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Dependence of accuracy of ESPRIT estimates on signal eigenvalues: the case of a noisy sum of two real exponentials

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Abstract

This paper is devoted to estimation of parameters for a noisy sum of two real exponential functions. Singular Spectrum Analysis is used to extract the signal subspace and then the ESPRIT method exploiting signal subspace features is applied to obtain estimates of the desired exponential rates. Dependence of estimation quality on signal eigenvalues is investigated. The special design to test this relation is elaborated.

1 Introduction

In this paper we address the task of estimation of signal parameters having noisy measurements when the signal is the sum of two real exponential functions. For this purpose we apply the method ESPRIT [1] originally aimed at estimation of parameters of complex-valued exponentials (cisoids).

The ESPRIT method exploits the subspace properties of the signal and this is what connects it with Singular Spectrum Analysis (SSA) [2]. SSA is the method of time series analysis aimed at decomposition of the time series into the sum of components, e.g. signal and noise. As an essential step, SSA includes the construction of a subspace which approximates the subspace generated in some way by the signal. ESPRIT uses a basis of this subspace for estimating the rates of the exponential components. After that, the coefficients before the exponentials can be evaluated using the conventional least squares method.

We introduce the signal subspace as follows. For the given series $S = (s_0, \dots, s_{N-1})$, we select the *window length* L , where $2 \leq L \leq (N+1)/2$. Then S is embedded into $\mathbb{R}^L$ constructing $S_j = (s_{j-1}, \dots, s_{j+L-2})^T \in \mathbb{R}^L$, $j = 1, \dots, K$, $K = N - L + 1$. The linear span of $\{S_j\}$ is called *L-trajectory space* of the series S . The *L-rank* (or simply *rank*) of the series is defined to be the dimension r of the trajectory space, naturally $r \leq L$. The rank of the series S is equal to (i) the rank of the *trajectory matrix* $\mathbf{S}$ with the column vectors $\{S_j\}$, and (ii) the number of non-zero eigenvalues of the matrix $\mathbf{S}\mathbf{S}^T$, i.e. the number of components of Singular Value Decomposition (SVD) of $\mathbf{S}$. We denote the non-zero eigenvalues as $\lambda_1(S) \geq \dots \geq \lambda_r(S)$, and $\mu_j(S) = \lambda_j(S)/K$.

If the signal S has rank $r < L$, then the trajectory subspace of S is called the *signal subspace* and determines the shape of the signal up to coefficients before summands. When the series contains both the signal and noise, the signal subspace can be found only approximately (e.g. by means of SSA) that leads to approximate estimation of parameters of the signal.

Algorithms of SSA and ESPRIT are briefly described in Section 2. The design of numerical tests aimed to reveal the relation between properties of ESPRIT estimates and the signal eigenvalues $\lambda_j(S)$ is constructed in Section 3. The results on mean square deviations of estimates are presented and discussed in Section 4.

2 Algorithms

Let us consider a noisy signal $\tilde{\mathcal{F}} = S + \mathcal{N} = (f_0, \dots, f_{N-1})$ and the window length selected is L . The first step of both SSA and ESPRIT is the construction of the trajectory matrix $\mathbf{F}$, having the columns $F_j = (f_{j-1}, \dots, f_{j+L-2})^T, j = 1, \dots, K, K = N - L + 1$. The second step is the SVD of the matrix $\mathbf{F}$ producing, in particular, the eigenvectors U_j of the matrix $\mathbf{F}\mathbf{F}^T$ ordered by decreasing the corresponding eigenvalues $\lambda_j = \lambda_j(\tilde{\mathcal{F}})$. The next step of SSA is the grouping (summation) of SVD components. For example, to extract the signal S from the given series $\tilde{\mathcal{F}}$, one can select a group of SVD components with indices $\mathcal{J}$, which produces an approximation $\hat{S}$ of the signal after the diagonal averaging, on condition that the signal is approximately separated from the rest [2]. As a rule, $\mathcal{J} = \{1, \dots, r\}$.

Under conditions of approximate separability, the space spanned by $\{U_j\}_{j \in \mathcal{J}}$ approximates the signal subspace. In general, the better is the reconstruction of the signal subspace, the more precise are the estimates of signal parameters.

Let us describe ESPRIT for the signal $S = \mathcal{A} + \mathcal{B}$ consisting of the sum of two exponential components $\mathcal{A}$ and $\mathcal{B}$ with elements $a_n = c_a \exp(\alpha n) = c_a r_a^n$ and $b_n = c_b \exp(\beta n) = c_b r_b^n$. The aim is to estimate the parameters r_a, r_b and the coefficients c_a, c_b . The rank of S equals two and its subspace is approximated by $\text{span}\{U_1, U_2\}$. We denote by $\mathbf{U}_\Delta$ the matrix $\mathbf{U}$ without its last row, by $\mathbf{U}^\nabla$ the matrix $\mathbf{U}$ without its first row, and by $\mathbf{U}_\Delta^+$ the Moore-Penrose pseudoinverse of the matrix $\mathbf{U}_\Delta$. Then the eigenvalues of the matrix $\mathbf{U}_\Delta^+ \mathbf{U}^\nabla$ of size 2×2 are the estimates of the parameters r_a and r_b of the signal, $\hat{r}_a$ and $\hat{r}_b$ [1]. Note that in the noiseless case $\tilde{\mathcal{F}} = S$ it holds that $\hat{r}_a = r_a$ and $\hat{r}_b = r_b$. Having the estimates $\hat{r}_a$ and $\hat{r}_b$, the coefficients c_a and c_b can be estimated, e.g. by the linear least-squares method.

3 Design of tests

Let us consider the signal $S = \mathcal{A} + \mathcal{B}$, define signal eigenvalues $\lambda_{\min} = \lambda_1(S), \lambda_{\max} = \lambda_2(S)$ and denote their divided by K versions as $\mu_{\min}$ and $\mu_{\max}$. Suppose that we observe $\tilde{\mathcal{F}} = S + \mathcal{N}$, where $\mathcal{N}$ is the white gaussian noise with variance σ^2 . Let $\mathbf{S}$ and $\mathbf{N}$ be the trajectory matrices of S and $\mathcal{N}$, correspondingly, and $\|\cdot\|$ is the Frobenius norm which is defined to be the square root of the sum of squares of the entries of the matrix. Hence $\|\mathbf{S}\|^2 = \lambda_{\min} + \lambda_{\max}$,

$\|\mathbf{N}\|^2 = \sum_{j=1}^L \lambda_j(\mathcal{N}) \approx LK \sigma^2$ and the signal-to-noise ratio $\|\mathbf{S}\|^2 / \|\mathbf{N}\|^2$ for the given L is specified by $(\mu_{\min} + \mu_{\max}) / (L\sigma^2)$, where the numerator is actually the weighted sum of squares of the elements of the signal. Below we refer to the Frobenius norm of the trajectory matrix as L -norm of the series. Note that the average of noise eigenvalues $\lambda_j(\mathcal{N}), j = 1, \dots, L$, is approximately equal to $K\sigma^2$. To obtain the proper approximation of the two-dimensional signal subspace, we need $\lambda_{\min}$ to be larger than the maximal of noise eigenvalues. Hence, the relation between $\mu_{\min}$ and σ^2 is the key point here.

Obviously, the eigenvalues depend on L . We choose $L = N/2$ in order to increase $\mu_{\min}$ and to improve also the separability between signal and noise following the recommendations of SSA theory [2]. Let us fix $N = 99$ and $L = 50$. We consider eight test sets of parameters producing exponential components with equal L -norms and the fixed signal L -norms: $\mu_1(\mathcal{A}) = \mu_1(\mathcal{B})$, $\mu_{\min} + \mu_{\max} = 2$. These test sets differ in the value of $\mu_{\min}$, see Table 1 with the test sets ordered by increasing $\mu_{\min}$. Note that we consider as the first exponential the one with the largest absolute value of rate: $|\alpha| \geq |\beta|$, i.e. the first exponential component changes not slower than the second one.

4 Numerical results

The statistical simulation was performed to estimate the parameters of the model. Two levels of noise were considered, $\sigma^2 = 1.e-2$ and $\sigma^2 = 1.e-4$. The signal-to-noise ratio is equal to 4 and 400, correspondingly. The two obtained ESPRIT estimates $\hat{r}_a$ and $\hat{r}_b$ were ordered by value to have the same order as r_a and r_b . Agreement of the results with the signal model was not checked, that is, if the eigenvalue was complex-valued or negative, we formally took its real part.

In our experiments, the noise variance σ^2 is relatively low for the signal to be reconstructed with good accuracy. However, based on the previous section, the accuracy of the parameter estimates is assumed to depend not only on σ^2 but on $\mu_{\min}$ also.

The simulation involving 1000 replicates confirmed this guess. Though the signal estimates are precise (not shown in Table 1), the parameters estimates are close to the original values only when $\mu_{\min}$ is several times larger than σ^2 , see error values in Table 1. Moreover, it turns out that the estimates of r_a and r_b are different in the sense that the value of $\mu_{\min}$ is crucial only for the former one, which corresponds to the faster changing exponential component.

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Table 1

Eight test sets of the parameters used in simulation and mean square deviations (MSD) for the parameters estimates. For better reading, the significantly large MSD values are highlighted in bold

α	β	r_a	r_b	c_a	c_b	μ_{min}	$MSD(\hat{r}_a)$ $\sigma = 0.1$	$MSD(\hat{r}_b)$ $\sigma = 0.1$	$MSD(\hat{r}_a)$ $\sigma = 0.01$	$MSD(\hat{r}_b)$ $\sigma = 0.01$
-0.01	0	0.990	1.000	1.57e-1	1.01e-1	5.27e-5	1.2e+0	5.6e-3	1.2e+0	4.9e-3
-0.03	-0.01	0.970	0.990	3.12e-1	1.59e-1	5.90e-4	1.3e+0	1.1e-2	4.2e-1	8.5e-3
-0.03	0	0.970	1.000	3.19e-1	1.04e-1	3.44e-3	1.2e+0	1.5e-2	6.4e-3	4.6e-3
-0.10	-0.03	0.905	0.970	9.70e-1	3.28e-1	1.07e-2	1.1e+0	3.1e-2	3.5e-3	2.1e-3
-0.03	+0.03	0.970	1.030	3.54e-1	1.87e-2	5.83e-2	4.0e-1	1.6e-2	1.3e-3	1.4e-3
-0.10	0	0.905	1.000	1.09e+0	1.20e-1	9.90e-2	9.0e-2	1.0e-2	1.4e-3	7.1e-4
-0.10	+0.03	0.905	1.030	1.21e+0	2.16e-2	3.80e-1	8.4e-3	5.2e-3	7.9e-4	5.4e-4
-0.10	+0.10	0.905	1.105	1.28e+0	7.09e-5	8.66e-1	6.3e-3	7.3e-3	6.0e-4	7.3e-4