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Advanced Theory of Driven Birdcage Resonator with Losses for Biomedical Magnetic Resonance Imaging and Spectroscopy

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Abstract

A complete time-dependent physics theory of symmetric unperturbed driven Hybrid Birdcage resonator was developed for general application. In particular, the theory can be applied for RF coil engineering, computer simulations of coil-sample interaction, etc. Explicit time dependence is evaluated for different forms of driving voltage. The major steps of the solution development are shown and appropriate explanations are given. Green's functions and spectral density formula were developed for any form of periodic driving voltage. The concept of distributed power losses based on transmission line theory is developed for evaluation of local losses of a coil. Three major types of power losses are estimated as equivalent series resistances in the circuit of the Birdcage resonator. Values of generated resistances in Legs and End-Rings are estimated. An application of the theory is shown for many practical cases. Experimental curve of B_1 field polarization dependence is measured for eight-sections Birdcage coil. It was shown, that the steady-state driven resonance frequencies do not depend on damping factor unlike the free oscillation (transient) frequencies. An equivalent active resistance is generated due to interaction of RF electromagnetic field with a sample. Resistance of the conductor (enhanced by skin effect), Eddy currents and dielectric losses are the major types of losses which contribute to the values of generated resistances. A biomedical sample for magnetic resonance imaging and spectroscopy is the source of the both Eddy current and dielectric losses of a coil. As demonstrated by the theory, Eddy currents losses is the major effect of coil shielding.

Keywords

birdcage; losses; damping; Kirchhoff; Green's functions; Laplace transform; perturbation; loss tangent

1. Introduction

The Birdcage type resonator has been introduced in [1] as a highly homogeneous radiofrequency (RF) coil for whole-body imaging.

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Two main approaches to the theoretical description of Birdcage resonator physics system have been published. One approach considers the system as a complicated oscillator [2–5]; another represents the Birdcage as a transmission line [6–9].

The term Hybrid Birdcage resonator implies a more complicated form of the Birdcage Coil which has both Low-Pass or High-Pass features [4,10,11]; however there are other types of Birdcage resonators such as Double-Tuned coils [12,13], etc.

The real and the imaginary parts of an impedance may be referred to as active and reactive resistances respectively. Here a Hybrid type Birdcage coil with active resistance elements in its electronic circuit (Fig. 1) will be discussed to describe a complete physics theory of a symmetric unperturbed Birdcage resonator.

Many previous publications [3,4] discuss the practical forms of Birdcage resonators like Low-Pass and High-Pass, which do not contain active resistance elements in their circuits. However, an active resistance is always present in real RF coils at room temperature. Any type of resonator loaded with a sample for experimental nuclear magnetic resonance (NMR) measurements due to inductive and capacitive losses has an equivalent resistance, and acts as an active series resistance element in an electronic circuit [14,15]. Hence, the loaded quality factor (Q) decreases essentially [16]. Also, there is a useful power transduction for nuclear spins excitation, which induces an equivalent resistance as well. The presence of an active resistance in any of above mentioned forms is a necessary condition of power transduction from a RF source to a resonator. Hence, it is appropriate to consider theory of a resonator with active resistance elements included in its circuit. Previously, there was a brief theoretical discussion of symmetric unperturbed driven Birdcage resonator with losses [2], though it is incomplete and mathematically inaccurate.

The physics system of the Birdcage resonator represents a hybrid between continuous and discrete systems, since it has both continuous and discrete variables. That is the reason why the solution of electronic motion equations can not be obtained straightforwardly by standard methods [17,18]. Evaluation of the solution involves mathematical methods such as Theory of Groups Representations, Vector Analysis, Matrices and Determinants, Differential Equations, among others.

2. Theoretical Solution Development

2.1 Setting equations of electronic motion

According to the Kirchhoff rule, the voltage balance of the n -th loop (Fig. 1) can be expressed by the equation:

$$\begin{aligned}
& 2L_1 \frac{d}{dt} I_n \\
& + 2R_1 I_n \\
& + \frac{2}{C_1} \int I_n dt \\
& + L_2 \frac{d}{dt} (I_n \\
& - I_{n-1}) + R_2 (I_n \\
& - I_{n-1}) + \frac{1}{C_2} \int (I_n \\
& - I_{n-1}) dt \\
& + L_2 \frac{d}{dt} (I_n - I_{n+1}) + R_2 (I_n \\
& - I_{n+1}) + \frac{1}{C_2} \int (I_n \\
& - I_{n+1}) dt = U_n(t)
\end{aligned} \tag{1}$$

To eliminate integral signs, it is reasonable to apply differential operation to both parts of the equation:

$$\begin{aligned}
& 2L_1 \frac{d^2}{dt^2} I_n \\
& + 2R_1 + \frac{d}{dt} I_n + \frac{2}{C_1} I_n + L_2 + \frac{d^2}{dt^2} (I_n \\
& - I_{n-1}) + R_2 \frac{d}{dt} (I_n \\
& - I_{n-1}) + \frac{1}{C_2} (I_n \\
& - I_{n-1}) + L_2 \frac{d^2}{dt^2} (I_n - I_{n+1}) + R_2 \frac{d}{dt} (I_n \\
& - I_{n+1}) + \frac{1}{C_2} (I_n \\
& - I_{n+1}) = \frac{d}{dt} U_n(t)
\end{aligned} \tag{2}$$

For convenience, assume time dependence: $I_n(t) \sim e^{\lambda t}$. Hence, the equation may be represented in the form:

$$2 \left[\lambda^2 (L_1 + L_2) + \lambda (R_1 + R_2) + \left(\frac{1}{C_1} + \frac{1}{C_2} \right) \right] I_n - \left[\lambda^2 L_2 + \lambda R_2 + \frac{1}{C_2} \right] (I_{n-1} + I_{n+1}) = \frac{d}{dt} U_n(t) \tag{3}$$

which is the same as:

$$aI(t, n) + b[I(t, n-1) + I(t, n+1)] = F(t, n) \tag{4}$$

where elements of the equation are:

$$a=2\left[\lambda^2(L_1+L_2)+\lambda(R_1+R_2)+\left(\frac{1}{C_1}+\frac{1}{C_2}\right)\right] \quad (5)$$

$$b=-\lambda^2L_2-\lambda R_2-\frac{1}{C_2} \quad (6)$$

$$I(t,n)=I_n \quad (7)$$

$$I(t,n-1)=I_{n-1} \quad (8)$$

$$I(t,n+1)=I_{n+1} \quad (9)$$

$$F(t,n)=\frac{d}{dt}U_n(t) \quad (10)$$

Note, the whole system of Eq. (4) may be represented as:

$$\hat{A}|I(t)\rangle=|F(t)\rangle \quad (11)$$

where $\hat{A}$ is a matrix operator and $|I(t)\rangle, |F(t)\rangle$ are column vectors:

$$\hat{A}=\begin{vmatrix} a & b & 0 & . & 0 & b \\ b & a & b & . & 0 & 0 \\ 0 & b & a & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & a & b \\ b & 0 & 0 & . & b & a \end{vmatrix} \quad |I(t)\rangle=\begin{vmatrix} I(t,1) \\ I(t,2) \\ . \\ . \\ . \\ I(t,N) \end{vmatrix} \quad |F(t)\rangle=\begin{vmatrix} F(t,1) \\ F(t,2) \\ . \\ . \\ . \\ F(t,N) \end{vmatrix} \quad (12)$$

The next step of the solution development is an evaluation of the matrix operator, $\hat{A}$. To evaluate an operator it is appropriate to find its eigenvalues and eigenfunctions.

2.2 Finding eigenvalues and eigenvectors of the space operator

Eigenvalues A_n of the matrix operator $\hat{A}$ must satisfy the equation:

$$\hat{A}|X_n\rangle=|X_n\rangle A_n \quad (13)$$

where $|X_n\rangle$ are the eigenvectors of the operator.

To find eigenvalues and eigenvectors of the matrix operator $\hat{A}$, it is appropriate to know the properties of a Circulant matrix [19–21]. A Circulant is a matrix of type C with constant diagonals,

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \cdot & c_{N-1} & c_N \\ c_N & c_1 & c_2 & \cdot & c_{N-2} & c_{N-1} \\ c_{N-1} & c_N & c_1 & \cdot & c_{N-3} & c_{N-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_3 & c_4 & c_5 & \cdot & c_1 & c_2 \\ c_2 & c_3 & c_4 & \cdot & c_N & c_1 \end{pmatrix} \quad (14)$$

which can be diagonalized by unitary transformation:

$$U^* C U = D \quad (15)$$

where U is a unitary matrix with the main element $\theta = e^{i(\frac{2\pi}{N})}$:

$$U = \frac{1}{\sqrt{N}} \begin{pmatrix} \theta & \theta^2 & \theta^3 & \cdot & \theta^{N-1} & 1 \\ \theta^2 & \theta^4 & \theta^6 & \cdot & \theta^{2(N-1)} & 1 \\ \theta^3 & \theta^6 & \theta^9 & \cdot & \theta^{3(N-1)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta^{N-1} & \theta^{2(N-1)} & \theta^{3(N-1)} & \cdot & \theta^{(N-1)(N-1)} & 1 \\ 1 & 1 & 1 & \cdot & 1 & 1 \end{pmatrix} \quad (16)$$

and D is the diagonal matrix. The diagonal elements d_{kk} of matrix D may be expressed by the formula:

$$d_{kk} = c_1 + c_2 \theta^k + c_3 \theta^{2k} + \dots + c_N \theta^{(N-1)k} \quad (17)$$

It is clear that the diagonal elements d_{kk} are equal to eigenvalues of both diagonal D and non-diagonal matrix C . Eigenvectors $|X_n\rangle$ of Circulant matrices are equal to normalized columns (or rows) of symmetric unitary matrix U . Hence for Circulant matrix operator $\hat{A}$ eigenvalues A_n are:

$$A_n = a + 2b \cos(2\pi \frac{n}{N}) \quad (18)$$

Row eigenvectors $\langle X_n|$ are equal to normalized rows (or columns) of symmetric unitary matrix U^* and satisfy the condition:

$$\langle X_n | X_m \rangle = \delta_{nm} \quad (19)$$

Also it is appropriate to apply the group representation theory here. Since the symmetry class of the system is C_N , the components of space eigenfunctions of the system must be equal to the characters of one-dimensional irreducible matrix representations of N -fold rotational Abelian group C_N . Indeed, the characters of irreducible representations of the

group C_N are equal to the elements of the symmetric unitary matrix U . Hence, $|X_n\rangle$ are the eigenvectors of the Circulant operator $\hat{A}$ and the space eigenfunctions of the whole system.

2.3 Separation of variables

In order to solve Eq. (11), it is necessary to separate variables of time and space. Though the time variable is continuous and the space variable is discrete, it is possible to separate variables using the standard Fourier method for partial differential equations [17,18].

Since the space eigenfunctions have been found, it is possible to calculate time-dependent projection coefficients $T_k(t)$ of function $|I(t)\rangle$ on the space eigenfunctions:

$$T_k(t) = \langle X_k | I(t) \rangle \quad (20)$$

The function $|I(t)\rangle$ may be represented in the form:

$$|I(t)\rangle = \sum_{k=1}^N |X_k\rangle T_k(t) \quad (21)$$

Where $T_k(t)$ is a scalar function of time, which is the same for all components of the vector $|X_k\rangle$, and k is the quantum number, which is an individual attribute of the function.

The same procedure may be applied to the function $|F(t)\rangle$:

$$f_k(t) = \langle X_k | F(t) \rangle \quad (22)$$

$$|F(t)\rangle = \sum_{k=1}^N |X_k\rangle f_k(t) \quad (23)$$

To separate time from space functions, apply the space eigenfunction $\langle X_k |$ to both parts of Eq. (11), then, by taking into account Eqs. (13,18,19,21,23), obtain the equation:

$$[a + 2b \cos(2\pi \frac{k}{N})] T_k(t) = f_k(t) \quad (24)$$

As a result, time functions are separated from space functions; however, it is necessary to evaluate time-dependent coefficients $T_k(t)$.

2.4 Evaluation of time dependence

Equation Eq.(24) may be considered as a time operator action. First, by taking into account Eq.(3), represent Eq. (24) in the form

$$(\lambda^2 + 2\gamma_k \lambda + \omega_k^2) T_k(t) = \alpha_k f_k(t) \quad (25)$$

Where γ_k , ω_k , and α_k are the quantum number dependent constants. Particularly, γ_k is a damping constant:

$$\gamma_k = \frac{1}{2} \frac{R_1 + R_2 [1 - \cos(2\pi \frac{k}{N})]}{L_1 + L_2 [1 - \cos(2\pi \frac{k}{N})]} \quad (26)$$

As it will be shown later, ω_k are the steady-state driven resonance frequencies:

$$\omega_k^2 = \frac{\frac{1}{C_1} + \frac{1}{C_2} [1 - \cos(2\pi \frac{k}{N})]}{L_1 + L_2 [1 - \cos(2\pi \frac{k}{N})]} \quad (27)$$

and α_k is a quantum number dependent constant:

$$\alpha_k = \frac{1}{2\{L_1 + L_2 [1 - \cos(2\pi \frac{k}{N})]\}} \quad (28)$$

The characteristic polynomial of the Eq. (25) represents differential time operator. Then, solve Eq. (25) as a standard differential equation of driven damped oscillation. The next step of the solution is finding the roots of the characteristic polynomial:

$$\lambda^2 + 2\gamma_k \lambda + \omega_k^2 = 0 \quad (29)$$

which gives two values of λ_k :

$$\lambda_{k1} = -\gamma_k + i\beta_k \quad (30)$$

$$\lambda_{k2} = -\gamma_k - i\beta_k \quad (31)$$

where β_k are the free oscillation (transient) frequencies:

$$\beta_k = \sqrt{\omega_k^2 - \gamma_k^2} \quad (32)$$

Solution of the inhomogeneous differential equation Eq. (25) can be obtained, as an option, by the method of constant variation [22,23]. Assuming

$$T_k(t) = C_1(t)e^{\lambda_{k1}t} + C_2(t)e^{\lambda_{k2}t} \quad (33)$$

the steady state solution for $T_k(t)$ can be obtained by integration of the Green's function:

$$T_k(t) = \alpha_k \int_0^t G_k(t, \tau) f_k(\tau) d\tau \quad (34)$$

Where the Green function is:

$$G_k(t, \tau) = \frac{1}{\beta_k} e^{-\gamma_k(t-\tau)} \sin[\beta_k(t - \tau)] \quad (35)$$

The Green's function presented above is not unique for that particular differential equation. There are other possible forms of the Green's function, which give the same solution.

In the case of the complicated form of periodic driving voltage, it is possible to represent the driving voltage as a linear combination of sine and cosine waves by expanding it to Fourier series. Since in this case the differential equation is linear, it is possible to obtain the solution for $T_k(t)$ as a linear combination of solutions for those sine and cosine waves.

2.5 The general steady state solution for loop currents

At this point, when space and time eigenfunctions have been evaluated it is possible to compose the general solution for loop currents. The solution for the column vector of loop currents $|I(t)\rangle$ can be obtained by taking into account Eqs. (21, 22, 34):

$$|I(t)\rangle = \sum_{k=1}^N |X_k\rangle \alpha_k \int_0^t G_k(t, \tau) \langle X_k | F(\tau) \rangle d\tau \quad (36)$$

which is the same as, in terms of original electromagnetic parameters:

$$|I(t)\rangle = \sum_{k=1}^N \frac{|X_k\rangle}{2\{L_1 + L_2[1 - \cos(2\pi \frac{k}{N})]\}} \int_0^t d\tau G_k(t, \tau) \left[\frac{d}{d\tau} \langle X_k | U(\tau) \rangle \right] \quad (37)$$

Where $|U(t)\rangle$ is the column vector of driving voltages. Components of the vector can be expressed as:

$$U(t, n) = U_n(t) \quad (38)$$

Also, the formula for the spectral density of loop currents may be obtained by Fourier transformation of both parts of Eq. [37]:

$$|I(\omega)\rangle = \sum_{k=1}^N \frac{i\omega |X_k\rangle \langle X_k | U(\omega) \rangle}{2\{L_1 + L_2[1 - \cos(2\pi \frac{k}{N})]\}(\omega^2 - \omega_k^2 + i2\gamma_k\omega)} \quad (39)$$

Where components of $|I(\omega)\rangle$ and $|U(\omega)\rangle$ vectors are the Fourier transform of the $|I(t)\rangle$ and $|U(t)\rangle$ vectors components:

$$I(\omega, n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega t} I(t, n) dt \quad (40)$$

$$U(\omega, n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega t} U(t, n) dt \quad (41)$$

It should be pointed out, that the formula, similar to Eq. (39), has been introduced in [2], though with a considerable mistake.

The quality factor Q_k in this case is also a quantum number dependent parameter:

$$Q_k = \frac{\omega_k}{2\gamma_k} \quad (42)$$

It can be concluded from Eq. (39), that the steady-state driven resonance frequencies ω_k do not depend on damping factor γ_k .

3. Application of the theory to practical cases

3.1 The case of harmonic driving voltage

For practical application there are required more simple Birdcage resonance circuits, than illustrated in Fig. 1. Ordinarily, it is sufficient to use only one driving voltage source for the whole circuit. However, those facts are not the problem for an application of the theory, presented in previous sections. In many cases it is possible to exclude a particular element from useful equations by zeroing or by extrapolating to infinity the physical value of the element.

Using the theory for a Birdcage resonator with a single sine-wave driving voltage source in the mesh of m -th number, we get:

$$U(t, m) = U_m \sin(\omega t) \quad (43)$$

By zeroing driving voltage values in all other components of the vector $|U(t)\rangle$, it is possible to obtain a formula for loop current of any mesh by calculating sums and integrals in Eq. (37). Hence, the complex loop current for the n -th mesh can be expressed as:

$$I(t, n) = \frac{U_m \omega}{2N} \sum_{k=1}^N \frac{[(\omega_k^2 - \omega^2) \cos(\omega t) + 2\omega \gamma_k \sin(\omega t)] e^{i2\pi k \frac{n-m}{N}}}{\{L_1 + L_2 [1 - \cos(2\pi \frac{k}{N})]\} [(\omega_k^2 - \omega^2)^2 + (2\gamma_k \omega)^2]} \quad (44)$$

In general, $2N$ steady-state resonance waves are present in the system. For every steady-state driven resonance frequency ω_k there are two waves. The first one is a cosine wave times

$(\omega_k^2 - \omega^2)$ and the second one is a sine wave times $2\omega \gamma_k$. The cosine wave expresses the power reflected from the circuit, while the sine wave expresses the power deposited into the

circuit. The multipliers $(\omega_k^2 - \omega^2) / \sqrt{(\omega_k^2 - \omega^2)^2 + (2\gamma_k \omega)^2}$ and $2\omega \gamma_k / \sqrt{(\omega_k^2 - \omega^2)^2 + (2\gamma_k \omega)^2}$ may be considered as the reflection and the absorption coefficients respectively. It is important to point out, that the real amplitudes of resonance waves are modulated by the

multiplier $\cos(2\pi k \frac{n-m}{N})$ due to the complex exponent in Eq. (44). Since all sine and cosine waves in Eq. (44) are of the same frequency, they may be represented as one sine wave with a particular amplitude and phase.

In a particular case when the driving voltage frequency is tuned to one of the steady-state resonance frequencies the amplitude of the sine resonance wave increases considerably with respect to other waves, hence the last ones may be neglected. In this case the complex loop current can be approximated as:

$$I(t, n) = \frac{U_m}{2N} \frac{\sin(\omega_k t)}{\{R_1 + R_2 [1 - \cos(2\pi \frac{k}{N})]\}} e^{i2\pi k \frac{n-m}{N}} \quad (45)$$

Note, in that case, the reflection coefficient for resonating frequency is equal to zero, while the absorption coefficient is equal to unity.

Finally, it is appropriate to consider the trivial cases of the quantum number k . The case $k=N$ corresponds to the absolutely symmetric oscillation mode:

$$I(t, n) = \frac{U_m}{2NR_1} \sin(\omega_N t) \quad (46)$$

In this case, all loop currents are equal and leg currents $I(t, n) - I(t, n-1)$ vanish. Obviously, in this case the circuit elements L_2 , C_2 , and R_2 do not include in loop currents and frequency ω_N value.

For even numbers of N there is the NMR useful anti-symmetric mode. That is a doubly degenerated mode, which includes quantum numbers $k=1$ and $k=N-1$. In this case the real loop currents in meshes are:

$$\text{Re}\{I(t, n)\} = \frac{U_m}{2N} \frac{\sin(\omega_1 t)}{\{R_1 + R_2 [1 - \cos(\frac{2\pi}{N})]\}} \cos(2\pi \frac{n-m}{N}) \quad (47)$$

Note, that the maximal currents will be present in the meshes number m and $m+N/2$, while zero currents are possible in some other meshes. Hence, for an anti-symmetric mode there is a correspondence:

$$I(t, n) = -I(t, n \pm N/2) \quad (48)$$

Due to the geometry of a birdcage coil, the anti-symmetric mode produces homogeneous magnetic field B_1 , useful for experimental NMR measurements. In this case the magnetic field B_1 will be linearly polarized in the direction orthogonal to the planes of current loops number m and $m+N/2$, which are opposite to each other for circular geometry of a coil.

3.2 The system's response to non-periodic driving voltage

It would be useful to evaluate the Birdcage loop currents upon the action of step-modulated sine-wave driving voltage, as for surface coils [24]. Assume a step-cosine wave driving voltage is applied to the mesh number m at the time moment τ :

$$U(t, m) = \begin{cases} 0 & (t < \tau) \\ U_m \cos(\omega t) & (t \geq \tau) \end{cases} \quad (49)$$

Otherwise:

$$U(t, m) = U_m \text{step}(t - \tau) \cos[\omega(t - \tau)] \quad (50)$$

Apply the Laplace transform operation to the function $T_k(t)$ of Eq.(25):

$$L[T_k(t)] = \int_0^{\infty} e^{-ts} T_k(t) dt \quad (51)$$

In this case the Laplace image of the function is:

$$L[T_k(t)] = \frac{U_m s^2 \exp(-\tau s)}{(s^2 + \omega^2)[(s + \gamma_k)^2 + \beta_k^2]} e^{-i2\pi k \frac{m}{N}} \quad (52)$$

Hence the final solution for loop currents Eq. (21) can be obtained by inverse Laplace transform of Eq.(52), involving residues calculus:

$$I(t, n) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{ts} L[I(t, n)] ds \quad (53)$$

Calculation of Eq.(53) gives:

$$\begin{aligned} I(t, n) = & \frac{U_m}{2N} \sum_{k=1}^N A_k(\omega) \cos[\omega(t - \tau)] e^{i2\pi k \frac{n-m}{N}} \\ & + \frac{U_m}{2N} \sum_{k=1}^N B_k(\omega) \sin[\omega(t - \tau)] e^{i2\pi k \frac{n-m}{N}} \\ & + \frac{U_m}{2N} \sum_{k=1}^N C_k(\omega) \sin[\beta_k(t - \tau)] e^{-\gamma_k(t - \tau) + i2\pi k \frac{n-m}{N}} \\ & + \frac{U_m}{2N} \sum_{k=1}^N D_k(\omega) \cos[\beta_k(t - \tau)] e^{-\gamma_k(t - \tau) + i2\pi k \frac{n-m}{N}} \end{aligned} \quad (54)$$

$$A_k(\omega) = \frac{\omega(\omega^2 - \omega_k^2)}{F_k(\omega)} \quad (55)$$

$$B_k(\omega) = \frac{2\omega^2 \gamma_k}{F_k(\omega)} \quad (56)$$

$$C_k(\omega) = \frac{\omega_k^2(\omega_k^2 - \omega^2) + 2\gamma_k^2 \omega^2}{\beta_k F_k(\omega)} \quad (57)$$

$$D_k(\omega) = -\frac{2\gamma_k \omega^2}{F_k(\omega)} \quad (58)$$

$$F_k(\omega) = \{L_1 + L_2[1 - \cos(2\pi \frac{k}{N})]\}[(\omega_k^2 - \omega^2)^2 + 4\gamma_k^2 \omega^2] \quad (59)$$

So, as it can be seen from Eqs.(54–59) the system's response to step-cosine driving voltage contains four terms. The first two terms compose the steady state solution, and the last two terms describe the transient behavior of the system.

Also it would be useful to know the system's response to step driving voltage:

$$U(t, m) = U_m \text{step}(t - \tau) \quad (60)$$

In this case the system's response can be easily found, assuming $\omega = 0$ in the solution Eq. (54):

$$I(t, n) = \frac{U_m}{2N} \sum_{k=1}^N \frac{1}{\beta_k} \sin[\beta_k(t - \tau)] e^{-\gamma_k(t - \tau) + i2\pi k \frac{n - m}{N}} \quad (61)$$

That is the system response to the step driving voltage in the mesh number m . In general, a single pulse of any shape and frequency can be represented as a linear combination of step-sine functions with different phase shifts.

Hence, using the Laplace transform method and its properties, it would be possible to evaluate the system's response to a single pulse of any shape, finite and infinite pulse trains, and other forms of driving voltage.

3.3 An influence of asymmetric elements in the coil circuit on the resonance frequencies

It is appropriate to consider the influence of asymmetric elements, like a tuning capacitor on the resonance frequencies of the Birdcage coil, since every NMR sample has a particular influence on the frequencies. In practice, for every particular NMR sample the coil should be tuned and matched again. Hence, in the case of single tuning and matching network, the symmetry inevitably will be broken. In the case of multiple networks, as in Fig. 1, it would be difficult to keep symmetry.

In this case, it is possible to use the standard perturbation theory of quantum mechanics [25]. The perturbation is time-independent, hence one can apply time-independent perturbation theory to space operators and space eigenfunctions Eq. (13). Perturbed equation Eq. (13) will take the form:

$$(\hat{A} + \hat{H})|\tilde{X}_n\rangle = |\tilde{X}_n\rangle \tilde{A}_n \quad (62)$$

where $\hat{H}$ is the perturbation operator, $|\tilde{X}_n\rangle$ are the perturbed eigenfunctions, and $\tilde{A}_n$ are the perturbed eigenvalues:

$$\tilde{A}_n = A_n + a_n^{(1)} + a_n^{(2)} + \dots \quad (63)$$

$$|\tilde{X}_n\rangle = |X_n\rangle + \sum_{\substack{k=1 \\ k \neq n}}^N c_k |X_k\rangle \quad (64)$$

$$c_k = c_k^{(1)} + c_k^{(2)} + \dots \quad (65)$$

Here, $a_n^{(1)}$ and $c_n^{(1)}$ are the first order corrections to eigenvalues and eigenfunctions, respectively, and $a_n^{(2)}, c_n^{(2)}$ are the second order corrections, etc. According to the perturbation theory the first order correction to the eigenvalues can be calculated as:

$$a_n^{(1)} = \langle X_n | \hat{H} | X_n \rangle \quad (66)$$

Using the last equations, it is possible to calculate values of the perturbed frequencies. Assuming a tuning capacitance C_m is introduced to the mesh number m , then the condition:

$$\frac{1}{C_m} \ll 2\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \quad (67)$$

must be satisfied according to the perturbation theory, since perturbation energy should be small with respect to the total energy of the system. In the case of Birdcage resonator reactive resistance of tuning capacitance should be much smaller than reactive resistance of the total capacitance of a mesh. In this case, only one element of $\hat{H}$ matrix operator will have nonzero value:

$$h_{nk} = \delta_{nm} \delta_{km} \frac{1}{C_m} \quad (68)$$

Calculating $a_n^{(1)}$ from Eq.(66):

$$a_n^{(1)} = \frac{1}{2NC_m} \quad (69)$$

the first order correction for any eigenvalue A_n has been obtained. The perturbed resonance frequencies can be expressed as:

$$\tilde{\omega}_n^2 = \omega_n^2 + \frac{1}{2NC_m \{L_1 + L_2 [1 - \cos(2\pi \frac{n}{N})]\}} \quad (70)$$

Hence, square values of all resonance frequencies will be shifted according to Eq.(70).

Consider the case of multiple networks, when every mesh has a tuning capacitance C_k . In this case, only diagonal elements of perturbation matrix operator $\hat{H}$ will have nonzero values:

$$h_{nk} = \delta_{nk} \frac{1}{C_k} \quad (71)$$

The first order correction to eigenvalues will take the form:

$$a_n^{(1)} = \frac{1}{2N} \sum_{k=1}^N \frac{1}{C_k} \quad (72)$$

Finally, obtain perturbed resonance frequencies for this case:

$$\tilde{\omega}_n^2 = \omega_n^2 + \frac{\sum_{k=1}^N \frac{1}{C_k}}{2N\{L_1 + L_2[1 - \cos(2\pi \frac{n}{N})]\}} \quad (73)$$

Again, square values of perturbed resonance frequencies will be shifted according to Eq. (73).

As a result, a consistency of standard perturbation method is shown for both cases of asymmetric networks.

3.5 Experimental B_1 field polarization dependence

As discussed in the section 3.1, B_1 field induced by Birdcage resonator with a single driving voltage source should be polarized according to Eq.(47), provide the resonator is operating on NMR doubly degenerate frequency mode. Polarization dependence of B_1 field was investigated inside eight-sections high-pass Birdcage resonator. The Birdcage resonator was designed for proton mouse brain imaging with 11.7T Bruker spectrometer. Working NMR frequency of the resonator was 500 MHz. Current was induced in a rotating frame, mounted inside the resonator. Induced voltage was measured for 16 positions of the rotating frame inside the resonator. An accuracy of measurement was within 5%.

Theoretical (dashed line) and experimental (solid line with experimental points) B_1 field polarization dependence curves in polar system are represented in Fig. 2. Some disagreement between theoretical and experimental curves can be explained by both voltage measurement inaccuracy and mechanical construction imperfection of the experimental set.

4. Interaction of the system with a sample

4.1 Distributed parameters and losses of RF coil conductor

An RF coil circuit includes distributed and lumped elements. Conductors are the distributed elements, while tuning and matching capacitors are the lumped elements of a circuit. As mentioned earlier, transmission line theory has been applied to the whole Birdcage coil construction (6–9), including lumped elements. However, the transmission line theory may be applied to a small piece of RF coil conductor as well, which does not contain lumped elements.

An RF coil, like a transmission line, has distributed parameters. Distributed parameters are not concentrated within a lumped circuit element, but are spread over the conductor of a coil.

Hence, a distributed parameter can be defined as a parameter of a coil conductor of a unit length at a particular point of the coil and can be represented as a function of a position along the conductor ($l(z)$ for a distributed inductance). However, the last definition does not mean, that the parameter value should be constant within a unit length of the conductor.

Distributed capacitance (also known as stray capacitance) is the capacitance between the conductor and the ground electrode of the circuit. If a circuit is not grounded, the zero ground potential can be assumed to be in the point of infinity. The total stray capacitance of a coil is the result of an integrated distributed capacitance.

There are three types of power losses that can be defined for an RF coil conductor. The first one is loss due to metal resistance increased by skin effect. The second type of loss is associated both with magnetic and electric fields induced by distributed inductance. In this case electric field lines make up closed loops, inducing Eddy currents. Finally, the third type of losses is associated with distributed capacitance of a coil. In the last case, electric field lines start from a conductor and finish on the other electrode of the distributed capacitance, which may be in the point of infinity.

The equivalent circuit of distributed losses can be represented similar to those previously described in [26,27]. A simplified equivalent circuit of distributed parameters is shown (Fig. 3a). By using the method of equivalent power losses [28,29], parallel active resistances may be represented as series active resistances. Hence, even more simplified circuit of equivalent distributed power losses is shown in (Fig. 3b). In the last case (Fig. 3b) it was assumed that only the voltage of a conductor reduces due to dielectric losses equivalent resistance $r_D(z)$, while distributed admittance $g_D(z)$ are assumed to be zero. In practice actual dielectric losses occur due to the current losses through distributed admittance $g_D(z)$ (Fig. 3a).

4.2 Distributed losses due to metal resistance

It is well-known that the distributed resistance of a conductor in the case of direct current can be expressed by the formula:

$$r_c(z) = \frac{1}{\sigma S(z)} \quad (74)$$

where σ is conductance and $S(z)$ is the conductor's cross section area.

In the case of RF current, the resistance will be increased by the skin effect associated with skin depth:

$$\Delta = \sqrt{\frac{2}{\sigma \omega \mu_0 \mu_r}} \quad (75)$$

Hence, the distributed resistance of the conductor in the case of RF current can be expressed as:

$$r_c(z) = \frac{1}{\sigma s(z) \Delta} = \frac{1}{s(z)} \sqrt{\frac{\omega \mu_0 \mu_r}{2\sigma}} \quad (76)$$

where $s(z)$ is the perimeter of the conductor's cross section. In the case of RF current, the distributed resistance depends on the total conductor's surface of a unit length.

4.3 Eddy current losses of a coil conductor

Distributed power losses for Eddy currents can be estimated by using magnetic vector potential $\vec{A}(\vec{r})$, induced by small piece of RF coil conductor, by the integral calculation:

$$P_M = \omega^2 \int_{SPACE} \vec{A}^2(\vec{r}) \sigma(\vec{r}) d\vec{r} \quad (77)$$

where ω is RF current frequency, and $\sigma(\vec{r})$ is the total conductance of the medium, which may be assumed to be zero for air unlike for other dielectric media. The magnetic vector potential of a current length dz can be estimated in three dimensional space as:

$$\vec{A}(r) = \frac{\mu_0 \mu_r}{4\pi} I \frac{dz}{r} (\cos\theta \hat{r} + \sin\theta \hat{\theta}) \quad (78)$$

$$|\vec{A}(\vec{r})| = \frac{\mu_0 \mu_r}{4\pi} I \frac{dz}{r} \quad (79)$$

A distributed equivalent power loss resistance $r_M(z)$ in Fig. 3(b) can be estimated as:

$$r_M(z) = \frac{P_M}{\langle I^2(t) \rangle} = \frac{\omega^2 \mu_0^2}{16\pi^2} \int_{SPACE} \frac{\mu_r^2(r) \sigma(\vec{r})}{r^2} d\vec{r} \quad (80)$$

The last integration should be performed over all objects in the space. However, remote from the current conductor objects contribute little to the integral value due to r^2 factor in the denominator Eq. (80). Hence, the contribution from remote objects can be ignored.

In general, magnetic vector potential $\vec{A}(\vec{r})$ should be calculated with an accounting of inverse influence of eddy currents, especially for strong conducting media. Those kind of evaluations are rather complicated for Birdcage coil geometry [30–32] and for many other possible geometries [33]. An example of strong conducting medium interaction will be discussed later for the case of a shielded coil.

4.4 Dielectric losses of a coil conductor

Distributed capacitance can be considered as imperfect leaking condensers in parallel to a coil conductor [28]. Dielectric permeability of a sample and air has both a real and an imaginary components $\varepsilon = \varepsilon' - i\varepsilon''$. The values ε'' and $\tan(\delta) = \varepsilon''/\varepsilon'$ can be referred to as loss and dissipation factors correspondingly. Hence, the total capacitance of the leaking condenser $\tilde{C}_D(z)$ must have the complex value:

$$\tilde{C}_D(z) = c_D(z)(1 - i \tan \delta) \quad (81)$$

As a result, the current phase through the capacitance will get an additional shift δ , which gives non-zero conductance and admittance. The imaginary component of dielectric permeability is responsible for conductance. A simple evaluation shows the connection between conductance, loss and dissipation factors:

$$\sigma = \omega \varepsilon_0 \varepsilon'' = \omega \varepsilon_0 \varepsilon' \tan \delta \quad (82)$$

Using Telegrapher's equations, current through the distributed capacitance c_D (Fig. 3(a)) can be calculated as:

$$i_D(z, t) = - \frac{\partial I(z, t)}{\partial z} = g_D(z) V(z, t) + c_D(z) \frac{\partial V(z, t)}{\partial t} \quad (83)$$

where g_D is a distributed admittance due to dielectric losses of the capacitance. As can be deduced from Eq. (83), an average power losses of the capacitance is:

$$P_D(z) = \langle V(z, t) i_D(z, t) \rangle = \langle V^2(z, t) \rangle \omega c_D(z) \tan \delta \quad (84)$$

which gives the value of distributed admittance for a coil conductor:

$$g_D(z) = \omega c_D(z) \tan \delta \quad (85)$$

Hence an equivalent dielectric power losses resistance can be expressed as:

$$r_D(z) = \frac{P_D(z)}{\langle I^2(z, t) \rangle} = \frac{\langle V^2(z, t) \rangle}{\langle I^2(z, t) \rangle} \omega c_D(z) \tan \delta \quad (86)$$

The tangent loss in the last equation is related to the distributed capacitance rather than to a particular material, since the dielectric material of distributed capacitance is rather heterogeneous.

4.5 The local power losses of the Birdcage Resonator

In general, the local power losses may be calculated for any type of RF coil. One can define the local losses of a coil as losses on a particular part of the coil conductor, which does not contain lumped elements and points of branch. A Birdcage coil has branch points, while solenoids and surface coils do not. Hence, the local power loss and the equivalent local resistance on the piece of the coil conductor between z_1 and z_2 points can be calculated as:

$$P_{12} = R_{12} \langle I^2(t) \rangle \quad (87)$$

$$R_{12} = \int_{z_1}^{z_2} [r_C(z) + r_M(z) + r_D(z)] dz \quad (88)$$

Earlier, it was assumed that only the voltage of a conductor is reduced due to dielectric losses (Fig. 3b). Hence, the current of a conductor stays constant on the local part of a coil.

In the case, where a Leg or an End Ring (ER) of a Birdcage resonator does not contain lumped elements, the active resistances R_1 and R_2 (Fig. 1) can be calculated as one integral:

$$R_1 = \int_{ER} [r_c(z) + r_m(z) + r_d(z)] dz \quad (89)$$

$$R_2 = \int_{LEG} [r_c(z) + r_m(z) + r_d(z)] dz \quad (90)$$

If a Leg or an ER contains lumped elements, the active resistances R_1 and R_2 can be calculated as the sum of local losses or particular Leg or ER of the coil conductor, despite the lumped elements.

4.6 Conducting and dielectric properties of materials

As previously discussed, dielectric materials with nonzero loss tangent have conducting properties in RF electric fields. Hence, the total conductance of materials consists of direct current conductance σ_{DC} and RF current component [34], due to loss tangent:

$$\sigma = \sigma_{DC} + \omega \epsilon_0 \epsilon' \tan \delta \quad (91)$$

For Eddy current losses, both components of Eq. (91) are active, while for dielectric losses only the second component is active, since the direct current circuit is interrupted. Though dielectric materials do not have direct current component of conductance, most of them have nonzero loss tangents (Table 1).

Based on the data of Table 1, the tangent loss of air may be assumed zero. Though air is present between the coil conductor and other objects, it does not nullify dielectric losses of the conductor. Even if a current conductor, transmitting RF current, is placed in vacuum, there will be dielectric energy losses as long as nonzero tangent loss objects are present in a reasonable proximity to the current conductor. In this case, energy will travel from current to the objects and heat them.

NMR samples such as biological tissues might have direct current component of conductance due to Eq. (91). Therefore, the most experimental data for RF current in biological tissues are given for RF currents [35–37], when the both components contribute to the total conductance (Table 2).

5. Influence of a shield in close proximity to the coil

A shield may be considered a highly conductive medium, connected to the ground potential electrode. As previously cited: “Such a shield increases the coil’s distributed capacitance and effective resistance, and reduces its inductance”[28]. An increase of dielectric losses of the coil can be deduced from Eqs. (84–86), since a grounded shield increases distributed capacitance of a coil conductor. Though according to Eq. (39), increased damping factor does not affect the steady state resonance frequencies of the Birdcage coil, the frequencies of a shielded coil can change due to increased distributed capacitance and decreased inductance of the coil conductor. An increased stray capacitance includes in capacitances C_1 and C_2 , while inductive coupling from induced image currents includes into inductances L_1 , L_2 in Eq. (27). Though increased stray capacitance and decreased inductance have the opposite influence on the trends in Eq.(27), experimental measurement shows, that the resonance frequencies of a shielded coil increase with respect to an unshielded one. Hence, the reduced inductance trend in Eq.(27) is stronger than the increased capacitance one.

Magnetic vector potential $\vec{A}(r)$ in the case of strong conducting medium may not be approximated by Eqs.(78,79) due to the inverse influence of Eddy currents, induced by the shield. The total sum of image currents induced in a shield is equal to the coil conductor current, though the image currents are localized within skin depth of the shield surface Eq. (75). The last fact allows to consider a shield as an infinite half-space, parallel to the current conductor, that would be an appropriate approximation for a small piece of the conductor.

If a coil current conductor is assumed to be a filament of infinitely small diameter placed on the distance b from the shield surface, the magnetic vector potential inside the shield medium can be evaluated as described [33]:

$$\vec{A}(x,y) = \left[\frac{\mu_0 \mu_r I}{\pi} \int_0^\infty \frac{e^{qy-mb}}{\mu_r m + q} \cos(mx) dm \right] \hat{z} \quad (92)$$

where $q = \sqrt{(m^2 + ip^2)}$, $p^2 = \mu_0 \mu_r \omega \sigma$.

The axis y in Eq.(92) is orthogonal to the shield surface, is directed to the filament and has the zero point on the surface, while z axis is parallel to the filament. The distributed power loss of the system in this case is given as [33]:

$$P_M(b) = \text{Re} \left\{ \frac{p^2 I^2}{2\pi \sigma} \int_0^\infty \frac{q^* e^{-2mb}}{(\mu_r m + q)(\mu_r m + q^*)} dm \right\} \quad (93)$$

where q^* is the complex conjugate of q .

Hence, an equivalent distributed magnetic losses resistance $r_M(z)$ can be found the same way, as in Eq.(80). It can be concluded from Eq.(93), that distributed power losses decrease exponentially with increasing the distance b between the coil conductor and the shield surface.

The reduced inductance of a shielded conductor can be explained a simple way. Since a grounded half-space generates opposite charges, generated image currents should be shifted by phase π . Hence, the mutual inductance between a current and its image should be negative. According to the Bios-Savart law, the mutual inductance in this case should decrease as $1/(2b)^2$ with the increasing distance b . So, the mutual inductance between a current and its image can be approximated by formula:

$$M = - \frac{f(L)}{(2b)^2} \quad (94)$$

where $f(L)$ is an unknown geometry factor function of the original inductance. Restrictions to M value are applied [28,29], ($M_{12} \leq \sqrt{L_1 L_2}$), which restricts $f(L)$ as:

$$0 \leq \frac{f(L)}{(2b)^2} \leq L \quad (95)$$

Hence, the effective distributed inductance $l(z)$ and the effective local inductances L_1, L_2 in Eq.(27) should be reduced by their mutual inductances as:

$$L_{eff} = L + M = L - \frac{f(L)}{(2b)^2} \quad (96)$$

Since stray capacitance is always small with respect to lapped capacitance of a circuit, the total capacitance of a circuit does not change considerably with increasing stray capacitance. Inductance of a coil, on the contrary, can be reduced considerably by shielding according to Eq.(96). This is the reason why the resonance frequencies of a shielded Birdcage resonator are increased with respect to an unshielded one.

It can be shown by simple evaluations that the distributed capacitance of a shielded coil will increase with respect to an unshielded one. A small piece of coil conductor can be approximated by a conductive sphere of corresponding radius r . Consider two concentric spherical shells of radii r and R . The capacitance of the smaller shell with respect to the bigger one is given by:

$$C_{rR} = 4\pi\epsilon_0\epsilon \frac{rR}{R-r} \quad (97)$$

The capacitance of the smaller shell with respect to the point of infinity may be obtained by extrapolation of the value R to infinity, so Eq.(97) reduces to:

$$C_{r\infty} = 4\pi\epsilon_0\epsilon r \quad (98)$$

The capacitance C_{rR} is larger than $C_{r\infty}$ at any $R > r$. The smaller the difference $(R-r)$, the larger the capacitance C_{rR} .

Again in the case when a conductor is close to a shield wall, the wall can be approximated as a grounded conductive half-space. The grounded surface generates a surface charge as if another sphere with opposite charge were on the opposite side of the surface. The total capacitance of a conductive sphere of radius r with respect to the grounded surface can be approximately expressed by the formula [38]:

$$C_{r\infty} = 4\pi\epsilon_0\epsilon r \left[1 + \frac{1}{2} \ln\left(\frac{b}{b-r}\right) \right] \quad (99)$$

The constant b in Eq.(99) is the distance from the sphere center to the shield surface. Again in this case, the capacitance $C_{r\infty}$ in Eq.(99) is increased with respect to the unshielded and ungrounded sphere in Eq.(98).

So, the distributed capacitance of a shielded coil conductor and the total stray capacitance of the shielded coil are increased with respect to the unshielded coil. However, dielectric losses of a shielded coil will not increase as much as magnetic losses, since there are no conductive or dielectric objects on the shortcut between the conductor and the shield. As mentioned, the loss tangent of air is close to zero.

Increased magnetic loss is the main problem of a shield in close proximity to a coil.

6. Conclusions

Finally, it should be pointed out the agreement of the theory with experimental results. The influence of an active resistance and of a sample load, the quality factor change, ranges of resonance frequencies, magnetic field polarization and other theoretical conclusions have been experimentally investigated. No obvious contradictions to the theory have been detected during the observations. Theoretical and experimental B_1 field polarization dependence curves of 8-sections Birdcage resonator are represented in Fig. 2. Induced voltage was measured for 16 positions of a rotating frame inside the resonator.

The theory can be upgraded for the case of inductive coupling between current loops [3]. For example, Eq.(27) upgraded for the case of inductive coupling looks similar to Eq.(7) in [3]:

$$\omega_k^2 = \frac{2}{\tilde{M}_k} \left\{ \frac{1}{C_1} + \frac{1}{C_2} [1 - \cos(2\pi \frac{k}{N})] \right\} \quad (100)$$

while the value of $\tilde{M}_k$ can be expressed as:

$$\tilde{M}_k = \sum_{m=0}^{N-1} M_m e^{i2\pi k \frac{m}{N}} \quad (101)$$

where M_m are mutual inductances between any two meshes separated by m other meshes. Eq. (101) follows from Eq.(17) for diagonal elements of matrix D . Hence, in the case of inductive coupling between current loops the formulas Eqs.(26–28, 37, 39) would be more cumbersome. This is the simple reason why inductive coupling is not considered here.

The perturbation method allows theoretical evaluation of more complicated tasks. For example, influence of heterogeneous NMR sample requires evaluation of both diagonal and non-diagonal elements of $\hat{H}$ perturbation matrix. The evaluated in this paper explicit time dependence of Birdcage resonator allows to solve time-dependent perturbation tasks, by using the standard time-dependent perturbation theory [25].

This theory may be easily reduced to practical cases like Low-Pass and High-Pass Birdcage coils and may be upgraded for more complicated cases.

As it follows from the presented theory, an induced active resistance in the circuit of Birdcage coil is very important component of the system. Explicit time dependence theory does not make sense without active resistances because of infinite resonance curves in Eqs. (39,44,45). The explicit time dependence theory in turn gives such important parameters of technical performance as transient frequencies in Eq.(32).

Though some researchers believe Birdcage resonator is already well-studied instrument, there are always possibilities to advance developed earlier theories and to improve technical performance of Birdcage coil. The presented theory gives a good example how new theoretical ideas can be used for better coil design.

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References

1. Hayes CE, Edelstein WA, Schenk JF, Mueller OM, Eash M. An efficient highly homogeneous radiofrequency coil for whole-body NMR imaging at 1.5 T. *J Magn Reson* 1985;63:622–8.
2. Tropp J. The theory of Birdcage Resonator. *J Magn Reson* 1989;82:51–62.
3. Leifer MC. Resonant modes of the Birdcage coil. *J Magn Reson* 1997;124:51–60.
4. Jin, J. *Electromagnetic Analysis and Design in Magnetic Resonance Imaging*. New York: CRC Press; 1999. p. 137-209.
5. Harpen M. Harmonic structure of cylindrical imaging coils. *Medical Physics* 1990;17:686–90. [PubMed: 2215415]
6. Watkins JC, Fukushima E. High-pass Birdcage coil for nuclear magnetic resonance. *Rev Sci Instrum* 1988;59:926–9.
7. Pascone R, Garcia BJ, Fitzgerald TM, Vullo T, Zipagan R, Cahill PT. Generalized electrical analysis of low-pass and high-pass Birdcage resonators. *Magn Reson Imaging* 1991;9:395–408. [PubMed: 1881259]
8. Vullo T, Zipagan R, Pascone R, Whalen JP, Cahill PT. Experimental design and fabrication of Birdcage resonators for magnetic resonance imaging. *Magn Reson Med* 1992;24:243–52. [PubMed: 1569864]
9. Pascone R, Vullo T, Farrelly J, Mancuso R, Cahill PT. Use of transmission line analysis for multi-tuning of Birdcage resonators. *Magn Reson Imaging* 1993;11:705–12. [PubMed: 8345785]
10. Pimmel P, Briguet A. A hybrid Birdcage resonator for sodium observation at 4.7T. *Magn Reson Med* 1992;24:158–62. [PubMed: 1313525]
11. Tropp, J. The hybrid Birdcage resonator. *Abstracts of Soc Magn Reson Med; Annual Meeting*; 1992. p. 4009
12. Joseph PM, Lu D. A technique for double resonant operation of Birdcage imaging coils. *IEEE Trans Med Imaging* 1989;8:286–94. [PubMed: 18230528]
13. Rath AR. Design and performance of double- tuned Bird-Cage coil. *J Magn Reson* 1990;86:488–95.
14. Hoult DI, Leuterbur PC. The sensitivity of the zeugmatographic experiment involving human samples. *J Magn Reson* 1979;34:425–33.
15. Gadian DG, Robinson FNH. Radiofrequency losses in NMR experiment on electrically conducting samples. *J Magn Reson* 1979;34:449–55.
16. Li SJ, Shen J. Integrated RF probe for in vivo multinuclear spectroscopy and functional imaging of rat brain using an 11.7 Tesla 89 mm bore vertical microimager. *MAGMA* 2005;18:119–27. [PubMed: 16007474]
17. Sommerfeld, A. *Partial Differential Equations in Physics*. New York: Academic Press; 1949. p. 166-235.
18. Sveshnikov, AG.; Bogolubov, AN.; Kravtsov, BB. *Lectures on Mathematical Physics*. Moscow: Moscow University; 1993. p. 40-64. p. 253-318.
19. Levine, IN. *Quantum Chemistry*. N.J. Prentice Hall; Englewood Cliffs: 1991. p. 545-93.
20. Muir, T. *A Treatise on the Theory of Determinants*. New York: Dover; 1960. p. 442-84.
21. Ortega, JM. *The second Course*. New York: Plenum Press; 1987. Matrix theory; p. 242-46.
22. Coddington, EA. *An Introduction to Ordinary Differential Equations*. Englewood Cliffs N.J: Prentice Hall; 1961. p. 49-101.
23. Tikhonov, AN.; Vasilieva, AB.; Sveshnikov, AG. *Differential Equations*. Moscow: Nauka; 1980. p. 70-107.
24. Barbara TM, Martin JF, Wurl JG. Phase transients in NMR probe circuits. *J Magn Reson* 1991;93:497–508.
25. Landau, LD.; Lifshitz, LM. *Non- Relativistic Theory*. New York: 1974. *Theoretical Physics. Quantum Mechanics*; p. 163-97.

26. Redpath TW, Hutchinson JMS. Estimating patient dielectric losses in NMR imagers. *Magn Reson Imaging* 1984;2:295–300. [PubMed: 6530932]
27. Heteren JG, Henkelman RM, Bronskill MJ. Equivalent circuit for coil-patient interactions in magnetic resonance imaging. *Magn Reson Imaging* 1987;5:93–99. [PubMed: 3586884]
28. Terman, FE. *Electronic and Radio Engineering*. New York: McGraw Hill; 1955. p. 24–43.
29. Terman, FE. *Radio Engineer's Handbook*. New York: McGraw Hill; 1943. p. 73–134.
30. Harpen MD. Eddy current distributions in cylindrical samples: Effect on equivalent sample resistance. *Phys Med Biol* 1989;34:1229–38. [PubMed: 2798558]
31. Harpen MD. Theoretical Description of the interaction between Birdcage coils and lossy samples. *Med Phys* 1991;18:1052–56. [PubMed: 1961146]
32. Harpen MD. Equivalent circuits of Birdcage resonators. *Magn Reson Med* 1993;29:263–8. [PubMed: 8429794]
33. Tegopoulos, JA.; Krieziz, EE. *Eddy Currents in Linear Conducting Media*. New York: Elsevier; 1985. p. 89–95.
34. Sueki M, Rinard GA, Eaton SS, Eaton GR. Impact of high-dielectric-loss materials on the microwave field in EPR experiments. *J Magn Reson* 1996;118A:173–88.
35. Stoy RD, Foster KR, Schwan HP. Dielectric properties of mammalian tissues from 0.1 to 100 MHz; a summary of recent data. *Phys Med Biol* 1982;27:501–13. [PubMed: 7089048]
36. Surowiec A, Stuchly SS, Eidus L, Swarup A. In vitro dielectric properties of human tissues at radiofrequencies. *Phys Med Biol* 1987;32:615–21. [PubMed: 3588674]
37. Stuchly MA, Stuchly SS. Dielectric properties of biological substances—tabulated. *J Microwave Power* 1980;15:19–26.
38. Hudlet S, Jean MS, Guthman C, Berger J. Evaluation of capacitance force between an atomic force microscopy tip and a metallic surface. *The European Physical Journal* 1998;B2:5–10.

Fig. 1.

Three consecutive meshes of Driven Hybrid Birdcage ladder network. All meshes of the network are driven by different periodic voltage sources $U_n(t)$ with low internal impedances.

Fig. 2. Theoretical (dashed line) and experimental (solid line with experimental points) B_1 field polarization dependence curves of eight-section Birdcage resonator in polar system. Induced voltage was measured for 16 positions of rotating frame inside the resonator.

Fig. 3.

a) A simplified equivalent circuit of distributed parameters for a small piece of an RF coil conductor.

b) A simplified equivalent circuit of distributed power losses for a small piece of an RF coil conductor. Additional equivalent distributed active resistances of the conductor $r_M(z)$ and $r_D(z)$ are included, while distributed admittances $g_M(z)$ and $g_D(z)$ are assumed zero.

TABLE 1

Dielectric properties of materials

Material	Dielectric permittivity ε'	Loss tangent $\tan \delta = \varepsilon''/\varepsilon'$	Reference
Air	1.00	0.000	[28]
Water	64.1	0.46	[34]
Wood	3–5	0.03–0.07	[28]
Polystyrene	2.4–2.9	0.0002	[28]
Polyethelene	2.3	0.00015–0.0003	[28]

TABLE 2

Dielectric properties of biological tissues

Tissue	Frequency (MHz)	Dielectric permittivity ϵ'	Relative loss factor ϵ''	Conductivity σ (mS cm ⁻¹)	Reference
Brain (Rat) <i>in vivo</i>	100	84	145	–	[37]
Brain (Human) <i>minced</i>	100	–	69	3.3–5.0	[35]
Liver (Human) <i>in vitro</i>	100	95	76–79	7.02	[36]
Kidney (Dog) <i>in vivo</i>	100	78.5	140	8.7	[35]
Muscle (Rat) <i>in vivo</i>	100	80	180	8.4	[35]