Published in final edited form as:

Comput Methods Biomech Biomed Engin. 2010 December ; 13(6): 795–802. doi:
10.1080/10255841003630629.

A method to determine whether a musculoskeletal model can resist arbitrary external loadings within a prescribed range

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Abstract

Computational models of the musculoskeletal system are prone to design errors. It is possible to create a model that cannot satisfy equilibrium conditions for a set of external loading conditions. A model is “loadable” if there exists a set of muscle forces that can resist an arbitrary applied force within a prescribed range. In this study, a novel mathematical method is introduced to determine whether models are loadable. In addition, an idealized musculoskeletal model is presented in order to develop the theory behind the mathematical method. The method uses the simplex algorithm to determine feasibility of the linear programming problem and can determine loadability for an arbitrary, continuous range of external forces. The method was applied to a three-dimensional model of the shoulder and correctly determined loadability for a range of externally applied forces.

Keywords

loadability; biomechanics; muscle force; shoulder; optimization; musculoskeletal modeling

1. Introduction

Musculoskeletal models are important for applications such as the prediction of joint function after surgery, the evaluation of assistive medical devices, and the study of musculoskeletal diseases. Computational models of the musculoskeletal system are commonly used to predict soft tissue and joint loading. Some models use empirically-derived muscle moment arms (Hughes and An 1996), but most modern models use mathematical representations of the anatomy to compute muscle lengths and moment arms (Hogfors *et al.* 1987; van der Helm 1994; Holzbaur *et al.* 2005; Damsgaard *et al.* 2006; Dickerson *et al.* 2007). These parameters are critical because muscle force predictions are very sensitive to them. The task of constructing mathematical models of musculature is not easy because it requires determining (1) how to represent large muscles, (2) origins and insertions, and (3) how the muscle wraps around bony anatomy (Hogfors *et al.* 1987; van der Helm *et al.* 1992; Charlton and Johnson 2001; Valero-Cuevas 2005; Valero-Cuevas *et al.* 2007). It is possible to create a model that will not satisfy mechanical conditions of equilibrium for a set of external loading conditions. Such a model would be “unloadable.” Clearly, models of healthy subjects should be able to be solved for tasks involving submaximal exertions. A model that has a solution for any externally applied force that lies in a prescribed range will be termed “loadable.” In other words, a musculoskeletal model is “loadable” for a prescribed range of external forces if and only if the model can provide the

necessary muscle force to remain in equilibrium and resist the application of any external force in the prescribed range.

Other methods have been created to determine whether models can resist arbitrary loads. Valero-Cuevas *et al.* have described a very efficient and powerful method that uses computational geometry to find the set of all muscle force magnitudes that are able to produce zero torque in the model (Valero-Cuevas *et al.* 1998). However, if the objective is only to determine model loadability for a given external force, it is not necessary to determine the actual muscle force magnitudes required to resist that force. It may require less computational effort if the goal is solely to determine loadability of a model. In addition, the method described by Valero-Cuevas *et al.* finds the entire set of biomechanically achievable force vectors that can be produced by the model with zero torque; it does not determine loadability of a model for a specific, continuous range of external forces. If the goal is to determine loadability solely for a specific, chosen set of external forces, it may be more computationally efficient to disregard the external force vectors outside the set of interest.

Therefore, the purpose of this project is to develop an efficient method for testing whether a musculoskeletal model is loadable for a specific, continuous range of external forces. The method can be used in the model development process to identify problems in describing the musculoskeletal anatomy.

2. Theory

Consider a static planar model of a hinge joint with two muscles, shown in Figure 1. The appendage has mass m and length r . Vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ correspond to muscle forces from muscles 1 and 2 acting across the joint, and vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ correspond to moment arms of their respective forces. There is an external force $\mathbf{F}$ applied to the end of the appendage. The moment arm of the external force is $\mathbf{r}$, and the moment arm of the force of gravity is $\mathbf{d}$. The acceleration of gravity is represented by the vector $\mathbf{g}$.

Static equilibrium for the idealized single-segment model in Figure 1 is formulated as

$$\sum_{i=1}^n \mathbf{a}_i \times \mathbf{f}_i = -\mathbf{r} \times \mathbf{F} - \mathbf{d} \times m\mathbf{g} \quad (1)$$

where n , the number of muscles acting across the joint, is 2. The equation in (1) defines the moment equilibrium conditions. Although the static, planar Figure 1 was used to develop the equilibrium conditions, the equation in (1) is applicable in three dimensions since the moments resulting from the cross products are three-dimensional. Let $\mathbf{x}_i$ be defined as the magnitude of the force vector $\mathbf{f}_i$ generated by muscle i , and let $\mathbf{\tau}_i$ be defined as a unit vector in the same direction as $\mathbf{f}_i$. Combined with upper and lower limits on muscle force, equation (1) can be rewritten to give the system:

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{0} &\leq \mathbf{x} \leq \mathbf{u} \end{aligned} \quad (2)$$

where $\mathbf{A}$ is a matrix whose i th column corresponds to $\mathbf{a}_i \times \mathbf{\tau}_i$ for muscle i , $\mathbf{x}$ is a column vector of length n containing force magnitudes for each muscle, $\mathbf{b}$ is a column vector containing the right side of equation (1), and $\mathbf{u}$ is a column vector of length n containing

magnitude upper bounds for each muscle. Although (1) and (2) were developed using a single-joint system, (1) and (2) also apply to multiple-joint systems in three dimensions. For example, for a two joint system, (1) can be specified using the appropriate moment arms for the first joint, and then another instance of equation (1) can be specified using the appropriate moment arms for the second joint. Solving the two equations would yield the muscle forces for static equilibrium of the two-joint system. Using the matrix notation of (2), additional rows are added to $\mathbf{A}$ for the degrees of freedom provided by the second joint. For example, if $\mathbf{A}$ was originally a 3-by- n matrix for a single joint, and the second joint provided an additional 3 degrees of freedom, $\mathbf{A}$ would now be a 6-by- n matrix where the first 3 rows correspond to $\mathbf{a}_i \times \boldsymbol{\tau}_i$ for the first joint, and rows 4 through 6 correspond to $\mathbf{a}_i \times \boldsymbol{\tau}_i$ for the second joint. The columns still each correspond to $\mathbf{a}_i \times \boldsymbol{\tau}_i$ for muscle i . The vector $\mathbf{x}$ would remain the same, while $\mathbf{b}$ would be extended to a length of 6 to include the gravity and external force torques applied to the second joint.

The system is considered arbitrarily loadable if there exists a solution $\mathbf{x}$ to (2) for any arbitrary $\mathbf{F}$. The formulation in (2) is sufficient to determine the loadability of a model for any single force vector $\mathbf{F}$. However, there are many instances in which it is advantageous to determine the loadability of a model for a specific and continuous range of forces. In the previous formulation, (2) would need to be solved for every $\mathbf{F}$ in the range of forces, which could require finding a solution to (2) an infinite number of times. Therefore, a computationally efficient method was developed as follows. In order to determine loadability for forces within a specified range, the user can specify an arbitrary number of vectors $\mathbf{h}_k$ so that these vectors describe the boundary of a convex hull of externally applied force vectors. Every force vector $\mathbf{F}$ that lies in the convex hull can be written as a non-negative linear combination of the vectors $\mathbf{h}_k$:

$$\begin{aligned} \mathbf{F} &= \sum_{k=1}^q \lambda_k \mathbf{h}_k \\ \sum_{k=1}^q \lambda_k &= 1 \quad \text{and} \quad \lambda_k \geq 0 \quad (k=1, 2, \dots, q) \end{aligned} \quad (3)$$

where λ_k is an arbitrary non-negative scalar multiplier, the $\mathbf{h}_k$ vectors specify the boundary of the user-chosen convex hull of external forces, and q is the number of $\mathbf{h}_k$ vectors.

The system is determined to be loadable for all vectors that lie in the convex hull if one or more solutions, $\mathbf{y}_k$, to the following equations

$$\begin{aligned} \mathbf{A}\mathbf{y}_k &= \boldsymbol{\beta}_k \\ \mathbf{0} &\leq \mathbf{y}_k \leq \mathbf{u} \\ (k &= 1, 2, \dots, q) \end{aligned} \quad (4)$$

exist for every k , where $\boldsymbol{\beta}_k = -\mathbf{r} \times \mathbf{h}_k - \mathbf{d} \times m\mathbf{g}$. See the Appendix for the proof of this fact. The equations in (4) were constructed by substituting each user-chosen $\mathbf{h}_k$ for $\mathbf{F}$ in (1). In other words, each equation in (4) is the same as in (1), except that $\mathbf{h}_k$ is used for the externally applied force vector. Note that (4) actually consists of q equations, each using a different $\boldsymbol{\beta}_k$. By solving (4) for each $\boldsymbol{\beta}_k$, loadability is determined for all the possible force vectors that lie in the convex hull specified by the $\mathbf{h}_k$ vectors. If no solutions to (4) exist, the system is unloadable since the equations in (4) are merely using each $\mathbf{h}_k$ as the external force vector.

However, for the purposes of determining loadability of a musculoskeletal model, only the feasibility of a solution $\mathbf{y}_k$ to (4) needs to be ascertained. If there are p degrees of freedom in the model, feasibility can be determined by solving the linear program:

$$\begin{aligned} &\text{minimize} && \sum_{j=1}^p v_j \\ &\text{subject to} && \mathbf{A}\mathbf{y}_k + \mathbf{v} = \boldsymbol{\beta}_k \\ &&& \mathbf{0} \leq \mathbf{y}_k \leq \mathbf{u}, \mathbf{0} \leq \mathbf{v} \quad (k=1, 2, \dots, q) \end{aligned} \quad (5)$$

The equations in (5) are the ones that determine loadability of the musculoskeletal model for the given range of external forces. The formulation in (5) is Phase 1 of the Two-Phase Simplex Method that solves the original problem in (4). In the Two-Phase Simplex Method, the solution to Phase 1 is used as the starting point for Phase 2. Phase 1 also determines whether a feasible solution exists for the original problem. In the case that Phase 1 cannot be solved with $\mathbf{v} = \mathbf{0}$, then no feasible solution exists for the original problem. Phase 1, represented by the equations in (5), can be solved using the Revised Simplex Method (Dantzig and Orchard-Hays 1954), a matrix-based implementation of the Generalized Simplex Method (Dantzig 1951; Dantzig *et al.* 1955). Therefore, each formulation in (4) has a solution if and only if each one in (5) has a solution with $\mathbf{v} = \mathbf{0}$. Consequently, solutions to (5) with $\mathbf{v} = \mathbf{0}$ are necessary and sufficient for loadability of the model. When solutions to (5) with $\mathbf{v} = \mathbf{0}$ cannot be found, the model does not provide the necessary muscle force or bony configuration to remain in equilibrium. This behavior could be due to an error in the model itself, or it could be a desired property that was designed into the model.

In summary, the method for musculoskeletal model loadability is as follows. First, specify vectors $\mathbf{h}_k$ that describe the boundary of the desired convex hull of external forces. If, for each $\mathbf{h}_k$, there is a solution to (5) with $\mathbf{v} = \mathbf{0}$, then the model is loadable for all external forces applied to it that lie in the convex hull specified by the $\mathbf{h}_k$ vectors. If no solutions to (5) with $\mathbf{v} = \mathbf{0}$ exist, then the model is unloadable.

3. Examples

The method of determining loadability was applied to two examples involving a three-dimensional shoulder model: one example that was determined to be loadable, and one that was determined to be unloadable. The shoulder model was constructed using the AnyBody Modeling System version 3.0, and the loadability method and resulting analysis was implemented in MATLAB.

3.1 Loadable shoulder model without the deltoid and supraspinatus muscles

The shoulder model was constructed using Repository 6.2 of the AnyBody Modeling System Version 3.0. The model used a total of 34 muscles, where each muscle was split into one or more segments, depending on the spread of the muscle. A table of the muscles used is shown in Table 1, and a graphical representation of the model is shown in Figure 2. The static posture used in the analysis is shown in Figures 2 and 3, and consists of a glenohumeral abduction angle of $\pi/6$, an elbow flexion angle of $\pi/2$, and an elbow pronation angle of $5\pi/18$.

The goal of this example was to determine whether shoulder muscles other than the deltoid and supraspinatus are able to produce shoulder abduction. If external forces are applied to the model in such a way to produce adduction of the shoulder, removing all the deltoid and supraspinatus muscles may cause the model to be unloadable since those muscles are the

major shoulder abductors. Therefore, a convex hull of external forces was chosen so that the forces pushed medially against the elbow, creating adduction of the shoulder. The convex hull of externally applied forces is specified by three vectors applied medially to the elbow, in addition to the zero vector applied at the elbow. This convex hull is shown in Figure 3. The AnyBody global coordinate convention is such that $+x$ is anterior, $+y$ is superior, and $+z$ is lateral to the right of the body. The three non-zero vectors describing the convex hull are directed so that they are at angles of $\pi/6$ to the z -axis. The three non-zero vectors were chosen to specify a convex hull of external forces that push medially against the elbow. There is no significance to why they are oriented specifically at angles of $\pi/6$ to the z -axis; they were just chosen to specify a triangular pyramid pointing to the elbow, as shown in Figure 3. The zero vector was included as the fourth vertex of the triangular pyramid, where the tips of the three non-zero vectors intersect.

The AnyBody Modeling System uses inverse dynamic analysis to determine the muscle configuration of its models (Damsgaard *et al.* 2006). For a given posture and externally applied force, the software produces a matrix similar to $\mathbf{A}$ in (4), and a vector similar to β_k in (4). Each column of the matrix similar to $\mathbf{A}$ contains the cross product of the moment arm and a unit vector in the same direction as the muscle force. In other words, each column of $\mathbf{A}$ contains the unit torque for a muscle. The vector similar to β_k contains the torque provided by the external force and gravity.

The method described by (5) was applied to determine if a shoulder model without the deltoid and supraspinatus muscles was in fact unloadable. In this example, there were four vectors, $\mathbf{h}_1 \dots \mathbf{h}_4$, used to specify the convex hull, as described above. The deltoid and supraspinatus muscles were removed from the model, which caused the AnyBody software to produce a matrix similar to $\mathbf{A}$ for the chosen posture and muscle configuration. The appropriate entries of this matrix were then extracted to construct the matrix $\mathbf{A}$ in (4) for use in a MATLAB program. By using $\mathbf{h}_1$ as the external force applied to the elbow, AnyBody produced a vector similar to β_1 . Again, appropriate entries of this vector were extracted for use in the MATLAB program. Since the goal of this example was to investigate whether shoulder muscles other than the deltoid and supraspinatus can produce abductive force, the upper bounds in the vector $\mathbf{u}$ were set to very large numbers in order to avoid finding a solution based on limitations in strength for different shoulder muscles. Phase 1 of the Two-Phase Simplex Method, specified by (5), was computed with the `linprog()` MATLAB function, using the $\mathbf{A}$ matrix and β_1 vector to find a solution $\mathbf{y}_1$ with $\mathbf{v} = \mathbf{0}$. The function was used with the 'LargeScale' option set to 'off' and the 'Simplex' option set to 'on'. This computation was repeated with the β_2 , β_3 , and β_4 vectors.

Surprisingly, this model was determined to be loadable since solutions with $\mathbf{v} = \mathbf{0}$ were found for all four $\mathbf{h}_k$ vectors. Although only the existence of solutions to (4) is necessary to determine loadability for the specified convex hull, a solution was solved for each of the three non-zero force vector vertices of the convex hull by solving Phase 2 of the Two-Phase Simplex Method. Phase 2 was computed using the `linprog()` function with the same options mentioned above, and solves for the optimal muscle force magnitudes $\mathbf{y}_k$ in equation (4). The solution for each of these vertices used an infraspinatus segment, a triceps (medial head) segment, and a pronator teres caput humeral segment to provide the necessary abductive force to counter the external force. Each solution used a different amount of force for each of those muscles, but all three solutions activated those same muscles.

3.2 Unloadable shoulder model

The next objective was to test the method on a model that was unloadable. The same shoulder model used in Example 3.1 was used in this example, with the same posture and same convex hull of external forces. However, in addition to taking out all the deltoid and

supraspinatus muscles, all the infraspinatus muscles and the medial heads of the triceps muscles were removed.

After applying the loadability algorithm as described in section 3.1, solutions to (5) with $\mathbf{v} = \mathbf{0}$ could not be found for each vector defining the vertices of the convex hull. Since (5) could not be solved with $\mathbf{v} = \mathbf{0}$, the shoulder model was determined to be unloadable.

4. Discussion

Leijnse (1996) considered a static planar model of a finger with two joints and three muscles. Static equilibrium was formulated in essentially the same manner as (1) and (2). The system was termed “loadable” if there existed finite $\mathbf{x}$ satisfying (2) for any arbitrary direction of load $\mathbf{F}$ applied to the distal phalanx in the sagittal plane. However, a more general formulation is to examine arbitrary loading at each joint. The rationale for this extension is that forces are applied to all three phalanges in multiple different planes during many tasks. Therefore, the system is considered arbitrarily loadable if there exists a solution $\mathbf{x}$ to (2) for any arbitrary $\mathbf{F}$ applied in any plane in three-dimensions. This method differs from that of Leijnse since Leijnse used a graphical method involving moment arm plots to determine loadability in a single plane. The method discussed in this paper uses optimization to determine loadability in three dimensions.

As mentioned in the Introduction, Valero-Cuevas *et al.* have created a very powerful method that uses computational geometry to find the entire set of muscle force magnitudes that are able to produce zero torque in the model, and subsequently, the entire set of biomechanically achievable force vectors that can be produced by the model with zero torque (Valero-Cuevas *et al.* 1998). The method described in Section 2 determines loadability of a model without having to find the actual muscle force magnitudes that are required for the model to resist an external force. In doing this, the method may be more computationally efficient in certain situations. Because the method in Section 2 uses Phase 1 of the Two-Phase Simplex Method, the complexity of the method depends largely on the particular linear programming problem it is applied to. In the worst case, the simplex method can require exponential time. However, the simplex method is typically much more efficient in practice, usually operating in polynomial time (Spielman and Teng 2001). The method described by Valero-Cuevas *et al.* uses the vertex enumeration algorithm presented by Avis and Fukuda. This vertex enumeration algorithm finds the v vertices in d -dimensional space defined by a nondegenerate system of n inequalities in time $O(ndv)$ (Avis and Fukuda 1992). In addition, the method in Section 2 determines loadability for a convex hull of external forces by examining only the vertices of the convex hull. In order to determine loadability for a prescribed convex hull of external forces, the method described by Valero-Cuevas *et al.* first determines the entire set of muscle force magnitudes that produce zero torque in the model, and then the entire set of biomechanically achievable forces the model can produce using those muscle force magnitudes. Then it would have to compare that set of biomechanically achievable forces to the prescribed convex hull, and determine how the two sets intersect. Because of the differences in computational complexity and overall objective of the two methods, it is difficult to compare the two methods as they have different applications. The method described by Valero-Cuevas *et al.* finds all possible solutions to equation (2) for all external forces that can be resisted by the model, while the method in Section 2 focuses more on loadability and less on finding actual solutions.

The method presented here is most suitable for analyzing three-dimensional musculoskeletal models having multiple joints. While the presentation has been organized around a static model, it could be applied to a dynamics model on a frame-by-frame basis. Simpler models,

such as one- or two-degree of freedom models, are likely to be amenable to analysis by direct observation of moment arm magnitudes and signs.

In example 3.1, an otherwise normal shoulder model had its deltoid and supraspinatus muscles removed. Although these muscles are the two major muscles responsible for abduction at the glenohumeral joint, other muscles also contribute to abduction. Consequently, the mathematical method found a solution to resist the convex hull of external forces by utilizing weaker shoulder abductors such as the infraspinatus combined with the triceps. The pronator teres muscle was activated in order to maintain the elbow flexion angle of $\pi/2$, and elbow pronation angle of $5\pi/18$ in the model posture. Hence, this model was determined to be loadable for the specified convex hull of forces.

Example 3.2 removes the infraspinatus muscles and the medial heads of the triceps in addition to the deltoid and supraspinatus. Removing these additional muscles eliminates all muscles with an abduction moment, and the shoulder is not able to abduct against the external forces. The model is clearly unloadable for the given convex hull of external forces.

5. Conclusion

An efficient mathematical method to determine loadability of a musculoskeletal model was presented. This method conveniently determines loadability for a continuous range of forces, specified as a convex hull of external force vectors. As musculoskeletal models become more realistic, their complexity also tends to increase. Shoulder models, for example, tend to have a large number of muscles, with each of those muscles subdivided into several lines of action. In addition, many design decisions are involved during model development, including the orientation of each muscle, and how they wrap around bones and other tissues. The mathematical method for loadability provides a useful tool to confirm that a model is able to resist a range of external forces. This tool can be used during model development to quickly verify appropriate anatomical functionality.

Acknowledgments

We would like to thank Christopher J. Gatti for his technical expertise. Alan Chu was supported by the Student Biomedical Research Program at the University of Michigan.

Appendix

Suppose there exist solutions, $\mathbf{y}_k$, to (4). Furthermore, suppose $\mathbf{F}^*$ is the external force vector applied to the model. Given $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_q$, which specify the vertices of the convex hull C

$$C = \{ \mathbf{c} | \mathbf{c} = \lambda_1 \mathbf{h}_1 + \lambda_2 \mathbf{h}_2 + \dots + \lambda_q \mathbf{h}_q, \sum_{k=1}^q \lambda_k = 1, (\lambda_1, \lambda_2, \dots, \lambda_q \geq 0) \} \quad \text{A.1}$$

the goal is to prove that if $\mathbf{F}^* \in C$, then there exists a solution $\mathbf{x}^*$ satisfying

$$\begin{aligned} A\mathbf{x}^* &= \mathbf{F}^* \\ \mathbf{0} &\leq \mathbf{x}^* \leq \mathbf{u} \end{aligned} \quad \text{(A.2)}$$

where $\mathbf{f}^* = -\mathbf{r} \times \mathbf{F}^* - \mathbf{d} \times m\mathbf{g}$. In other words, the goal is to prove that if the model is able to resist each of the vertex vectors of a convex hull, then the model can resist any force vector inside that convex hull. Multiplying both sides of equation (4) by λ_k yields

$$\begin{aligned} A\lambda_k \mathbf{y}_k &= -\lambda_k \mathbf{r} \times \mathbf{h}_k - \lambda_k \mathbf{d} \times m\mathbf{g} \\ \mathbf{0} &\leq \mathbf{y}_k \leq \mathbf{u} \quad (k=1, 2, \dots, q) \end{aligned} \quad (\text{A.3})$$

Adding together all q equations in (A.3) produces

$$A(\lambda_1 \mathbf{y}_1 + \lambda_2 \mathbf{y}_2 + \dots + \lambda_q \mathbf{y}_q) = -\sum_{k=1}^q [\lambda_k \mathbf{r} \times \mathbf{h}_k] - \sum_{k=1}^q [\lambda_k \mathbf{d} \times m\mathbf{g}] \quad (\text{A.4})$$

By the definition of a convex hull, $\sum_{k=1}^q \lambda_k = 1$, so (A.4) can be written as

$$A(\lambda_1 \mathbf{y}_1 + \lambda_2 \mathbf{y}_2 + \dots + \lambda_q \mathbf{y}_q) = -\sum_{k=1}^q [\lambda_k \mathbf{r} \times \mathbf{h}_k] - \mathbf{d} \times m\mathbf{g} \quad (\text{A.5})$$

Since $\mathbf{F}^* \in C$, there exists $\lambda_1, \lambda_2, \dots, \lambda_q$, such that

$$\mathbf{F}^* = \sum_{k=1}^q \lambda_k \mathbf{h}_k \quad (\lambda_1, \lambda_2, \dots, \lambda_q \geq 0)$$

Therefore,

$$-\mathbf{r} \times \mathbf{F}^* = -\mathbf{r} \times (\lambda_1 \mathbf{h}_1 + \lambda_2 \mathbf{h}_2 + \dots + \lambda_q \mathbf{h}_q)$$

Since the cross product is distributive over addition,

$$-\mathbf{r} \times \mathbf{F}^* = -\lambda_1 (\mathbf{r} \times \mathbf{h}_1) - \lambda_2 (\mathbf{r} \times \mathbf{h}_2) - \dots - \lambda_q (\mathbf{r} \times \mathbf{h}_q) = -\sum_{k=1}^q [\lambda_k \mathbf{r} \times \mathbf{h}_k] \quad (\text{A.6})$$

By letting $\mathbf{x}^* = \lambda_1 \mathbf{y}_1 + \lambda_2 \mathbf{y}_2 + \dots + \lambda_q \mathbf{y}_q$ and substituting (A.6) into (A.5),

$$A\mathbf{x}^* = -\mathbf{r} \times \mathbf{F}^* - \mathbf{d} \times m\mathbf{g} = \mathbf{f}^*$$

Since λ_k and $\mathbf{y}_k$ are non-negative from (A.1) and (4), respectively, it follows that $\mathbf{x}^*$ is also non-negative. Moreover, $\mathbf{x}^* \leq \mathbf{u}$ because

$$\sum_{k=1}^q \lambda_k \mathbf{y}_k \leq \mathbf{u} \sum_{k=1}^q \lambda_k$$

and $\mathbf{x}^*$ is a convex combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_q$, and each $\mathbf{y}_k$ is bounded by $\mathbf{u}$ from (A.3). Therefore, given that there exist solutions to (4) and that $\mathbf{F}^* \subset C$, there exists a solution $\mathbf{x}^*$ satisfying (A.2). In other words, if the model can resist each vertex vector of a convex hull, and the externally applied force vector is inside that convex hull, then the model can resist that externally applied force vector.

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Nomenclature

$\mathbf{a}_i$	moment arm of force vector $\mathbf{f}_i$
$\mathbf{A}$	3-by- n matrix whose i th column corresponds to $\mathbf{a}_i \times \boldsymbol{\tau}_i$ for muscle i
$\mathbf{b}$	column vector of length 3 that equals $-\mathbf{r} \times \mathbf{F} - \mathbf{d} \times m\mathbf{g}$
$\boldsymbol{\beta}_k$	column vector of length 3 that equals $-\mathbf{r} \times \mathbf{h}_k - \mathbf{d} \times m\mathbf{g}$
$\boldsymbol{\beta}^*$	column vector of length 3 that equals $-\mathbf{r} \times \mathbf{F}^* - \mathbf{d} \times m\mathbf{g}$.
$\mathbf{C}$	convex hull of externally applied force vectors
$\mathbf{d}$	moment arm of the force of gravity acting upon the appendage
$\mathbf{f}_i$	force vector of muscle i acting across the joint
$\mathbf{F}$	external force vector applied to the appendage
$\mathbf{F}^*$	external force vector within the convex hull $\mathbf{C}$
$\mathbf{g}$	acceleration of gravity
$\mathbf{h}_k$	vector describing part of the boundary of a convex hull of externally applied force vectors
i	muscle number
k	index of the $\mathbf{h}_k$ vectors. ($k = 1, 2, \dots, q$)
λ_k	arbitrary non-negative scalar multiplier
m	mass of the appendage
n	number of muscles acting across the joint
p	degrees of freedom in the model
q	number of $\mathbf{h}_k$ vectors
r	length of the appendage
$\mathbf{r}$	moment arm of the external force vector $\mathbf{F}$
$\boldsymbol{\tau}_i$	unit vector in the same direction as force vector $\mathbf{f}_i$
$\mathbf{u}$	column vector of length n containing magnitude upper bounds for each muscle
$\mathbf{v}$	dummy column vector of length 3
$\mathbf{x}$	column vector of length n containing force magnitudes for each muscle
x_i	magnitude of the force $\mathbf{f}_i$ generated by muscle i . An element of the vector $\mathbf{x}$.
$\mathbf{x}^*$	column vector of length n containing force magnitudes for each muscle. Solution to $\mathbf{A}\mathbf{x}^* = \boldsymbol{\beta}^*$.
$\mathbf{y}_k$	column vector of length n containing force magnitudes for each muscle. There are $k = 1, 2, \dots, q$ of these vectors.

Biographical Notes

Richard Hughes, Ph.D., is an Associate Professor of Orthopaedic Surgery, Biomedical Engineering, and Industrial & Operations Engineering at the University of Michigan. He completed his undergraduate education at Princeton University and received his Ph.D. from the University of Michigan in 1991. He was a post-doctoral fellow at the Mayo Clinic and

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Figure 1.
Idealized static planar model with two muscles.

Figure 2.

Anybody shoulder model used in the examples. The posture has a glenohumeral abduction angle of $\pi/6$, an elbow flexion angle of $\pi/2$, and an elbow pronation angle of $5\pi/18$.

Figure 3.

A convex hull of externally applied forces included with the Anybody shoulder model. The AnyBody global coordinate convention is such that $+x$ is anterior, $+y$ is superior, and $+z$ is lateral to the right of the body. In coordinate notation, the four vectors making up the vertices are $(0, -1, -\sqrt{3})$, $(\sqrt{3}/2, 1/2, -\sqrt{3})$, $(-\sqrt{3}/2, 1/2, -\sqrt{3})$, and $(0, 0, 0)$.

Table 1

Muscles used in the AnyBody shoulder model and the number of segments that compose each muscle.

Muscle name	Number of segments
Wrist Flexion	1
Wrist Extension	1
Wrist Abduction	1
Wrist Adduction	1
Biceps Brachii Caput Breve	1
Biceps Brachii Caput Longum	1
Coracobrachialis	6
Deltoideus Scapular	6
Deltoideus Clavicular	6
Infraspinatus	6
Latissimus Dorsi	5
Levator Scapulae	3
Pectoralis Major Thoracic Part	5
Pectoralis Major Clavicular Part	5
Pectoralis Minor	4
Rhomboideus	3
Serratus Anterior	6
Subscapularis	6
Supraspinatus	6
Teres Major	6
Teres Minor	6
Trapezius Scapular	6
Trapezius Clavicular	6
Brachialis	2
Triceps (long head)	2
Triceps (medial head)	2
Triceps (lateral head)	2
Brachioradialis	2
Anconeus	2
Pronator Teres Caput Humeral	2
Pronator Teres Caput Ulnare	1
Supinator Humerus	2
Supinator Ulna	2
Pronator Quadratus	2