Published in final edited form as:

J Exp Psychol Learn Mem Cogn. 2010 November ; 36(6): 1389–1398. doi:10.1037/a0020212.

Evidence for direct retrieval of relative quantity information in a quantity judgment task: Decimals, integers, and the role of physical similarity

Dale J. Cohen

University of North Carolina Wilmington

Abstract

Participants' reaction times in numerical judgment tasks in which one must determine which of two numbers are greater generally follow a monotonically decreasing function of the numerical distance between the two presented numbers. Here, I present three experiments that assess the relative influences of numerical distance and physical similarity in just such a task using integers and decimals as stimuli. The data reveal that numerical distance is the primary feature controlling participants' RTs when integers are presented. However, the physical similarity between the decade place of the standard and the probe is the primary feature controlling participants' RTs when decimals are presented. I conclude that the unique qualities of decimals do not lend themselves to share the place coding representation of integers, thus a direct retrieval mechanism for judging the relative quantity of decimals has developed.

The numerical cognition literature is virtually universal in its acceptance of the conclusion that the numerical distance between two numbers is the primary mechanism influencing participants' reaction times (RTs) in a numeral judgment task whereby participants are asked to judge the larger of two numerals (here termed a *numeral quantity judgment task*). Here, I present three experiments that challenge this assumption by examining the relative influences of numerical distance and physical similarity in a numeral quantity judgment task for Arabic digits representing decimals and positive integers.

In what has become a classic paper, Moyer and Landauer (1967) examined the process by which adults judge the relative quantities of two Arabic digits representing positive integers. They presented several alternative hypotheses, including (1) the direct retrieval of relative quantity information from memory and (2) a process in which the observer compares analogue representations of the quantities symbolized by the digits. To test their hypotheses, Moyer and Landauer presented two digits side-by-side and asked participants to judge which of the two digits was larger. The data revealed a monotonically decreasing function relating participants' RTs and the numerical distance between the two quantities symbolized by the presented digits (termed the *numerical distance effect*). Moyer and Landauer showed that the numerical distance effect was described by the Welford (1960) function:

Please address correspondence to Dale J. Cohen, Ph.D. cohend@uncw.edu. Phone: 910.962.3917. Fax: 910.962.7010.
Dale J. Cohen, Department of Psychology, University of North Carolina Wilmington, 601 South College Road, Wilmington, NC 28403-5612.

Publisher's Disclaimer: The following manuscript is the final accepted manuscript. It has not been subjected to the final copyediting, fact-checking, and proofreading required for formal publication. It is not the definitive, publisher-authenticated version. The American Psychological Association and its Council of Editors disclaim any responsibility or liabilities for errors or omissions of this manuscript version, any version derived from this manuscript by NIH, or other third parties. The published version is available at www.apa.org/pubs/journals/xlm.

$$RT = a + k * \log [L / (L - S)], \quad (1)$$

where a and k are constants, L is the larger quantity and S is the smaller quantity. Because the Welford function also described the relation between RT and judgments of natural quantities such as line lengths and dot patterns (see also, Buckley & Gillman, 1974; Moyer & Landauer, 1973), Moyer and Landauer concluded that an analogueⁱ representation of numerical information is the most parsimonious explanation of the numerical distance effect. Moyer and Landauer's (1967) results have been replicated under a variety of conditions (e.g., Buckley & Gillman, 1974; Dehaene, 1989; Dehaene & Akhavein, 1995; Dehaene, Depoux, & Mehler, 1990; Duncan, & McFarland, 1980; Hinrichs, Yurko, & Hu, 1981; Moyer, & Landauer, 1967; Moyer, & Landauer, 1973; Schwartz, & Stein, 1998; Shepard, et al., 1975; Takahashi, & Green, 1983).

Hinrichs et al. (1981) extended the numerical distance effects to two-digit integers. The authors were interested in determining whether observers evaluated two-digit integers holistically, or whether they evaluated the unit and the decade place separately. The authors presented participants with the numbers 11–99 and asked them to determine whether the presented number was greater or less than 55. The data were well fit by both the Welford and a similar but symmetric Log D function [$RT = a * \log(L - S) + b$]. Although there were some slight abnormalities in the data, the authors concluded that the data were well predicted by the numerical distance between the quantities symbolized by the standard and the probes. The authors concluded that observers processed two-digit integers holistically. Dehaene, et al. (1990) replicated these results with french students.

More recently, evidence suggests that the decade and unit values of two-digit integers are processed, at least in part, separately rather than holistically (e.g., Ganor-Stern, Tzelgov, & Ellenbogen, 2007; Nuerk, Weger, & Willmes, 2001; Ratinckx, Brysbaert, & Fias, 2005; Vergots & De Moor, 2005). In a clever experiment, Nuerk et al. (2001) conducted a typical two-digit quantity judgment task similar to Hinrichs et al. (1981). Nuerk et al., however, varied the unit-decade-compatibility, whereby the numeral comparison was defined as compatible if both the unit and the decade provided the same greater than/less than answer. The numeral comparison was defined as incompatible if the unit and the decade provided different greater than/less than answers. The authors showed a robust and consistent effect of compatibility on RT, indicating that participants judged the decade and unit separately to some degree. Ganor-Stern, et al. (2007) found a similar compatibility effect when assessing the Size Congruency Effect of two-digit numbers. Perhaps most unexpectedly, in a two-digit naming experiment with priming, Ratinckx et al. (2005) found that compatibility of the prime to the target influenced RTs on a non-semantic level. That is, the compatibility effect discovered by Ratinckx et al. (2005) revealed a "... direct, nonsemantic route from Arabic input to verbal output ..." (p. 1158). Taken together, these experiments provide important evidence that two-digit numerals are not processed solely on a holistic level, and Ratinckx et al. (2005) showed that perhaps the unit and decade components are not solely processed on a semantic level either.

One strength of the numerical distance effect is the consistency with which it is identified in numeral quantity judgment tasks. One weakness, however, is the conclusion that numerical distance is the primary factor controlling participants' RTs. Numerical distance is only one of several features that are correlated with the order of the numbers on the number line (Cohen,

ⁱHere I use the term *analogue* to reference numerical representations that share some properties of physical numerosity, but are not necessarily on a continuum. I use the term *analog* to reference numerical representations in which numerosity is represented on a continuum.

2009a). Therefore, although numerical distance is correlated with participants' RTs, it may not be the controlling factor of participants' RTs. Unfortunately, numerical cognition researchers rarely consider plausible alternatives to the numerical distance hypothesis. This can be seen in the extent to which researchers assess various forms of the numerical distance hypothesis (e.g., Welford vs. Log D), without entertaining other reasonable hypotheses (e.g., Hinrichs et al., 1981; Dehaene, et al. 1990).

One feature that is correlated with both numerical distance and the order of numbers on the number line is the physical similarity of Arabic single digits (see Cohen, 2009b). Numerical quantity and physical similarity may be correlated because numerical symbols are likely distorted representations of the actual quantity they represent (see Ifrah, 2000). Cohen (2009b) operationalized the similarity between two Arabic digits in relation to the seven line, figure 8 structure of integers used on old alarm clocks. Specifically, all 10 digits (0–9) can be represented by highlighting specific lines on that figure 8 structure. Physical similarity is defined as,

$$PS=O/D,$$

2

where O is the number of lines that both to-be-compared digits share, and D is the number of remaining lines required to compose both to-be-compared digits.

Cohen (2009b) demonstrated that the physical similarity between the target and distractor, rather than the numerical distance, was the controlling factor in participants' responses in a numeral same/different task. Cohen presented participants the Arabic integers 1 through 9 and asked them to judge whether the presented integer was a 5. Cohen assessed whether the RT data were better predicted by the Welford function or the Physical Similarity function. The data demonstrated that (1) numerical distance exerted no control over participant's responses, and (2) physical similarity exerted virtually complete control over participant's responses. Cohen concluded that, because the semantic information associated with the numerical symbols did not influence responses, semantic information is not automatically activated by a numerical symbol.

Cohen's (2009b) demonstration of the influence of physical similarity on participant's judgments in the numeral same/different task raises the possibility that physical similarity also exerts an influence in the numeral quantity judgment task. If the process by which relative quantity judgments are made relies on the comparison of analogue representations, then the influence of physical similarity should be relatively small to non-existent. At best, physical similarity may exert a slight inhibitory effect whereby the relevant numerical symbols that share important features are slightly slowed. Nevertheless, the analogue comparison process will exert the primary control over the participants' RTs. Given the robust nature of the numerical distance effect when numerals symbolizing integers are compared, this pattern is predicted when the relative quantity of integers are judged.

If, however, physical similarity is the primary controlling factor of participants' RTs in the numeral quantity judgment task, then one unavoidable conclusion is that the relative quantity symbolized by two numerals can be judged through a process other than analogue comparison. One reasonable alternative process is the direct retrieval hypothesis mentioned by Moyer and Landauer (1967). As stated by Moyer and Landauer (1967), "... if humans judge differences between numerals in the same way as differences along physical continual, reaction times should be inversely proportional to the difference between two numbers and smaller for smaller numbers. If, instead, the process involves direct memory look-up, there is little reason to expect such relations." (p. 1520). To specify more precisely: if numerical symbols are directly associated with greater than/less than information relative to other numerical symbols, then

symbol identification becomes the primary task of the quantity judgment process. Because a numeral quantity judgment task requires participants to simultaneously consider two presented numerals, and the identity (and therefore the structure) of the numerals is primary, it is reasonable to assume that some processing interference will occur between the two numerals when encoding the structure of one or both numerals. As a result, the symbol identification process may be influenced by the physical similarity of the two presented numerals. Once identification occurs, however, relative quantity information should be directly available, so little or no influence of numerical distance should be present.

There is reason to hypothesize that the numerical distance effect will not be present for the judgement of decimals. The preponderance of evidence supporting an analogue representation of integers suggests that this representation is primarily one of *place coding* rather than *magnitude coding* (e.g., Dehaene, Bossini, & Giraux, 1993; Verguts, Fias, & Stevens, 2005). A place coding representation is one in which the particular quantity is represented as a point on a quantity continuum, such as a number line. In contrast, a magnitude representation is one in which quantity is represented as a virtual mass whereby greater quantities have greater virtual mass. Integers are efficiently portrayed by a place coding representation, as is demonstrated by the use of number lines depicting integers in elementary schools. One feature of integers that is conducive to number lines is the relation between number of digits and magnitude. That is, a one digit numeral is always less than a two digit numeral, etc. As such, there is a finite number of integers between any two other integers. This quality leads to perfect predictability and perfect identifiability of all integers along any sub-section of the number line continuum. Decimals do not have this quality. The number of digits in a decimal is related to precision rather than magnitude. Therefore, between any two points there can always be a third point of an arbitrary number of digits. One cannot identify all decimals along any sub-section of the number line continuum - no matter how small the sub-section. The difficulty of devising a place coding representation of decimals that captures this unique quality is apparent by the relatively complicated number “line” used for fractional values. The typical number “line” representation of fractional values is (1) two-dimensional and (2) uses fractions rather than decimals. To conclude, if the brain evolved a place coding system of numerical representation, decimals would not easily fit into that system. It is therefore not unreasonable to assume that the mental representation of decimals is different from that of integers. My colleagues and I have previously demonstrated the uniqueness of decimal representation by revealing that the mental representation of decimals is different from and perhaps independent of the mental representation of relative frequencies (Cohen, Ferrell, & Johnson, 2002). In addition, there is some evidence that fractions are processed by the individual integer components, rather than the holistic proportion they represent (Bonato, Fabbri, Umiltà, & Zorzi, 2007). This provides more evidence that proportions are treated differently than integers.

In the present set of experiments, I conduct standard numeral quantity judgment tasks using Arabic digits symbolizing integers and decimals. When analyzing the data, I simultaneously assess the relative influences of numerical distance and physical similarity.

Experiment 1

Experiment 1 was a typical numeral quantity judgment task whereby participants were presented an integer between 1 and 99 and asked to judge whether the integer was larger or smaller than 55. The standard of 55 was used because it provided a single digit (5) to which probe integers would be compared for physical similarity. Participants' RTs were recorded.

Method

Participants—Thirty participants volunteered to participate for class credit in General Psychology.

Apparatus and Stimuli—All stimuli were presented on a 15-inch color monitor with a 60-Hz refresh rate controlled by a Pentium microcomputer using a DOS operating system. The resolution of the monitor was 1024×768 .

A trial consisted of an integer presented in the center of the screen. The integers were selected randomly from the integers 1–99, excluding 55 in Arabic notation. Similar to Cohen (2009b), all integers were presented in Ariel font. The individual numbers between 1–9 subtended 1.33 degrees visual angle.

Procedure—Participants were tested individually in a small, dark room. The participants were instructed that on every trial they would be presented an integer between 1 and 99 and that they were to press one button on the keyboard if the integer represented a quantity greater than 55 and another button if the integer represented a quantity less than 55. Half the participants were told to press the “d” key if an integer greater than 55 was presented and the “l” key otherwise. The keys were reversed for the remaining participants. The participants were instructed that speed was important, but accuracy was essential.

Each trial consisted of the presentation of a stimulus and the participant's response. On every trial, the stimulus was randomly chosen from a uniform distribution of integers between 1 and 99, excluding 55. The stimulus remained on the screen until the participant responded. The computer recorded the participant's RT. There was a 500 ms delay between trials. Every 130 trials the experiment would pause and ask the participant if he or would like to take a self-timed break. The experiment resumed when the participant hit the return key.

Each session, consisting of 16 practice trials and 588 experimental trials, lasted about 50 minutes. To collect enough data across the entire distribution of integers, participants were run in three sessions, each on a separate day, resulting in 1764 trials per participant.

Results

Prior to analyzing the data, I trimmed the data to remove all RTs less than 200 ms and greater than 5000 ms (about 0.4% of the data). In addition, I removed all incorrect responses (about 2.5% of the data).

To assess the relative contributions of the Welford and the Physical Similarity functions, I computed a simultaneous regression with the average RT for each probe as the criterion variable and the Welford and Physical Similarity functions as the predictor variables. To fully capture Physical Similarity, I included three Physical Similarity predictor variables: one for the decade of the probe, one for the units of the probe, and one for the interaction between the two (computed by multiplying the Physical Similarity of the decade by the Physical Similarity of the unit). To assess the relative contribution of each predictor, I standardized all variables:

$$z(\bar{RT}_{probe}) = z(Welford) + z(PS_{Decade}) + z(PS_{Unit}) + (z(PS_{Decade}) * z(PS_{Unit})) \quad 3$$

Although less parsimonious than the single Welford function variable, this set up allows for the discovery of how participants' perception of the decades and units might interact. When no decade was present (i.e., the numbers 1–9), Physical Similarity of the decade equals 0 because there are no overlapping lines. Because I am assessing the physical similarity of decades and units separately, it is necessary to assign the number 5 a similarity measure. To eliminate the singularity that occurs when calculating the similarity of 5 to itself, I modified Cohen's (2009b) original PS formula by adding a constant to the difference term. The constant equalled

$1/\max(D)$, where $\max(D)$ is the maximum D estimate of the set of probes. Thus, the modified PS formula is,

$$PS = O / (D + (1/\max(D)))$$

4

Table 1 presents the correlations between the variables of the equation. As stated in the introduction, the Welford function and the Physical Similarity of the decade are highly correlated. Despite the high correlation, the predicted pattern of responses are quite unique in the case of two digit integers (see Figure 1). Specifically, the Welford function predicts a smooth increase in RTs as the probes approach the standard. In contrast, the Physical Similarity of the decade predicts relatively flat RTs with a distinct jump at the decade of the standard. Among other discrepancies, the 70's decade is lower than the 60's and 80's decade and there is no effect of the units on RT. Thus, the Welford function and the Physical Similarity function should be both statistically and visually distinct.

The overall regression was significant, $F(4,93)=276$, $p < .01$, $r^2_{adj} = .92$, see Figure 1. The slopes for the Welford function (slope=0.78, $t=16.4$, $p < .01$) and the Physical Similarity of the Decades (slope=0.20, $t=3.1$, $p < .01$) were significant. There were no other significant coefficients. The standardized slopes indicate that the Welford function accounts almost 15 times more unique variance than the Physical Similarity function. Table 2 displays the average RTs for each decade of the probe. This information is presented to compare absolute RTs across experiments.

Discussion

The data from Experiment 1 demonstrate that numerical distance (i.e., the Welford function) is the primary controlling factor in participants' RT responses for the numerical distance task of two-digit integers. This finding is consistent with previous numerical distance two-digit integer tasks (Hinrichs et al., 1981; Dehaene, et al. 1990). The data also reveal, however, that physical similarity of the decade position between the standard and the probe exerts some control over participants' responses. These data suggest that participants attend selectively to the decade position of the integer in the current task, and the physical similarity between the integer in that position and the standard influences their response. This influence, however, is in conjunction with the overriding influence of numerical similarity described by the Welford function.

Experiment 2 assesses the same effects in the perception of decimals between .01 and .99. Decimals greater than .1 are identical to integers, with the exception that they are preceded by a period (e.g., .55 vs 55). In addition, two-digit decimals between .01 and .99 are ordinarily related in the same way as integers between 1 and 99. Finally, if one assumes that the place coding representation has a logarithmic structure (whether by mean shifts or scaler variance), then the difference between very small proportions should be as salient as larger numbers (i.e., the salience of the difference between .1 and 1 is equivalent to that of 1 and 10, which is equivalent to that of 10 and 100). Thus, if all numerical formats are represented similarly, one would expect decimals and integers to show similar patterns - i.e., large numerical distance effects and small physical similarity effects.

In contrast to the above hypothesis, there are logical arguments (outlined in the introduction) that decimals would not be efficiently represented in the same way as integers. Furthermore, Cohen et al. (2002) has demonstrated that people's understanding of decimals are seemingly independent of their understanding of relative frequencies (e.g., 50 in 120). This reveals a close link between numerical format and quantity representation, at least in the case of decimals and

relative frequencies. Such a link would predict that observers will demonstrate a different pattern of results when comparing decimals than when comparing integers.

Experiment 2

Experiment 2 assesses the relative influence of numerical distance and physical similarity on participants' RTs in a numeral quantity judgment task with decimals as stimuli. This is the first instance in which decimals have been used as stimuli in a numeral quantity judgment task.

Method

Participants—Thirty participants volunteered to participate for class credit in General Psychology.

Apparatus and Stimuli—The apparatus and stimuli were identical to Experiment 1 with the two exceptions: (1) the standard to which all probes were compared was .55 rather than 55 and (2) the probes were randomly chosen from a uniform distribution of decimals between .01 and .99, excluding .55.

Procedure—The procedure was identical to Experiment 1.

Results

Prior to analyzing the data, I trimmed the data to remove all RTs less than 200 ms and greater than 5000 ms (about 0.5% of the data). In addition, I removed all incorrect responses (about 2.6% of the data).

To assess the relative contributions of the Welford and the Physical Similarity functions, I computed a regression analysis identical to that of Experiment 1. Table 3 presents the correlations between the predictors of the equation. The correlations between predictors are slightly different than that of Table 1 because the zero decade in decimals is more similar to the 50's decade than the 10's, 20's, 40's, and 70's decades. Thus, the Physical Similarity function predicts a jump in RTs at the zero decade for decimals (see Figure 3). This jump at the zero decade provides a distinct advantage for the Welford function if numerical distance is the controlling factor of participants' RTs. Thus, the Welford function and the Physical Similarity function should be both statistically and visually distinct.

The overall regression was significant, $F(4,93)=77.4$, $p < .01$, $r^2_{adj} = .76$, see Figure 2. Both the slopes for the Welford function (slope=0.30, $t=3.9$, $p < .01$) and the Physical Similarity of the Decades (slope=0.62, $t=5.8$, $p < .01$) were significant. There were no other significant coefficients. The standardized slopes indicate that the Welford function accounts for about 75% less unique variance than the Physical Similarity function. Table 2 displays the average RTs for each decade of the probe.

As Figure 2 shows, participants' RTs to probes with 0s in the units place were unusually elevated for the decades 60–90. This elevation may have influenced the regression, increasing the apparent influence of physical similarity. I therefore re-ran the regression excluding the 0s in the units place for the decades 60–90. The results of the analysis showed a decreased influence of numerical distance (slope=0.23, $t=3.1$, $p < .01$) and an increased influence of Physical Similarity (slope=0.69, $t=6.9$, $p < .01$). There were no other significant predictors. The overall regression accounted for more variance than the original regression ($F(4,89)=92$, $p < .01$, $r^2_{adj} = .8$). Thus, the primary role of physical similarity predicting the pattern of results is confirmed in the present analysis.

Discussion

The data from Experiment 2 reveal that the physical similarity of numbers influence participants' responses at least as much as the numerical distance when they are comparing decimals in a numerical distance task. This finding is particularly surprising because the only visual difference between the integers used in Experiment 1 and the decimals used in Experiment 2 was the decimal point in front of the numbers in Experiment 2ⁱⁱ. Thus, participants could have successfully completed the task by simply ignoring the decimal point and treating the numbers as integers. The fact that the results of Experiments 1 and 2 were different suggests that the decimal point triggered some automatic interpretation of the numbers that was, as the data indicate, different to some degree from their interpretation of integers. The strong influence of physical similarity in Experiment 2 suggests that participants may have a weaker association between decimal symbols and any analogue representation of quantity relative to their association between integer symbols and their the quantity they represent. I will discuss this in more detail in the General Discussion.

I conducted Experiment 3 to more stringently assess decimals in the numeral quantity judgment task and discourage participants from ignoring the decimal point.

Experiment 3

Experiment 3 expands on the findings of Experiment 2 by assessing the relative influence of numerical distance and physical similarity on participants' RTs in a numeral quantity judgment task with decimals as stimuli. Participants in Experiment 2 could simply ignore the decimal point and treat the stimuli as integers. In Experiment 3, I created the stimuli in such a way to that this strategy would lead to the wrong answer. Specifically, for 1/3rd of the stimuli, the decimals were presented only to a single decimal place. For 1/3rd of the stimuli, the decimals were presented to two decimal places. Finally, for 1/3rd of the stimuli, the decimals were presented to three decimal places. The standard remained .55. Here, if the participant ignored the decimal point, he or she would interpret all the single place decimals as single unit integers and respond to all of them as less than the standard. In addition, he or she would interpret the three place decimals as numbers in the hundreds and respond to all of them as greater than the standard. In each case, however, approximately half are greater than and half are less than the standard. Thus, the strategy of ignoring the decimal point will fail in Experiment 3.

Method

Participants—Thirty seven participants volunteered to participate for class credit in General Psychology.

Apparatus and Stimuli—The apparatus and stimuli were identical to Experiment 2 with three exceptions (1) the probes were randomly chosen from a uniform distribution of decimals between .001 and .999, excluding .55, (2) that 1/3rd of the stimuli were rounded to one decimal place, 1/3rd were rounded to two decimal places and 1/3rd were rounded to three decimal places, and (3) there were 567 trials per session rather than 588 (this change was made simply to shorten each testing session).

Procedure—The procedure was identical to Experiment 1.

ⁱⁱThis is true for all integers greater than 9 and all decimals greater than .09. For decimals less than .10, a zero was between the decimal point and the integer. If one ignored the decimal point, the zero in front of the integer would obviously look different from an integer without the zero. Importantly, the physical similarity of the leading 0 in the decimal experiments is more similar to a 5 than a 1, 2, 4, or 7. Therefore, the leading Physical Similarity hypothesis predicts the jump in RT found when participants compared values between 0–.1 to the standard of .55.

Results

Prior to analyzing the data, I trimmed the data to remove all RTs less than 200 ms and greater than 5000 ms (about 0.5% of the data). In addition, I removed all incorrect responses (about 2.4% of the data). Because there were not enough responses per three digit probe for these probes' influence to be individually assessed, I truncated the last digit off the three digit probes, and treated all the data as two-digit probes. Because this truncation has minimal effect on these probes numerical distance from the standard, it should likewise have little effect on the predictive ability of the Welford function. Furthermore, because Experiment 2 revealed that participants' attention was directed at the probe's decade place, the truncation should also have relatively little effect on the predictive ability of the physical similarity function.

To assess the relative contributions of the Welford function and the Physical Similarity function, I computed a regression analysis identical to that of Experiment 1. Table 4 presents the correlations between the variables of the equation. The overall regression was significant, $F(4,92)=113$, $p < .01$, $r^2_{adj} = .82$, see Figure 3. Only the Physical Similarity of the Decades (slope=0.81, $t=8.6$, $p < .01$) was significant. There were no other significant coefficients. Table 2 displays the average RTs for each decade of the probe.

Because there was no overall effect of numerical distance, I tested two additional alternative hypotheses that treated the numbers within the decade of the standard (.50–.59) differently than the numbers outside of the decade of the standard (.01–.49 and .60–.99). The first hypothesis combined the influences of physical similarity and numerical distance. Specifically, for all values outside of the decade of the standard, physical similarity of the decade was the controlling factor, but within the decade of the standard, numerical distance was the controlling factor. The second hypothesis was analogous, but only assessed the influences of physical similarity. Specifically, for all values outside of the decade of the standard, physical similarity of the decade was the controlling factor, but within the decade of the standard, physical similarity of the unit was the controlling factor. Both the first hypothesis, $F(1,95)=438$, $p < .01$, $r^2_{adj} = .82$, and the second hypothesis, $F(1,95)=424$, $p < .01$, $r^2_{adj} = .82$, were significant. Neither, however, accounted for more variance than the model that only contains the influence of the physical similarity of the decade, $F(1,95)=429$, $p < .01$, $r^2_{adj} = .82$. Thus, the physical similarity of the decade of the standard to the probe appears to be the most parsimonious predictor of the data.

Discussion

The results of Experiment 3 reveal that decimals' numerical information did not influence participants' responses in the numerical distance task. Participants completed the task based primarily on physical structure of the numerical symbol. This suggests that decimals, once physically distinguished from the standard, directly permit a greater than/less than decision without reference to an analogue representation. This is quite different from our current understanding of how people encode and process integers. The reduced influence of the Welford function in Experiment 3 relative to Experiment 2 is likely because Experiment 3 discouraged the strategy of ignoring the decimal point and treating the stimuli as integers. I discuss these issues below in the General Discussion.

General Discussion

Experiments 1–3 assessed the relative influences of numerical distance and physical similarity in a standard numeral quantity judgment task using Arabic digits representing integers and decimals as stimuli. The data confirms that numerical distance is the primary controlling factor when judging whether a probe is greater or less than a standard for two-digit integers. Nevertheless, physical similarity of the standard and probe decade account some unique

variance in the task. The data reveals a quite different pattern for decimals: Physical similarity of the standard and probe decade is the primary controlling factor when judging whether a probe is greater or less than a standard for decimals. Numerical distance has a minor influence on responses when decimals are consistently rounded to two decimal places (Experiment 2) and thus can be treated as integers. When that strategy is inhibited, numerical distance has no influence on participants' responses. This is the first experiment to demonstrate that a numeral quantity judgment task can be completed without exhibiting the numerical distance effect.

Observers must process the physical form of the numerical symbol to some degree for the symbol to associate with the quantity it represents. This stage of the perceptual process has been relatively ignored in the numerical cognition literature. The lack of attention is likely because the strong and robust correlation between numerical distance and observers' RTs appear to obviate the need to explore the encoding stage of the numerical cognition process. However, as stated above, numerical distance is merely one feature of numerical symbols that is correlated with their order on the number line. Physical similarity, which is likely central to the encoding stage, is also correlated with the order of integers on the number line. Because physical similarity is correlated with numerical distance, the influence of the encoding stage on participants' responses may be greater than initially believed. The present set of experiments tested this influence.

The data from Experiment 1 suggest that two additive processes are taking place in the comparison of two-digit integers: (1) the comparison of the semantic content of the two-digit probes to that of the standard and (2) the selective comparison of the physical structure of the decade of the probe to that of the standard. The present data are consistent with previous findings addressing two-digit integers (e.g., Dehaene, et al. 1990; Hinrichs et al., 1981; Nuerk et al., 2001). The analogue processing of the two-digit integers is suggested because the data follow the Welford function quite closely. Similar to previous explorations of two-digit integers, I found some discontinuities at the changes between decades, especially the decade of the standard (e.g., between 49 and 50). The previous attempts to explain these discontinuities have revolved around different incarnations of the numerical distance effect (Hinrichs et al., 1981; Dehaene, et al. 1990), including the separate relative quantity comparisons of the decade and unit (e.g., Ganor-Stern et al., 2007). The present results reveal that these discontinuities are well explained by adding a small influence of the physical similarity of the decade of the standard and the probe. The added influence of physical similarity suggests a second process whereby participants are attending selectively to the decade of the probe. This process slows participants' responses to the degree that the decade of the probe is similar to that of the standard. This non-semantic comparison of the components of a two-digit number is consistent with the results of Ratinckx et al. (2005). It is unclear from the present data whether these two processes (semantic comparison and structural comparison) are (1) sequential or simultaneous, and (2) automatic or intentional. Furthermore, it is unclear whether this second process (the structural comparison of the decades) contributes to the relative quantity judgment made by the participants. Future research should address these issues.

The data from Experiment 3, in which participants were presented decimals and they could not simply ignore the decimal point, revealed no effect of numerical distance. These data are the first to demonstrate that participants can complete a numeral quantity judgment task without numerical distance influencing their RTs. Any analogue representation theory, such as the analog number line suggested by Dehaene and colleagues (e.g., Dehaene, 1992, 2002; Dehaene & Akhavein, 1995; Dehaene, Piazza, Pinel, & Cohen, 2003), would predict that numerical distance would be the controlling factor in participants RTs when judging the ordinal relation of numbers. This is because, with such an analog representation, numerically close numbers share more features than numerically distant numbers. The lack of a numerical distance effect

suggests that relative quantity judgments of decimals can be made without reference to an analogue representation.

The data from Experiment 3 reveal the primary feature controlling variability in RT across probes was the physical similarity of the decade of the standard to that of the probeⁱⁱⁱ. This pattern suggests that the key feature in judging the relative quantity of two unrounded decimals resides in the symbol identification process, rather than the quantity comparison process. Once the symbols are identified, there appears to be no variability in RTs across probes resulting from the comparison process. Such a pattern (i.e., large influence of symbol identification and no influence of comparison processes) is the signature of a direct retrieval process. Thus, the data from Experiment 3 argue that participants judge the relative quantity of unrounded decimals based on the direct retrieval of relative quantity information.

The direct retrieval of relative quantity information is controversial, given the robust nature of the numerical distance effect. Nevertheless, the unique structure of decimals may have imposed enough restrictions that a general purpose quantity representation was unattainable and direct retrieval of quantity information was the most efficient solution. One limitation of direct retrieval is the infinite bits of information required if one was to store quantity information individually for all possible decimals. This important limitation may explain why the physical structure of the decade position provided the observable interference. The importance of the decade position may arise from the fact that the first number after the decimal point in semantic representation of a quantity less than one provides the most information about the size of the presented quantity. As one moves successively away from the decimal point, less information about the quantity is provided, though greater precision is attained. It is possible that people have learned this correlation and thus allocate the majority of their attention to the initial number after the decimal point. Limiting direct retrieval to successive decades first has the advantage of limiting the number of bits of information required to be stored in memory^{iv}.

The finding that integers and decimals rely on different underlying representations is not completely unexpected. As stated in the introduction, the preponderance of evidence suggests that the underlying quantity representation of integers contains *place* information rather than *magnitude* information (e.g., Dehaene, Bossini, & Giraux, 1993; Verguts, Fias, & Stevens, 2005). An analog linear place coding representation, such as the number line, is relatively efficient when a finite number of identifiable points exist between any two identifiable end points. This is the case with integers. Decimals have the property that the number of digits is related to precision rather than magnitude. As such, an infinite number of points can exist between any two other points (endpoints or not). Such a property is substantially more difficult to depict as a place coding representation. This is why the fractional number line is depicted as a two dimensional “number line” of fractional values. Thus, an analog linear place coding representation may not be an efficient representation for decimals.

The lack of reference to a place coding representation of quantity by decimals may have implications for understanding the form of the place coding representation of integers. These implications can be gleaned most clearly in the comparison of the three experiments. The data from Experiment 1, in which integers were presented, show a clear numerical distance effect. The data from Experiment 2, in which decimals were presented but the decimal point could be ignored, showed a weak numerical distance effect. The data from Experiment 3, in which decimals were presented but the decimal point could not be ignored, showed no numerical

ⁱⁱⁱThis set of experiments does not identify, nor is its goal to identify, the process by which the visual structure of the standard interferes with that of the probe. Nevertheless, there is abundant evidence in the extant literature that the mere presence of a stimulus will inhibit the identification of similar stimuli (e.g., flanker tasks, repetition blindness, etc.).

^{iv}It may also be that people also allocate (less) attention to successive numbers, but my experiments were not sensitive enough to pick up the effects of this.

distance effect. In all three cases, a single representation that referenced an analog (i.e., continuous) number line would be adequate for successfully completing the task. Such a representation would predict a numerical distance effect for both decimals and integers. The data do not support this prediction. Alternatively, one might hypothesize that integers reference an analog number line, but decimals do not. Although such a hypothesis can account for the data, it is not parsimonious. In particular, integers do not require an analog number line to be understood (an ordinal number line is sufficient), yet decimals do. It is not parsimonious to theorize that a system would evolve an analog number line for a numerical system that does not require one and prevent a numerical system that does require one from referencing it. In contrast, a representation that contained the information inherent in an ordinal number line would be useful in many situations, including all those involving integers *and* those involving decimals rounded to a specific digit (and thus could be treated as integers). The *ordinal number line* is a number line composed of successive integers, but the quantity difference between each successive integer *may not* be known or equal. If we assume that the numerical representation of integers references the ordinal number line, rather than an analog (i.e., continuous) number line, then it is logical for the numerical representation of decimals and integers to differ^V. Because the ordinal number line is a useful representation with respect to integers, the numerical representation of integers may strongly reference it. In contrast, because the ordinal number line is much less relevant to decimals (because the number of digits that compose a decimal is unrelated to quantity), the numerical representation of decimals may not reference it, or do so only weakly. The ordinal number line hypothesis, therefore, would predict reference to the ordinal number line when dealing with integers and (depending on strategy) rounded decimals, but not with unrounded decimals. This is exactly what the present data showed. Thus, the ordinal number line hypothesis can parsimoniously explain the data.

The finding that integers exhibit data that are consistent with an analogue representation and decimals do not, provide further evidence that the representation associated with a particular numerical notation is specific for that notation, rather than universal across notations. The evidence against abstract numerical representations is increasing (e.g., Cohen et al., 2002; Cohen Kadosh & Walsh, 2009; Gonzalez & Kolers, 1982). Recently Cohen Kadosh and Walsh (2009) reviewed this evidence and hypothesize a dual code system in which intentionality and task demands can influence the representation of numbers. Although the present experiments do not directly address this issue, the data provide some support for Cohen Kadosh and Walsh's hypothesis. Specifically, the difference in the pattern of participants' responses to decimals in Experiment 2 and Experiment 3 suggest that task demands had some influence on their numerical representation. In both cases, the numerical system was the same (i.e., decimals). The only difference was the set of decimals used. It appears that participants adjusted their representations according to the demands of the task, thus demonstrating different patterns of responses.

There are some limitations to the current set of experiments. First, the numerical distance effect that I refer to in the integer data is merely correlated with the ordinal structure of the number line. Although the present analysis has demonstrated that the controlling factor in the experiment with integers is not physical similarity (as it appears to be with decimals), I have not distinguished the effects of numerical distance from other possible correlates with the ordinal structure of the numbers on the number line. Second, in the present set of experiments, the Physical Similarity function was undefined when the standard contained one of the numbers of the probe. I resolved this by adding a constant to the denominator of the Physical Similarity equation. This solution, however, may not be optimal. Therefore, an examination of Physical Similarity functions that are never undefined would be useful. Finally, models of

^VBecause an analog number line is relevant to both decimals and integers, if such a representation existed, it would be reasonable for both decimals and integers to reference it.

automatization (e.g., Logan, 1988) often assume that direct retrieval is a consequence of high levels of practice. This assumption appears to be in opposition to my finding - that direct retrieval is a consequence of weak learning in the case of decimals. It is possible, however, that the propensity to refer to a number line representation when viewing integers is related to the early and repeated exposure to a physical number line in schools, etc. Similarly, the lack of reference to a number line representation when viewing decimals may be related to the lack of exposure to such a physical “number line” representation of proportions. Indeed, people may be more likely to imagine proportions in the form of “pieces of a pie” than “points on a line” because of they have had more experience dividing pies than lines. Nevertheless, the relation between practice and direct retrieval of integers and decimals should be further explored.

In sum, the present set of experiments assessed the relative influence of numerical distance and physical similarity on integers and decimals in a numerical quantity judgment task. The data revealed that numerical distance was the controlling factor for participants responses to the task when integers were used as stimuli. However, when decimals were used as stimuli, physical similarity was the controlling factor for participants responses. This is the first demonstration of a numerical distance task being completed without the influence of numerical distance on the data. I hypothesize that the numerical distance effect will be present when the ordinal number line is useful for completing the task, but will not be present when it is irrelevant.

Acknowledgments

This work supported by NIH grant RO1HD047796.

References

- Bonato M, Fabbri S, Umiltà C, Zorzi M. The mental representation of numerical fractions: real or integer? *Journal of Experimental Psychology: Human Perception and Performance* 2007;33(6):1410–1419. [PubMed: 18085953]
- Cohen DJ. Numerical representations are neither abstract nor automatic. *Behavior and Brain Sciences* 2009a;32:332–333.
- Cohen DJ. Integers do not automatically activate their quantity representation. *Psychonomic Bulletin and Review* 2009b;16:332–336. [PubMed: 19293103]
- Cohen D, Ferrell J, Johnson N. What Very Small Numbers Mean. *Journal of Experimental Psychology: General* 2002;131:424–442. [PubMed: 12214756]
- Cohen Kadosh R, Walsh V. Numerical representation in the parietal lobes: Abstract or not abstract? *Behavior and Brain Sciences* 2009;32:313–328.
- Buckley PB, Gillman CB. Comparisons of digits and dot patterns. *Journal of Experimental Psychology* 1974;103:1131–1136. [PubMed: 4457588]
- Dehaene S. The psychophysics of numerical comparison: A reexamination of apparently incompatible data. *Perception & Psychophysics* 1989;45:557–566. [PubMed: 2740196]
- Dehaene S. Varieties of numerical abilities. *Cognition* 1992;44:1–42. [PubMed: 1511583]
- Dehaene, S. Verbal and nonverbal representations of numbers in the human brain. In: Galaburda, A.; Kosslyn, S.; Yves, C., editors. *The language of the brain*. Harvard University Press; Cambridge, MA, US: 2002. p. 179–190.
- Dehaene S, Akhavein R. Attention, automaticity, and levels of representation in number processing. *Journal of Experimental Psychology: Learning, Memory, and Cognition* 1995;21:314–326.
- Dehaene S, Bossini S, Giraux P. The mental representation of parity and number magnitude. *Journal of Experimental Psychology: General* 1993;122:371–396.

- Dehaene S, Dupoux E, Mehler J. Is numerical comparison digital? Analogical and symbolic effects in two-digit comparison. *Journal of Experimental Psychology: Human Perception and Performance* 1990;16:626–641. [PubMed: 2144576]
- Dehaene S, Piazza M, Pinel P, Cohen L. Three parietal circuits for number processing. *Cognitive Neuropsychology* 2003;20:487–506. [PubMed: 20957581]
- Duncan EM, McFarland CE. Isolating the effects of symbolic distance and semantic congruity in comparative judgments: An additive-factors analysis. *Memory and Cognition* 1980;8:612–622.
- Ganor-Stern D, Tzelgov J, Ellenbogen R. Automaticity and two-digit numbers. *Journal of Experimental Psychology: Human Perception and Performance* 2007;33:483–496. [PubMed: 17469981]
- Gonzalez E, Kolers P. Mental Manipulation of Arithmetic Symbols. *Journal of Experimental Psychology: Learning, Memory, and Cognition* 1982;8:308–319.
- Hinrichs JV, Yurko DS, Hu J. Two-digit number comparison: Use of place information. *Journal of Experimental Psychology: Human Perception and Performance* 1981;7:890–901.
- Ifrah, G. *The universal history of numbers*. John Wiley and Sons; New York: 2000.
- Logan GD. Toward an instance theory of automatization. *Psychological Review* 1988;95:492–527.
- Moyer RS, Landauer TK. Determinants of reaction time for digit inequality judgments. *Bulletin of the Psychonomic Society* 1973;1:167–168.
- Moyer RS, Landauer TK. Time required for judgments of numerical inequality. *Nature* 1967;215:1519–1520. [PubMed: 6052760]
- Nuerk H, Weger U, Willmes K. Decade breaks in the mental number line? Putting the tens and units back in different bins. *Cognition* 2001;82:B25–B33. [PubMed: 11672709]
- Ratinckx E, Brysbaert M, Fias W. Naming two-digit arabic numerals: Evidence from masked priming studies. *Journal of Experimental Psychology: Human Perception and Performance* 2005;31:1150–1163. [PubMed: 16262504]
- Schwartz W, Stein F. On the temporal dynamics of digit comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition* 1998;24:1275–1293.
- Shepard RN, Kilpatrick DW, Cunningham JP. The internal representation of numbers. *Cognitive Psychology* 1975;7:82–138.
- Takahashi A, Green D. Numerical judgements with Kanji and Kana. *Neuropsychologia* 1983;21:259–263. [PubMed: 6877579]
- Verguts T, Fias W, Stevens M. A model of exact small-number representation. *Psychonomic Bulletin & Review* 2005;12:66–80. [PubMed: 15945201]
- Verguts T, De Moor W. Two-digit comparison: Decomposed, Holistic, or Hybrid? *Experimental Psychology* 2005;52:195–200. [PubMed: 16076067]
- Welford AT. The measurement of sensory-motor performance: Survey and reappraisal of twelve years' progress. *Ergonomics* 1960;3:189–230.

Figure 1.
Predicted RT pattern for each probe in the integer numerical distance task of Experiment 1 for the Welford Function (gray dashes) and the Physical Similarity function (black diamonds).

Figure 2. Participants' average $z(RT)$ for each probe in the integer numerical distance task of Experiment 1 (black diamonds) and the best fit predicted model (gray dashes).

Figure 3.

Predicted RT pattern for each probe in the decimal numerical distance task of Experiments 2 and 3 for the Welford Function (gray dashes) and the Physical Similarity function (black diamonds).

Figure 4. Participants' average $z(RT)$ for each probe in the decimal numerical distance task of Experiment 2 (black diamonds) and the best fit predicted model (gray dashes).

Figure 5.

Participants' average $z(RT)$ for each probe in the decimal numerical distance task of Experiment 3 (black diamonds) and the best fit predicted model (gray dashes).

Table 1

Correlation between variables in multiple regression run in Experiment 1

	Integers			
	RT	Welford Function	Decade Similarity	Unit Similarity
Welford Function	0.95			
Decade Similarity	0.83	0.79		
Unit Similarity	-0.02	-0.08	-0.10	
Decade * Unit Similarity	-0.47	-0.39	-0.57	-0.57

Table 2

Average RT for each decade by experiment. RTs increase smoothly for Experiment 1 when participants judged integers, but RTs flattened and exhibited a stronger Physical Similarity function when participants judged decimals in Experiments 2 and 3.

Decade Integer (Decimal)	Exp 1	Exp 2	Exp 3
(.0)	644	764	827
1 (.1)	675	731	791
2 (.2)	694	734	787
3 (.3)	725	760	786
4 (.4)	751	797	817
5 (.5)	874	958	1026
6 (.6)	767	798	829
7 (.7)	721	768	784
8 (.8)	716	753	794
9 (.9)	691	740	780

Table 3

Correlation between variables in multiple regression run in Experiment 2.

	Rounded Decimals			
	RT	Welford Function	Decade Similarity	Unit Similarity
Welford Function	0.78			
Decade Similarity	0.86	0.77		
Unit Similarity	-0.07	-0.08	-0.10	
Decade * Unit Similarity	-0.49	-0.39	-0.58	-0.58

Table 4

Correlation between variables in multiple regression run in Experiment 3.

	Unrounded Decimals			
	RT	Welford Function	Decade Similarity	Unit Similarity
Welford Function	0.72			
Decade Similarity	0.90	0.77		
Unit Similarity	0.01	-0.07	-0.10	
Decade * Unit Similarity	-0.61	-0.41	-0.58	-0.57