m \in (0, 1)$, rejects the hypotheses $H_{(e)}$, $e = 1, \dots, k_1$, such that k_1 is the largest number for which $y_{k_1} \leq b_{k_1}$, and accepts all others. For $n \in \{0, 1, \dots, m\}$, assume that the null holds for $H_1, H_2, \dots, H_n$ and that the alternative holds for $H_{n+1}, H_{n+2}, \dots, H_m$. Let $S_1 = \{X_1, X_2, \dots, X_n\}$ be the set of p -values that correspond to the null hypotheses, and $S_2 = \{X_{n+1}, X_{n+2}, \dots, X_m\}$ be the set of p -values for which the alternative holds. Then j is the number of null hypotheses that are rejected, which is equal to the number of p -values corresponding to null hypotheses that fall in the interval $[0, b_{k_1}]$.

Under the assumption that the p -values for which the alternative holds have the same distribution, one can use the methods of this paper to find the joint distribution of j and k_1 .

For each value of k_1 and m , Glueck *et al.* [6] pointed out that the rejection regions for the Benjamini and Hochberg (1995) procedure can be decomposed into disjoint sets of events. These events correspond to certain numbers of order statistics falling into sets of intervals, defined by the numbers b_i . Details about the decomposition of the rejection regions into these events are given in Glueck *et al.* [6]. The general case is too complicated to detail here. However, as an example, we calculate the probabilities that with $m = 2$ hypotheses, and $n = 1$ null hypotheses, the Benjamini and Hochberg [2] procedure rejects $k_1 = 1$ hypotheses and that j , the number of false rejections, is either 0 or 1.

Suppose we wish to test $m = 2$ hypotheses. Specifically, we wish to test hypotheses about the location of the sample mean. We plan to conduct a two-sided test. We assume that we have two large populations, with known variances (both σ^2), and that the variables of interest, say ε_1 and ε_2 , are normally distributed, so that $\varepsilon_1 \sim N(\mu_1, \sigma^2)$ and $\varepsilon_2 \sim N(\mu_2, \sigma^2)$. We wish to test two hypotheses $H_1: \mu_1 = \mu_0$, and $H_2: \mu_2 = \mu_0$, with the alternative hypothesis for both populations the same, so $H_A: \mu = \mu_A$. We sample N_i random variables from each population, say $\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{iN_i}$. For convenience, we will assume that the random sample is of the same size for each hypothesis test, so $N_1 = N_2 = N$.

With

$$\bar{\varepsilon}_i = N^{-1} \sum_{\delta=1}^N \varepsilon_{i\delta}, \quad (36)$$

the test statistics are given by

$$Z_i = \left(\frac{\sigma}{\sqrt{N}} \right)^{-1} (\bar{\varepsilon}_i - \mu_0), \quad (37)$$

and the two sided p -values are [7, p. 244]

$$X_i = \begin{cases} 2\Phi(Z_i) & Z_i \leq 0 \\ 2[1 - \Phi(Z_i)] & Z_i > 0 \end{cases}, \quad (38)$$

where Φ is the cumulative distribution function of the standard normal (mean = 0 and variance = 1). Let ϕ be the probability density function of the standard normal.

Suppose that in truth, we have $\varepsilon_1 \sim N(\mu_0, \sigma^2)$, so that the null holds for H_1 , and $\varepsilon_2 \sim N(\mu_A, \sigma^2)$, so that alternative holds for H_2 . Define $S_1 = \{X_1\}$, and $S_2 = \{X_2\}$. Then the number of p -value for which the null holds, $n = 1$. For H_1 , the hypotheses for which the null holds, the p -value has a uniform distribution on the interval $[0, 1]$, so for $x_1 \in [0, 1]$,

$$F_{x_1}(x_1) = x_1. \quad (39)$$

For H_2 , the alternative holds. When we conduct the hypothesis test, we are unaware of the truth. We always calculate the p -value under the null. However, since the alternative actually holds,

$$\begin{aligned}
\Pr[Z_2 \leq z_2] &= \Pr\left[\frac{\bar{e}_i - \mu_0}{\sigma/\sqrt{N}} \leq z_2\right] \\
&= \Pr\left[\frac{e_i - \mu_A}{\sigma/\sqrt{N}} \leq z_2 + \frac{\mu_0 - \mu_A}{\sigma/\sqrt{N}}\right] \\
&= \Phi\left[z_2 + \frac{\mu_0 - \mu_A}{\sigma/\sqrt{N}}\right].
\end{aligned} \tag{40}$$

Finally,

$$\begin{aligned}
F_{x_2}(x_2) &= \Pr(X_2 < x_2) \\
&= \Pr(\{X_2 < x_2\} \cap \{Z_2 \leq 0\}) + \Pr(\{X_2 < x_2\} \cap \{Z_2 > 0\}) \\
&= \Pr(\{2\Phi(Z_2) < x_2\}) + \Pr(\{2[1 - \Phi(Z_2)] < x_2\}) \\
&= \Pr\left(\left\{Z_2 \leq \Phi^{-1}\left(\frac{x_2}{2}\right)\right\}\right) + 1 - \Pr\left(\left\{Z_2 \leq \Phi^{-1}\left(\frac{1-x_2}{2}\right)\right\}\right) \\
&= \Phi\left[\Phi^{-1}(x_2/2) + \frac{\mu_0 - \mu_A}{\sigma/\sqrt{N}}\right] + 1 - \Phi\left[\Phi^{-1}(1 - x_2/2) + \frac{\mu_0 - \mu_A}{\sigma/\sqrt{N}}\right],
\end{aligned} \tag{41}$$

where the last step follows by substitution from Equation (40).

Now, as a specific example, we fix $\mu_0 = 0$, $\mu_A = 1$, $\sigma^2 = 1$, $\alpha = 0.05$. We wish to calculate the probability that $k_1 = 1$ and that $j = 0$ or $j = 1$. With $c_1 = 0$, $d_1 = \alpha/2$, $c_2 = \alpha$, $d_2 = 1$. This is the probability that of the two hypotheses, we reject exactly one, and it is H_1 , the hypothesis for which the null holds. When $j = 0$, the rejection we make is of the hypothesis for which the alternative holds, and when $j = 1$, the rejection we make is of the null hypothesis, a false rejection.

We calculated the probability using our methodology, and by a simulation using a sample of 100,000 variables. Recall that k_1 is the number of order statistics that is less than b_1 , and j is the number in Set 1, and less than b_1 . The results are shown in Table 2.

Notice that the simulation differs from the theory only in the fourth decimal place. The theory is exact. Software that implements this method in Mathematica is available from the authors upon request.

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References

1. Glueck DH, Karimpour-Fard A, Mandel J, Muller KE. On the probability that order statistics fall in intervals. 2008 in review.
2. Benjamini Y, Hochberg Y. Controlling the false discovery rate: A practical and powerful approach to multiple testing. J. R. Stat. Soc. Ser. B Stat. Methodol 1995;57:289–300.
3. David, HA. Order Statistics. 2nd ed. New York: Wiley; 1981.
4. Bapat RB, Beg MI. Order statistics for non-identically distributed variables and permanents. Sankhya Ser. A 1989;51:79–93.
5. Glueck DH, Karimpour-Fard A, Mandel J, Hunter L, Muller KE. Fast computation by block permanents of cumulative distribution functions of order statistics from several populations. Commun. Stat. Theory Methods. 2008 to appear.
6. Glueck DH, Muller KE, Karimpour-Fard A, Hunter L. Expected power for the false discovery rate with independence. Commun. Stat. Theory Methods 2008;37(12)

7. Rosner, B. Fundamentals of Biostatistics. 6th ed. New York: Brooks-Cole; 2006.
8. Ross, S. A First Course in Probability. 2nd ed. New York: Macmillan Publishing Company; 1984.

Table 1

Numbers of order statistics from the sets S_1 and S_2 in the interval $(0, d_1)$ and outside the interval $(0, d_1)$, in the event B .

	$< d_1$	$\geq d_1$	
S_1	j	$n - j$	n
S_2	$k_1 - j$	$(m - n) - (k_1 - j)$	$(m - n)$
Total	k_1	$m - k_1$	m

Table 2

Comparison of simulation and theory.

k_1	j	Theory	Simulation	Difference
1	0	0.472982	0.47388	0.000898
1	1	0.00978051	0.0095	0.00028051

Note: Recall that k_1 is the number of hypotheses that were rejected, and j is the number of null hypotheses that were rejected. We had two hypotheses and one null hypothesis.