Higher-order phase shift reconstruction approach

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Purpose: Biological soft tissues encountered in clinical and preclinical imaging mainly consists of atoms of light elements with low atomic numbers and their elemental composition is nearly uniform with little density variation. Hence, x-ray attenuation contrast is relatively poor and cannot achieve satisfactory sensitivity and specificity. In contrast, x-ray phase-contrast provides a new mechanism for soft tissue imaging. The x-ray phase shift of soft tissues is about a thousand times greater than the x-ray absorption over the diagnostic x-ray energy range, yielding a higher signal-to-noise ratio than the attenuation contrast counterpart. Thus, phase-contrast imaging is a promising technique to reveal detailed structural variation in soft tissues, offering a high contrast resolution between healthy and malignant tissues. Here the authors develop a novel phase retrieval method to reconstruct the phase image on the object plane from the intensity measurements. The reconstructed phase image is a projection of the phase shift induced by an object and serves as input to reconstruct the 3D refractive index distribution inside the object using a tomographic reconstruction algorithm. Such x-ray refractive index images can reveal structural features in soft tissues, with excellent resolution differentiating healthy and malignant tissues.

Methods: A novel phase retrieval approach is proposed to reconstruct an x-ray phase image of an object based on the paraxial Fresnel–Kirchhoff diffraction theory. A primary advantage of the authors' approach is higher-order accuracy over that with the conventional linear approximation models, relaxing the current restriction of slow phase variation. The nonlinear terms in the auto-correlation equation of the Fresnel diffraction pattern are eliminated using intensity images measured at different distances in the Fresnel diffraction region, simplifying the phase reconstruction to a linear inverse problem. Numerical experiments are performed to demonstrate the accuracy and stability of the proposed approach.

Results: The proposed reconstruction formula is a generalization of the transport of intensity equation (TIE). It has the second-order accuracy compared to the linear model used in the conventional phase retrieval approach. The numerical experiments demonstrate that the accuracy and stability of the proposed phase reconstruction method outperforms the TIE-based reconstruction method.

Conclusions: A novel approach has been proposed to retrieve an x-ray phase shift image induced by an object from intensity images measured at different distances in the Fresnel diffraction region. The authors' approach has the second-order accuracy and is able to retrieve the phase shift of an object stably, overcoming the restriction of slow phase variation assumed by the conventional phase retrieval techniques. © 2010 American Association of Physicists in Medicine.

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I. INTRODUCTION

Biological soft tissues encountered in clinical and preclinical imaging mainly consist of atoms of light elements with low atomic numbers and their elemental composition is nearly uniform with little density variation. Hence, the x-ray attenuation contrast is relatively poor and cannot achieve satisfactory sensitivity and specificity.^{1,2} In contrast, x-ray phase-contrast provides a new mechanism for soft tissues imaging. The x-ray phase shift of soft tissues is about a thousand times greater than the x-ray absorption over the diagnostic x-ray

energy range, yielding a higher signal-to-noise ratio than the attenuation contrast counterpart. Thus, phase-contrast imaging is a promising technique to reveal detailed structural variation in soft tissues, offering a high contrast resolution between healthy and malignant tissues.^{3,4} Moreover, the phase-contrast imaging does not intrinsically rely on x-ray absorption in tissues. As x-ray energy increases, photoelectric absorption of tissues decreases as $1/E^3$, while tissue phase shift decreases much slower only as $1/E$. Therefore, with an appropriate x-ray energy level, phase imaging would significantly reduce the deposited radiation dose in tissues

than attenuation contrast imaging.^{5,6} Nevertheless, unlike the measurement of x-ray intensity, x-ray phase shift through an object cannot be directly measured and a phase retrieval method is required to reconstruct the phase information from the intensity measurements.

In-line phase-contrast imaging does not require particularly demanding characteristics of monochromaticity and spatial coherence. This imaging modality can be realized with not only synchrotron radiation but also x-ray microfocus sources, and allows a relatively simple implementation.⁶ In this scenario, x-ray phase contrast is formed from the propagation of the wave field after interaction with an object, creating a quantitative correspondence between the object and the recorded images, which can be used to retrieve the phase shift induced by the object. Teague^{7,8} proposed the transport of intensity equation (TIE) for the phase retrieval, which was derived from the free space Helmholtz wave equation in the paraxial wave approximation. Based on the assumption of weak absorption and slow phase variation, TIE can be also derived from the linear approximation of the transmittance function in the autocorrelation equation of the Fresnel diffraction pattern.⁹ The solution uniqueness of TIE is shown when the spatial intensity is strictly positive¹⁰ and different methods were proposed to solve TIE, such as the Fourier transform method¹¹ and Green function method.^{8,12} However, TIE includes the intensity derivative along the optic axis on the object plane, which cannot be directly measured. The intensity derivative must be approximated using the difference between intensity images on two close planes, which is in contradiction with a relatively large separation between the planes for the good phase-contrast requirement in the presence of noise.

In this paper, we propose a novel approach to reconstruct x-ray phase shift. In Sec. II, we present our phase reconstruction method based on the paraxial Fresnel–Kirchhoff diffraction theory, which has a higher-order accuracy relative to the current linear approximation approach. The nonlinear terms in the autocorrelation equation of the Fresnel diffraction pattern are eliminated using diffraction images measured at different distances in the Fresnel diffraction region, simplifying the phase reconstruction to a linear inverse problem. In Sec. III, we evaluate the proposed approach in numerical simulations. Finally, in Sec. IV, we conclude the paper.

II. PHASE RETRIEVAL ALGORITHM

For in-line phase-contrast x-ray imaging, a partially coherent x-ray beam illuminates an object, the coherent x-ray-tissue interaction causes phase changes of the x-ray wave because of both x-ray diffraction and refraction effects, deforming the wavefront of x-ray beam propagation. When a detector is located directly behind the object, the conventional attenuation image is obtained, while at greater distances from the object, various phase-contrast images are formed. The amount of the phase change is determined by the dielectric susceptibility or, equivalently, by the refractive index of the tissue. Thus, the object is characterized by its complex-valued x-ray refractive index distribution $n=1-\delta$

+ $i\beta$, where the parameters δ characterizes the x-ray phase shift and β is related to absorption. It has been shown that δ (10^{-6} – 10^{-8}) is about 1000 times greater than β (10^{-9} – 10^{-11}) in the biological soft tissue over the 10–100 keV range.¹³ This implies that significant improvement can be achieved in terms of the sensitivity of x-ray imaging of biological soft tissues if x-ray phase information is utilized.

The wave-object interaction can be described by a transmittance function

$$u_0(\mathbf{r}) = A(\mathbf{r})\exp[i\Phi(\mathbf{r})], \quad (1)$$

where $\mathbf{r}$ denotes the transverse coordinates (x, y) in the transverse plane along the propagation direction z and $A(\mathbf{r})$ is the amplitude modulus of the wave and can be expressed as $A(\mathbf{r}) = \exp[-B(\mathbf{r})]$. After x-ray passing through an object, the absorption $B(\mathbf{r})$ and phase shift $\Phi(\mathbf{r})$ are projections through a complex refractive index distribution, and the phase change and attenuation of the x-ray beam are described by^{14,15}

$$B(\mathbf{r}) = \frac{2\pi}{\lambda} \int \beta(\mathbf{r}, z) dz; \quad \Phi(\mathbf{r}) = -\frac{2\pi}{\lambda} \int \delta(\mathbf{r}, z) dz, \quad (2)$$

where λ is the x-ray wavelength. According to the Fresnel–Kirchhoff diffraction theory, the relationship between wave amplitudes of the transverse plane in free space propagation is given by the Fresnel transformation formula¹⁴

$$u_z(\mathbf{r}) = \frac{\exp(ikz)}{i\lambda z} u_0(\mathbf{r}) ** \exp(ik|\mathbf{r}|^2/2z), \quad (3)$$

where $k=2\pi/\lambda$ and $**$ denotes convolution over the transverse coordinates $\mathbf{r}$. From Eq. (3), the Fourier transform of the Fresnel diffraction pattern can be written as¹⁶

$$\begin{aligned} & \iint |u_z(\mathbf{r})|^2 \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \\ &= \iint \left\{ A\left(\mathbf{r} - \frac{\lambda z}{2} \mathbf{w}\right) A\left(\mathbf{r} + \frac{\lambda z}{2} \mathbf{w}\right) \right. \\ & \quad \times \exp\left[i\Phi\left(\mathbf{r} + \frac{\lambda z}{2} \mathbf{w}\right) - i\Phi\left(\mathbf{r} - \frac{\lambda z}{2} \mathbf{w}\right)\right] \Big\} \\ & \quad \times \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r}. \end{aligned} \quad (4)$$

Because the hard x-ray wavelength λ is very short, we have the following approximation formulas under near field condition of Fresnel diffraction:

$$\begin{aligned} & \Phi\left(\mathbf{r} + \frac{\lambda z}{2} \mathbf{w}\right) - \Phi\left(\mathbf{r} - \frac{\lambda z}{2} \mathbf{w}\right) \approx \lambda z \mathbf{w} \cdot \nabla \Phi(\mathbf{r}) \\ & A\left(\mathbf{r} - \frac{\lambda z}{2} \mathbf{w}\right) A\left(\mathbf{r} + \frac{\lambda z}{2} \mathbf{w}\right) \\ & \approx A^2(\mathbf{r}) + \left(\frac{\lambda z}{2}\right)^2 [A(\mathbf{r}) \mathbf{w} \cdot \nabla^2 A(\mathbf{r}) \cdot \mathbf{w} \\ & \quad - (\mathbf{w} \cdot \nabla A(\mathbf{r}))^2]. \end{aligned} \quad (5)$$

Taking the second-order approximation of the phase term in Eq. (4), we obtain

$$\int \int |u_z(\mathbf{r})|^2 \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} - \frac{\lambda^2 z^2}{2} (\mathbf{w} \cdot \nabla \Phi)^2 \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r}. \quad (6)$$

$$= \int \int \left\{ \left[A^2(\mathbf{r}) + \left(\frac{\lambda z}{2} \right)^2 [A(\mathbf{r}) \mathbf{w} \cdot \nabla^2 A(\mathbf{r}) \cdot \mathbf{w} - (\mathbf{w} \cdot \nabla A(\mathbf{r}))^2] \right] \left[1 + i\lambda z \mathbf{w} \cdot \nabla \Phi \right] \right\} \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r}.$$

Using the measured intensity images at four different distances $z_i (i=1, 2, 3, 4)$ from the object in the Fresnel diffraction region, the corresponding four equations can be established from Eq. (6).

$$\mathbf{M} \cdot \begin{pmatrix} i\lambda \int \int (A^2 \mathbf{w} \cdot \nabla \Phi) \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \\ \frac{\lambda^2}{4} \int \int [A \mathbf{w} \cdot \nabla^2 A \cdot \mathbf{w} - (\mathbf{w} \cdot \nabla A)^2 - 2(A \mathbf{w} \cdot \nabla \Phi)^2] \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \\ i\frac{\lambda^3}{4} \int \int [A \mathbf{w} \cdot \nabla^2 A \cdot \mathbf{w} - (\mathbf{w} \cdot \nabla A)^2] (\mathbf{w} \cdot \nabla \Phi) \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \\ \frac{\lambda^4}{8} \int \int [(\mathbf{w} \cdot \nabla A)^2 - A \mathbf{w} \cdot \nabla^2 A \cdot \mathbf{w}] (\mathbf{w} \cdot \nabla \Phi)^2 \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \end{pmatrix} = \begin{pmatrix} \int \int (|u_{z_1}|^2 - A^2) \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \\ \int \int (|u_{z_2}|^2 - A^2) \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \\ \int \int (|u_{z_3}|^2 - A^2) \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \\ \int \int (|u_{z_4}|^2 - A^2) \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} \end{pmatrix}, \quad (7)$$

where $\mathbf{M}$ is a Vandermonde matrix of order 4,

$$\mathbf{M} = \begin{bmatrix} z_1 & z_1^2 & z_1^3 & z_1^4 \\ z_2 & z_2^2 & z_2^3 & z_2^4 \\ z_3 & z_3^2 & z_3^3 & z_3^4 \\ z_4 & z_4^2 & z_4^3 & z_4^4 \end{bmatrix}.$$

The linear system Eq. (7) can be exactly solved by inverting the matrix $\mathbf{M}$ and the high order terms in Eq. (7) are eliminated to obtain a linear integral equation with respect to the phase shift Φ ,

$$i\lambda \int \int [A^2(\mathbf{r}) (\mathbf{w} \cdot \nabla \Phi)] \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} = \int \int [\text{Row}_1(\mathbf{M}^{-1}) \cdot \mathbf{P}(\mathbf{r})] \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r}, \quad (8)$$

where $\text{Row}_1(\mathbf{M}^{-1})$ denotes the first row of the matrix $\mathbf{M}^{-1}$ and the vector $\mathbf{P}(\mathbf{r}) = \{|u_{z_1}|^2 - A^2, |u_{z_2}|^2 - A^2, |u_{z_3}|^2 - A^2, |u_{z_4}|^2 - A^2\}^T$. To filter the measurement noise, a matrix $\mathbf{T}$ of order 4 can be used to transform the measured intensity data, for example,

$$\mathbf{T} = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

Then, applying the singular value decomposition method, the matrix $(\mathbf{T} \cdot \mathbf{M})^{-1}$ can be decomposed into a diagonal matrix $\mathbf{D}$ with nonnegative diagonal elements in the decreasing order and unitary matrices $\mathbf{U}$ and $\mathbf{V}$ to stabilize the numerical computation. Thus, Eq. (8) is reduced to

$$i\lambda \int \int [A^2(\mathbf{r}) (\mathbf{w} \cdot \nabla \Phi)] \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r} = \int \int [\text{Row}_1(\mathbf{U}) \cdot \mathbf{D} \cdot \mathbf{V} \cdot \mathbf{T} \cdot \mathbf{P}(\mathbf{r})] \exp(i2\pi \mathbf{r} \cdot \mathbf{w}) d\mathbf{r}. \quad (9)$$

Equation (9) is also equivalent to the following differential equation in terms of the relationship between the spatial derivation and its Fourier transform:

$$\nabla \cdot (A^2(\mathbf{r}) \nabla \Phi) = -\frac{2\pi}{\lambda} \text{Row}_1(\mathbf{U}) \cdot \mathbf{D} \cdot \mathbf{V} \cdot \mathbf{T} \cdot \mathbf{P}(\mathbf{r}). \quad (10)$$

Equation (10) is a generalization of the TIE. It is derived by taking advantage of a second-order approximation to the phase term, which helps relax the limitation of slow phase variation assumed by the conventional linear approximation model, making the phase reconstruction more accurate and stable. The phase shift Φ can be solved based on Eq. (10) using the Fourier transform method¹¹ or Green function method.^{8,12} Furthermore, the phase shift Φ of an x-ray beam through object is described as a projection, the 3D reconstruction of the refractive index $\delta(\mathbf{r}, z)$ based on Eq. (2) can be achieved using a classic tomographic algorithm,¹⁷ such as the filter backprojection algorithm.

III. NUMERICAL EXPERIMENTS

We carried out a series of numerical simulation experiments to demonstrate the feasibility of our proposed phase retrieval approach. Two typical examples are presented as follows. The results show that our proposed method yielded a higher precision and was more robust against data noise

FIG. 1. Intensity images at propagation distances (a) 50, (b) 300, (c) 550, and (d) 800 mm, respectively.

than the TIE-based reconstruction method. In the first numerical test, an image of biological tissues (http://www.med.harvard.edu/JPNM/TF00_01/Oct3/CT.gif) was used to mimic a phase distribution on the object plane, as shown in Fig. 2(a). The image had 360×282 pixels, with the pixel size $5 \mu\text{m}$. The radiation wavelength was set to $\lambda = 0.3 \text{ \AA}$, corresponding to hard x rays. The intensity distribution was uniform on the object plane to mimic the homogeneous attenuation of object. Using the Fresnel transform Eq. (3), we calculated the intensity distributions $|u_{z_i}|$, ($i = 1, 2, 3, 4$) on the image planes at propagation distances $z_1 = 50$, $z_2 = 300$, $z_3 = 550$, and $z_4 = 800$ mm, respectively, as shown in Figs. 1(a)–1(d). Poisson noise was then added to these intensity images to simulate the real experimental condition. The proposed phase retrieval algorithm was applied to reconstruct the phase image. The reconstructed result was shown in Fig. 2(b). Furthermore, the phase reconstruction was performed using the TIE-based method. The TIE is derived from the paraxial approximation of the wave equation and it relates the phase to the derivative of image intensity along the wave propagation direction

$$\nabla \cdot [A^2(\mathbf{r}) \nabla \Phi(\mathbf{r})] = -\frac{2\pi}{\lambda} \frac{\partial A^2(\mathbf{r})}{\partial z}. \quad (11)$$

The intensity derivative along the optic axis on the object plane cannot be directly measured. The difference between the intensity image on the image plane and the intensity image on the object plane, $||u_z(\mathbf{r})|^2 - A^2(\mathbf{r})|/z$, can be used as an approximation of the intensity derivative in the propagation direction z . Theoretically, the smaller the distance apart between two intensity images is, the more accurate the difference of these two intensity images can be approximated to estimate the derivative. However, the intensity measurement always contains noise, which would significantly corrupt the phase contrast between the two intensity images separated by a small distance. Hence, in practical computation, appropriately increasing the distance between two intensity images

FIG. 2. Reconstruction of a tissue phase image. (a) The true phase image; (b) the phase image reconstructed with the proposed method; and [(c) and (d)] the phase images reconstructed from the TIE-based method with different intensity derivative estimates at distances of 5 and 50 mm, respectively.

would improve image contrast and enhance the phase shift reconstruction quality.^{5,18} On the other hand, if the detection distance is large enough to give a high contrast, the resolution becomes too poor to obtain a good visible image. Thus, both resolution and contrast should be balanced for the best visibility of a phase-contrast image.¹⁸ We tested different distances between two intensity images to estimate the intensity derivative for the phase reconstruction based on TIE. The distance of 50 mm gave optimal reconstruction results, as shown in Fig. 2(d), while using a closer distance of 5 mm between two intensity images to estimate the intensity derivative, the associated reconstruction in Fig. 2(c), was not satisfactory.

In the second example, the well-known “Lena” image was used as the phase image on the object plane, as shown in Fig. 3(a). The Lena image had a variety of features on different resolution scales. The image had 512×512 pixels with the pixel size $5 \mu\text{m}$. The intensity distribution on the object plane was also assigned the same as the Lena image to present the highly attenuation variation in the imaged object. Using the Fresnel transform Eq. (3), we calculated the intensity distributions $|u_{z_i}|$, ($i = 1, 2, 3, 4$) on the image planes at propagation distances $z_1 = 50$, $z_2 = 300$, $z_3 = 550$, and $z_4 = 800$ mm, respectively. Poisson noise was again added to

FIG. 3. Reconstruction of the Lena phase image. (a) The true phase image, (b) the phase image reconstructed with the proposed method, and (c) the phase image reconstructed with the TIE-based method.

these intensity images to simulate the experimental condition. The proposed phase retrieval algorithm was applied to reconstruct the phase image. The reconstruction result was shown in Fig. 3(b). Comparatively, we performed the numerical simulation experiment with same setting based on Eq. (11). The same Poisson noises were added to the intensity image on the object plane and an intensity image at a plane 50 mm away from the object plane. Then, the difference of the two intensity images was employed to approximate the intensity derivative in the propagation direction z for the phase reconstruction from Eq. (11). The reconstruction was shown in Fig. 3(c).

IV. CONCLUSIONS

In summary, we have proposed a novel phase retrieval approach to reconstruct x-ray phase shift induced by an object according to the paraxial Fresnel–Kirchhoff diffraction theory. A primary advantage of our approach is higher-order accuracy relative to the conventional linear approximation model, relaxing the restriction of slow phase variation. The nonlinear terms in the autocorrelation equation of the Fresnel diffraction pattern are eliminated using intensity images measured at different distances in the Fresnel diffraction region, simplifying the phase reconstruction to a linear inverse problem. Theoretically, our method can be extended to a higher-order approximation for the phase term at the cost of intensity measurement at more distances. Numerical experiments have demonstrated superior accuracy and stability of our proposed phase retrieval approach relative to the TIE-based reconstruction approach. It is expected that this new phase retrieval method will help improve x-ray phase-contrast imaging and find significant biomedical applications.

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