Buckling of sheared and compressed microfibrils

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In this paper, we study the stability of an initially straight elastic fibril clamped at one end, while the other end is subjected to a constant normal compressive force and a prescribed shear displacement. We found the buckling load of a sheared fibril to be always less than the Euler buckling load. Furthermore, if the end of the fibril loses adhesion, then the buckling load can be considerably less. Our result suggests that the static friction of microfibre arrays can decrease with increasing normal compressive load and, in some cases, friction force can actually become negative.

Keywords: buckling; fibrils; shear; friction

1. INTRODUCTION

Many small animals and insects use fine hairs on their feet to climb and to stick to surfaces. Inspired by these micro- and nanostructures, many research groups have fabricated synthetic mimics using various polymers or carbon nanotubes to create arrays of microfibrils. Typically, fibrils in these arrays are terminated with either a thin film or a spatula to enhance contact and adhesion (Gorb *et al.* 2006; Kim & Sitti 2006; Aksak *et al.* 2007; del Campo *et al.* 2007; Greiner *et al.* 2007; Reddy *et al.* 2007; Schubert *et al.* 2007). These fibrils are part of a backing layer that is made up of the same material, typically a soft elastomer such as poly(dimethylsiloxane) or polyurethane. These biologically inspired surfaces have adhesion considerably greater than that of a flat surface of the same material (Noderer *et al.* 2007). More recently, the friction behaviour of these microfibril arrays has been investigated by different research groups (e.g. Majidi *et al.* 2006; Bhushan & Sayer 2007; Kim *et al.* 2007; Varenberg & Gorb 2007; Yao *et al.* 2007; Shen *et al.* 2008, 2009; Gravish *et al.* 2009; Vajpayee *et al.* 2009). In this work, we focus on synthetic bioinspired surfaces instead of fibrillar structures in small animals such as gecko. These natural systems are more complex (e.g. fibrils have natural curvatures and exhibit directional adhesion and friction properties). Readers interested in these systems should refer to Zhao *et al.* (2008) and Autumn *et al.* (2006) and the references within.

Figure 1 shows the schematics of a typical experiment to measure static and sliding friction. A hard and smooth surface, such as the surface of a spherical glass indenter, is brought into contact with the surface of the microfibril array by the application of a normal compressive force. In a standard friction test, this

compressive force is *kept constant* as shear displacement is applied to move the indenter horizontally.

A difficulty in interpreting experimental results is that fibrils tend to buckle and collapse during experiments. While buckling increases compliance, it reduces adhesion by breaking contact between fibril ends and the indenter (Hui *et al.* 2007). As a result, fibril buckling is usually detrimental to adhesion, as demonstrated by Glassmaker *et al.*'s (2004) experiments. In some cases, collapsed fibrils can lead to greater contact and can increase adhesion and friction (e.g. Ge *et al.* 2007). To avoid buckling of fibrils, many shear experiments are carried out with a fixed normal indenter displacement (Kim *et al.* 2007; Zhao *et al.* 2008). These experiments show very high 'static' friction. Typically, in these experiments, a very small compressive force is applied to bring the fibrils in good contact with the substrate, then the fibrils are sheared keeping the normal indenter displacement fixed; as a result, the longitudinal force along the fibril changes from compression to tension. In these experiments, the normal force acting on the fibril array changes during shear. These behaviours are quite different from a friction test in which the normal force is maintained to be constant throughout; in this case, common sense suggests that buckling of fibrils will play an important role in the static friction behaviour of these arrays. For example, the experiment of Shen *et al.* (2009) clearly shows initial buckling of fibrils when the indenter is displaced laterally at a fixed normal load. It is interesting to note that this difference between normal displacement controlled and normal force controlled friction tests is much less significant for typical structural materials where the surfaces are non-fibrillar. This motivates us to analyse fibril buckling as a step towards developing contact and friction models for bioinspired fibrillar structures.

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Figure 1. Schematic of a typical indentation and friction experiment. (a) A rigid, smooth, spherical indenter is brought into contact with a microfibril array. The compressive force is sufficiently large to cause fibrils in the contact zone to buckle. (b) A smaller compressive force is applied to the indenter, which is sheared to the right, causing the fibrils attached to the indenter to shear. For a sufficiently large compressive force, the shear force at the ends of a fibril acts in a direction opposite to the imposed shear displacement.

Figure 2. Undeformed and deformed fibril in the inertial reference frame. The insert is a free body diagram of an infinitesimal element, $\delta s'$.

Intuitively, it is easy to understand why fibrils tend to collapse in shear experiments even though the compressive force is below the buckling load—a sheared fibril bends readily. However, it is non-trivial to calculate the deflection of a buckled fibre and to determine how buckling load is affected by shear. Since classical buckling theory deals with buckling about an *initially* straight state (straight fibril in our case), a question arises how one defines the buckling load of a sheared fibril. This question will be addressed both numerically and theoretically in this work.

Fibrils under combined normal and shear loads can exhibit very counterintuitive behaviour. Shen *et al.* (2009) observed that, during a shearing test, the fibrils shear in the direction *opposite* to the shearing force, as shown schematically in the inset in figure 1b. Note that this behaviour occurs initially during the test. We will demonstrate that this counterintuitive behaviour can occur under low applied compressive loads in systems with weak adhesion.

In our previous work (Liu *et al.* 2009; Shen *et al.* 2009), we successfully used a nonlinear rod theory in which the rod can stretch and bend but cannot shear to study the deformation of typical fibrils in an array subjected to shear and normal loads. Nonlinear rod theory is also required to study the post-buckling

behaviour. As is well known, the governing equations for *static* equilibria of a nonlinear rod can have many solutions corresponding to the same loading and boundary conditions. These equilibrium solutions have different deformed shapes which correspond to different energy states, many of which are stable. A difficulty with the static equilibria calculations is that it is not easy to identify which one of these stable states the rod will converge to. A simple way to circumvent this difficulty is to use a dynamic rod model to compute the deformation of a fibril. In this paper both approaches are used.

The plan of this paper is as follows. The dynamic rod model is briefly summarized in §2. The numerical results are given in §3. The results of stability analysis are presented in §4. Section 5 discusses and summarizes our results.

2. DYNAMIC ROD MODEL

The undeformed rod is assumed to be straight and the arc length coordinate of a material point on the centreline of the undeformed rod is specified by s (figure 2). The position vector of the material point s after deformation (at time t) is denoted as $\mathbf{r}(s,t)$. A local

coordinate system with orthonormal basis vectors $\{\hat{\mathbf{a}}_i\}$ is attached at each cross section, with $\hat{\mathbf{a}}_3$ aligned with the centreline tangent, while the other two vectors are aligned with the principal flexure axes. This body-fixed frame describes the orientation of a cross section with respect to the inertial frame, $\{\hat{\mathbf{e}}_i\}$. The dynamic state of a rod's cross section is represented by four field variables: linear velocity ($\mathbf{v}$), angular velocity ($\boldsymbol{\omega}$), curvature ($\boldsymbol{\kappa}$) and force ($\mathbf{f}$), defined along the rod's centreline. All field variables are functions of both s and t . The angular velocity $\boldsymbol{\omega}$ of a cross section is defined as the rotation of body-fixed frame $\{\hat{\mathbf{a}}_i\}$ per unit time relative to the inertial frame, $\{\hat{\mathbf{e}}_i\}$, i.e.

$$\left(\frac{\partial \hat{\mathbf{a}}_i}{\partial t}\right)_{\{\hat{\mathbf{e}}_i\}} = \boldsymbol{\omega} \times \hat{\mathbf{a}}_i. \quad (2.1)$$

Here, the subscript $\{\hat{\mathbf{e}}_i\}$ specifies that the derivative is taken relative to the inertial frame. Similarly, the curvature vector $\boldsymbol{\kappa}$ is the rotation of the body-fixed frame per unit arc length relative to the inertial frame,

$$\left(\frac{\partial \hat{\mathbf{a}}_i}{\partial s}\right)_{\{\hat{\mathbf{e}}_i\}} = \boldsymbol{\kappa} \times \hat{\mathbf{a}}_i. \quad (2.2)$$

The internal moment $\mathbf{q}$ is related to the curvature by a constitutive law for bending. For this paper, the initial rod curvature is zero and we use the standard linear relation

$$\mathbf{q}(s, t) = \mathbf{B}\boldsymbol{\kappa}(s, t). \quad (2.3)$$

Since we have chosen $\{\hat{\mathbf{a}}_i\}$ to coincide with the principal torsion–flexure axes of a cross section, the torsion–flexure stiffness tensor $\mathbf{B}$ is diagonal with respect to the body-fixed frame.

Extensibility of the centreline is added to the field equations by means of a kinematic relationship between ds , the length of an infinitesimal undeformed material element, and ds' , the same element after extension as $ds' = \lambda ds$, where λ represents the longitudinal stretch ratio. A nonlinear constitutive model is used to relate λ to the longitudinal force f_3 by assuming that the material is neo-Hookean. Under uniaxial tension, the engineering stress σ and the stretch ratio λ satisfies

$$\sigma_{33} = \frac{E}{3} \left(\lambda - \frac{1}{\lambda^2} \right). \quad (2.4)$$

It can be shown that there is only one positive real root for λ in equation (2.4). The longitudinal force f_3 is thus given by

$$f_3 = \frac{EA_0}{3} \left(\lambda - \frac{1}{\lambda^2} \right), \quad (2.5)$$

where A_0 is the undeformed cross-sectional area of a fibril.

Vector quantities such as $\mathbf{f}$ can be expressed in terms of their components once we fix the basis. In the following, quantities such as $\vec{f}, \vec{v}, \vec{q}, \vec{\omega}, \vec{\kappa}$ are three tuples consisting of the three components of these vectors with respect to the body-fixed frame $\{\hat{\mathbf{a}}_i\}$. In the body-fixed reference frame, the linear momentum and

the angular momentum balance equations for an extensible rod are (Goyal *et al.* 2005)

$$\frac{\partial \vec{f}}{\partial s} + \vec{\kappa} \times \vec{f} = m \left(\frac{\partial \vec{v}}{\partial t} + \vec{\omega} \times \vec{v} \right), \quad (2.6)$$

$$\frac{\partial \vec{q}}{\partial s} + \vec{\kappa} \times \vec{q} = \mathbf{I} \frac{\partial \vec{\omega}}{\partial t} + \vec{\omega} \times \mathbf{I} \vec{\omega} + \lambda \vec{f} \times \vec{a}_3, \quad (2.7)$$

where $\vec{a}_3 = (0, 0, 1)$. In equations (2.6) and (2.7), m is the mass per unit arc length and $\mathbf{I}$ is the mass moment of inertia matrix per unit arc length with respect to the body-fixed frame. Additionally, we have the following equation enforcing unshearability, i.e. line elements perpendicular to the centreline remain so even when the rod is deformed,

$$\frac{\partial \vec{v}}{\partial s} + \vec{\kappa} \times \vec{v} = \vec{a}_3 \frac{\partial \lambda}{\partial t} + \lambda \vec{\omega} \times \vec{a}_3, \quad (2.8)$$

and a compatibility condition between $\vec{\kappa}$ and $\vec{\omega}$ that enforces continuity of the body-fixed reference frame $\{\hat{\mathbf{a}}_i\}$ along the arc length coordinate s and time t , i.e.

$$\frac{\partial \vec{\omega}}{\partial s} + \vec{\kappa} \times \vec{\omega} = \frac{\partial \vec{\kappa}}{\partial t}. \quad (2.9)$$

The deformation of the fibrillar array while undergoing indentation and friction tests is assumed to be planar, i.e. independent of the out-of-plane coordinate $\hat{\mathbf{e}}_2$. This is also consistent with the loading conditions in shearing and indentation experiments. As a result of this simplification, $f_1, f_3, v_1, v_3, \omega_2, \kappa_2$ are the only non-zero physical quantities. Thus, the angular momentum balance reduces to a scalar equation and the only relevant element in the matrix $\mathbf{I}$ is I_2 , which we denote simply as I . Furthermore, since in our case there is no rotation about the $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$ axes, the only element of $\mathbf{B}$ that comes into play is the bending stiffness EJ about the principal flexure axis along $\hat{\mathbf{a}}_2$. Here, E is the small strain Young's modulus of the rod and J is the second moment of area. To damp out oscillations of elastic waves, we include the term $h v_i$ in the linear momentum balance equations. Here, h is the damping coefficient. The following six scalar equations survive from their three-dimensional vector counterparts in equations (2.6)–(2.9).

$$\frac{\partial f_1}{\partial s} + \kappa_2 f_3 = m \left(\frac{\partial v_1}{\partial t} + \omega_2 v_3 \right) + h v_1, \quad (2.10)$$

$$\frac{\partial f_3}{\partial s} - \kappa_2 f_1 = m \left(\frac{\partial v_3}{\partial t} - \omega_2 v_1 \right) + h v_3, \quad (2.11)$$

$$I \frac{\partial \omega_2}{\partial t} - \lambda f_1 = \frac{\partial q_2}{\partial s}, \quad (2.12)$$

$$\frac{\partial v_1}{\partial s} + \kappa_2 v_3 = \lambda \omega_2, \quad (2.13)$$

$$\frac{\partial v_3}{\partial s} = \kappa_2 v_1 + \frac{\partial \lambda}{\partial t} \quad (2.14)$$

$$\text{and} \quad \frac{\partial \omega_2}{\partial s} = \frac{\partial \kappa_2}{\partial t}. \quad (2.15)$$

We define the following normalized variables:

$$\left. \begin{aligned} s = LS, t = \tau T, q_2 = \frac{EJ}{L} Q_2, f_i = \frac{EJ}{L^2} F_i, \\ \kappa_2 = \frac{K_2}{L}, v_i = \frac{L}{\tau} V_i \text{ and } \omega_2 = \frac{1}{\tau} \Omega_2 \quad i = 1, 3, \end{aligned} \right\} \quad (2.16)$$

where L is the length of the undeformed rod. Note that time is normalized by $\tau \equiv \sqrt{mL^4/EJ}$, which is proportional to the time for a flexure wave to traverse the length of a rod. Also, with this normalization, the Euler buckling load for a non-stretchable clamped–clamped rod is $-4\pi^2$. Similarly, the Euler buckling load for a non-stretchable pinned–pinned rod is $-\pi^2$.

The normalized governing equations for an extensible rod are, accordingly,

$$\frac{\partial F_1}{\partial S} + K_2 F_3 = \frac{\partial V_1}{\partial T} + HV_1 + \Omega_2 V_3, \quad (2.17)$$

$$\frac{\partial F_3}{\partial S} - K_2 F_1 = \frac{\partial V_3}{\partial T} + HV_3 - \Omega_2 V_1, \quad (2.18)$$

$$\alpha^2 \frac{\partial \Omega_2}{\partial T} - \lambda F_1 = \frac{\partial Q_2}{\partial S}, \quad (2.19)$$

$$\frac{\partial V_1}{\partial S} + K_2 V_3 = \lambda \Omega_2, \quad (2.20)$$

$$\frac{\partial V_3}{\partial S} = K_2 V_1 + \frac{\partial \lambda}{\partial T}, \quad (2.21)$$

$$\frac{\partial \Omega_2}{\partial S} = \frac{\partial K_2}{\partial T} \quad (2.22)$$

$$\text{and} \quad F_3 = \frac{A_0 L^2}{3J} \left(\lambda - \frac{1}{\lambda^2} \right). \quad (2.23)$$

Note that these normalized equations are governed by two dimensionless parameters,

$$\alpha = \sqrt{\frac{I}{mL^2}} \quad \text{and} \quad H = \frac{hL^2}{\sqrt{mEJ}}, \quad (2.24)$$

where α is the slenderness ratio defined as the ratio of the radius of gyration of the rod to its length and H is the normalized damping coefficient. Since H controls damping of oscillations, the equilibrium solution depends only on α .

We employ a typical boundary condition that is encountered in shearing and indentation experiments. Specifically, the end of the fibril (O in figure 2) which is attached to the backing is assumed clamped, whereas the end that is in contact with the indenter is constrained from rotation but allowed to shear. It is important to note that the last two boundary conditions are satisfied only for fibrils that are well adhered to the indenter. Most investigators use fibrils that have structures at their tips to improve contact and adhesion; for example, Varenberg & Gorb (2007) and Kim & Sitti (2006) used terminal contact plates, whereas Glassmaker *et al.* (2007) and Yao *et al.* (2007) used a terminal continuous thin film. For these fibrils, it is expected that the aforementioned boundary conditions would be satisfied. However, these boundary conditions will not be met for fibrils with no special structures at their tips. The adhesion of these fibrils is found to be weak and they tend to buckle easily as

Figure 3. Example of a loading history prescribed at the top end of a fibril in dynamic simulation with a final normalized shear displacement of 0.05 and a normalized (compressive) load of $-4\pi^2$.

their tips lose adhesion (Hui *et al.* 2007). We will discuss this later in §4.

To solve the nonlinear partial differential equations (2.17)–(2.23) for the unknowns κ_2 , Ω_2 , F_1 , F_3 , V_1 and V_3 , we use Keller's box method (1971) to discretize these equations in both space and time as well as numerically integrating them. This method has second-order accuracy. The resulting nonlinear difference equations are implicit and their solution is required to satisfy the boundary conditions described above (see appendix A for some details).

3. RESULTS

The numerical results presented in this section are valid for fibrils of any dimension as long as they have the same slenderness ratio, α . To give an idea of typical dimensions, in the samples used by Shen *et al.* (2008), the fibrils have a square cross section of $10 \mu\text{m}$ in width. Fibril heights range from 30 to $100 \mu\text{m}$. Typical materials used to create these arrays are elastomers such as poly(dimethylsiloxane) (PDMS; Sylgard 184, Dow Corning) and polyurethane that has a shear modulus of the order of 1 MPa. The slender ratio of the samples used by Shen *et al.* (2008) is 1.054×10^{-3} . The numerical results below are obtained using this value.

To analyse the buckling behaviour of a fibril subjected to a compressive normal force and a shear displacement at the top end, we prescribe axial force f_3 (compressive in nature, negative) and shear displacement Δ_s at the top end. The kinematic boundary conditions are imposed by enforcing zero linear and angular velocities at the bottom end of the fibril. In each case, the desired shear displacement is simulated by prescribing a shear velocity at the top end (as shown in figure 3). Simultaneously, we increase the normalized compressive force F_n from zero to its maximum value using a hyperbolic tangent function. The notations F_n , F_s denote values of F_3 and F_1 , respectively, at the top end.

Figure 4. Normal tip displacement (normalized by L) plotted against the compressive load for six different values of Δ_s . For each prescribed shear displacement, there is a cusp at which the slope is discontinuous.

Figure 5. Normal tip displacement (normalized by L) versus compressive force of a fibril with a normalized shear tip displacement of 0.05. Note that the number of inflection points in the deformed shape changes from one to two as the cusp is reached.

How does one define the buckling load of a sheared fibril? A zoom-in view of the variation of the normal tip displacement with the compressive load for six fixed values of shear displacement Δ_s is shown in figure 4. For each Δ_s , there is a distinct point at which the slope of the normal tip displacement curve changes abruptly, i.e. a cusp is formed. Figure 5 shows the shape of a fibril just before and after this cusp. Note that the number of inflection points in the deformed shape changes from one to two as the cusp is reached.

As is well known, a cusp is a bifurcation point at which multiple solutions exist; one of these solutions has to satisfy the condition of continuity of slope. In our case, this solution may be unstable. If this is the case, then this bifurcation point corresponds to buckling instability or fibril collapse. The existence of an

Figure 6. Buckling load as a function of normalized shear displacement at the tip of a fibril for the nonlinear dynamic and stability analysis. Dashed line with asterisks, extensible-dynamic; solid continuous line, extensible-stability; dotted line, inextensible-stability.

unstable solution will be demonstrated in the next section. Assuming for now that an unstable solution does exist, we define the compressive force at this cusp point to be the buckling load of a sheared fibril.

We summarize our result in figure 6, where the normalized compressive force at the cusp F_n^* is plotted against the normalized shear displacement at the top end. As expected, F_n^* decreases as the shear displacement at the tip is increased. The results of the dynamics analysis (asterisks in figure 6) suggest the following quadratic dependence:

$$F_n^* = -0.00182 \left(\frac{\Delta_s}{L} \right)^2 + \frac{0.00232 \Delta_s}{L} + 41.275. \quad (3.1)$$

3.1. Shear constraint violation

The dependence of shear force on shear displacement is particularly interesting. Figure 7 shows two different regimes. For compressive forces less than π^2 , the slope of the shear displacement versus shear force curve is positive. That is, the shear force increases in the same direction as the shear displacement. However, when the compressive force is greater than π^2 , the slope becomes negative, i.e. the shear force is in the *opposite* direction to the shear displacement! This phenomenon is actually observed by the experiments of Shen et al. (2008), as mentioned in §1.

The fact that the shear force is in the opposite direction to the shear displacement suggests that if a fibril were to overcome the shear constraint at the boundary (recall that the fibrils are assumed to be adhered firmly to the rigid indenter) then the fibril will become unstable and will collapse at a lower compressive force of π^2 . This loss of boundary constraint is illustrated schematically in figure 8a,b (see also caption for explanation). Physically, this means that sheared fibrils can

Figure 7. Shear force versus shear displacement and stability/bifurcation diagram for an inextensible rod. Solid line, stable; dotted line, one negative eigenvalue (unstable); dashed line, two negative eigenvalues (unstable).

Figure 8. (a) Shear constraint on the upper end of the fibril; (b) an equivalent boundary condition for the critical condition when the shear force becomes zero. The buckling condition for (b) is the same as that for the equivalent pinned–pinned rod between two points of inflection A and B as denoted in (c). In experiments, some fibrils have weak shear constraints. Violation of the right side shear constraint (dashed grey) in (a) beyond a critical condition leads to dynamic collapse equivalent to that in (b) or (c).

collapse at much lower compressive loads if they are poorly adhered to the indenter.

Why does this instability occur at $F_n = -\pi^2$? Here, we offer a simple physical explanation. Figure 8b shows that when the shear constraint is removed and the top end of the fibril is allowed to shear without friction, i.e. $F_s = 0$, the resulting structure becomes equivalent to the pinned–pinned configuration as shown in figure 8c. The normalized buckling load for this case is exactly $-\pi^2$. It is well known that fibrils

without an attachment mechanism lose adhesion readily in *normal* indentation tests (no shear displacement applied) when they buckle. It has been shown experimentally and theoretically that buckling of the fibril causes the top end of a fibril which is attached to the indenter to rotate; as a result, it behaves as a pinned support instead of a clamped support when in contact with the indenter (Hui *et al.* 2007). This experimental and theoretical result is consistent with the fact that the shear instability occurs at $\pi^2 EI/L^2$.

3.2. Stability analysis

In the following, we analyse the stability of static equilibrium configurations of the sheared fibril as we vary the compressive load as well as the shear displacement. Recall that the fibril is assumed to be stretchable but unshearable. To analyse the stability of a given configuration, the fibril must be perturbed about the given equilibrium configuration such that the perturbed configuration also satisfies the unshearability constraint. For a stable configuration, all ‘nearby’ and admissible perturbed configurations will have higher potential energy. Numerical determination of stability, especially in the presence of a point-wise unshearability constraint, is a non-trivial problem and was discussed in detail in the work of Kumar & Healey (2010). This work also shows that the minimum potential energy method is equivalent to the linearized dynamics stability criterion for conservative problems, as is the case here. Here, we use their technique to determine whether a given configuration is stable or not.

Figure 6 plots the results of the dynamics analysis (extensible rod) as well as the stability results based on static analysis for both an extensible and inextensible rod. Figure 6 shows a decrease in the buckling load as the fibril is sheared. Also, the buckling load corresponding to an inextensible fibril is lower than that

for an extensible fibril. As expected, this difference becomes less pronounced as the normalized compressive load is reduced since the fibril starts to behave more as an inextensible rod. This indicates that extensibility does not affect the buckling load greatly. We mentioned that the dynamics result is slightly above the static result. We believe that the static result is more accurate since numerically it is more straightforward, so finer discretization can be used without sacrificing computation time. Figure 7 shows the stability–bifurcation diagram for an inextensible rod. All stable configurations are shown as solid black lines. As discussed earlier and shown in figure 7, shear force and shear displacement are of the same sign for compressive loads below π^2 . Figure 7 also shows that all equilibrium solutions are stable for normalized compressive loads less than 30. This means that the change in sign of the shear force is not related to the onset of buckling for a rod whose tip is constrained against shear displacement. However, if this constraint is violated, the fibril will become unstable as was described in the previous section. By examining the energy landscape of the rods for the normalized compressive loads of 30 and 35, we found the rods lose stability at a critical value of shear displacement. These critical shear forces and displacements correspond to buckling points. The presence of multiple equilibria (bifurcation) is indicated by the different dotted curves emitted from these bifurcation points. These bifurcated solutions are found to be unstable, indicating the absence of a stable equilibrium configuration. Physically, this corresponds to collapse of fibrils in this regime, which also confirms that the cusp observed in our dynamic simulation corresponds to the same bifurcation point.

4. DISCUSSION AND SUMMARY

Using dynamic rod model and stability analysis, we define and analyse the buckling of a fibril that is subjected to prescribed shear displacement and a constant normal compressive force. Our result is summarized in figure 6, which shows that buckling load decreases with the prescribed shear displacement. The buckling load is bounded from above by the Euler buckling load of a clamped–clamped rod, which is $4\pi^2 EI/L^2$. It should be noted that our rod model allows for non-uniform stretching of the centreline of the rod. This feature is included since there are situations in which fibril stretching is not negligible, e.g. when a fibril is sheared under a fixed normal displacement (see fig. 3 in Kim *et al.* 2007). However, for the boundary conditions used in this work, extensibility plays a small role in the determination of stability, as shown in figure 6.

What is unexpected is that instability can occur at or above a compressive load of $\pi^2 EI/L^2$, which is the Euler buckling load of a pinned–pinned rod. This load is exactly one quarter of the Euler buckling load of a clamped–clamped rod. Specifically, we found that, when a fibril subjected to a normal compressive load exceeding $\pi^2 EI/L^2$ is displaced laterally, the shear

Figure 9. Normalized shear force, F_s , versus normalized compressive force, F_n , for several normalized shear displacements, Δ_s/L .

force acting on the fibril will be in the opposite direction to the imposed shear displacement. This has the following interesting implications. Consider the rigid indenter in figure 1b, which is perfectly adhered to a set of m identical fibrils. Assume that the indenter has a very large radius of curvature, so it is a flat block. If this block is subjected to a normal compressive force greater than $m \times \pi^2 EI/L^2$ (e.g. by adding weights on top of the block), then it is unstable in shear. This means that a fibrillar interface has a negative friction coefficient, and energy is transferred from the fibrils to the loading device. This prediction is actually observed by Shen *et al.* (2009). In the initial part of their experiment, they observed that buckling of fibrils and the slope of the shear force versus shear displacement curve under a fixed normal compressive load becomes negative (see fig. 11 in Shen *et al.* 2009).

What happens if the applied normal compressive load is less than $\pi^2 EI/L^2$? In this case, the system is stable. However, intuitively, one expects that, for a fixed shear displacement Δ_s , the shear compliance of a fibril should increase with increasing normal load F_n . In other words, the shear force at a fixed Δ_s should decrease with increasing compression. Figure 9 shows this is indeed the case. Note that the compliance of the system is positive as long as $F_n > -\pi^2 EI/L^2$.

The fact that the shear force on a fibril decreases with normal load ($F_n > -\pi^2 EI/L^2$) suggests that *static friction should also decrease with normal force* as long as the boundary constraints on the fibrils are maintained. It is important to note that static friction is defined in different ways in the literature. In this paper, static friction corresponds to maximum shear load when the block is subjected to a *fixed normal load*. Since the normal load is fixed as we shear a fibril, the vertical displacement of the fibril end is not constrained. For example, a rigid block resting on a fibrillar interface consisting of a uniform matt of identical fibrils will move vertically down as it is sheared. Note that the maximum shear load may occur after a

Figure 10. Stencil for the Keller box method.

buckling instability since it is possible that: (i) the ends of the fibrils are still well adhered to the indenter and (ii) collapsed fibrils can support more shear. Static friction is defined differently in the work of Zhao *et al.* (2009) and Kim *et al.* (2007). In their tests, the vertical displacement of the contactor is fixed during shear after a compressive preload is applied to bring the contactor into contact with the fibrils. In this set-up, if the fibrils are well adhered to the contactor, they will be stretched and the normal load will change with shear and actually become tensile (see fig. 3 in Kim *et al.* 2007). In this case, static friction corresponds to adhesive failure of the fibril ends.

To summarize, assuming that fibrils in the array are identical and the number of fibrils does not change as we shear (e.g. the indenter is flat), our theory predicts the following.

- If the applied compressive normal force on each fibril is less than $\pi^2 EI/L^2$, then static friction should decrease with increasing normal load.
- If the applied compressive normal force on each fibril is greater than $\pi^2 EI/L^2$, then the friction force becomes negative if adhesion is weak, resulting in the violation of shear constraint.
- If shear constraint is not violated, then the fibrillar interface will collapse at a normal load (per fibril) given by figure 6.

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APPENDIX A: NUMERICAL METHODS

Briefly, in Keller's box method (Keller 1971), a rectangular grid is created along the space and time axis, as shown in figure 10. We discretize the differential equations by finite difference. For example, a dependent variable u is approximated by its value at the midpoint of the box by

$$u_{j-1/2}^{n-1/2} = \frac{1}{4} \left(u_j^n + u_j^{n-1} + u_{j-1}^n + u_{j-1}^{n-1} \right). \quad (\text{A } 1)$$

Here, n is the time index and j is the space index. Also, the partial derivatives of u with respect to s or t are given by

$$\left(\frac{\partial u}{\partial s} \right)_{j-1/2}^{n-1/2} = \frac{1}{2} \left[\frac{u_j^{n-1} - u_{j-1}^{n-1}}{\Delta s} \right] + \frac{1}{2} \left[\frac{u_j^n - u_{j-1}^n}{\Delta s} \right] \quad (\text{A } 2)$$

and

$$\left(\frac{\partial u}{\partial t} \right)_{j-1/2}^{n-1/2} = \frac{1}{2} \left[\frac{u_{j-1}^n - u_{j-1}^{n-1}}{\Delta t} \right] + \frac{1}{2} \left[\frac{u_j^n - u_j^{n-1}}{\Delta t} \right]. \quad (\text{A } 3)$$

Thus, a dependent variable and its partial derivatives are evaluated at the midpoint of the box in terms of two known nodal values (the dark circles in figure 10) and two unknown nodal values (the open circles). Boundary conditions are imposed by specifying their nodal values at the boundary nodes. The resulting nonlinear algebraic equations can be solved using Newton's method.

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