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On the structure of solutions to a class of quasilinear elliptic Neumann problems. Part II

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Abstract

We continue our work (Y. Li, C. Zhao in *J Differ Equ* 212:208–233, 2005) to study the structure of positive solutions to the equation $\varepsilon^m \Delta_m u - u^{m-1} + f(u) = 0$ with homogeneous Neumann boundary condition in a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 2$). First, we study subcritical case for $2 < m < N$ and show that after passing by a sequence positive solutions go to a constant in $C^{1,\alpha}$ sense as $\varepsilon \rightarrow \infty$. Second, we study the critical case for $1 < m < N$ and prove that there is a uniform upper bound independent of $\varepsilon \in [1, \infty)$ for the least-energy solutions. Third, we show that in the critical case for $1 < m \leq 2$ the least energy solutions must be a constant if ε is sufficiently large and for $2 < m < N$ the least energy solutions go to a constant in $C^{1,\alpha}$ sense as $\varepsilon \rightarrow \infty$.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) be a smooth bounded domain. We continue our former work [8] to investigate the structure of positive solutions to a class of quasilinear elliptic Neumann problems as follows:

$$\begin{cases} \varepsilon^m \Delta_m u - u^{m-1} + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1) \quad (2) \quad (3)$$

where $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2} \nabla u)$, $1 < m < N$ and n is the unit outer normal of $\partial\Omega$. The coefficient $\varepsilon > 0$ is a parameter.

The problem (1)–(3) appears in the study of non-Newtonian fluids, chemotaxis, and biological pattern formation. For example, in the study of non-Newtonian fluids, the quantity m is a characteristic of the medium. Media with $m > 2$ are called dilatant fluids and those with $m < 2$ are called pseudoplastics. If $m = 2$, they are Newtonian fluids (see [2] and its bibliography). In this special case $m = 2$, (1)–(3) is also known as the stationary equation of the Keller–Segal system in chemotaxis (see [11]) or the limiting stationary equation of the so-called Gierer–Meinhardt system in biological pattern formation [19].

For the case $m = 2$, the asymptotic behavior of mountain-pass solutions to the system (1)–(3) as $\varepsilon \rightarrow 0^+$ was studied by Lin, Ni, and Takagi in a series of remarkable papers [11,13,14]. First, Lin et al. [11] applied the mountain-pass lemma [1] to show the existence of a

mountain-pass solution u_ε to (1)–(3). Under the assumption that $\frac{f(t)}{t}$ is increasing on $\mathbb{R}^+$ it was shown [14] that every mountain-pass solution is a least-energy solution, by which it is meant that u_ε has the least energy among all the solutions to (1)–(3), with the energy functional defined by

$$I_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - F(u_+) \right) dx,$$

where $u_+ = \max\{u, 0\}$ and $F(u_+) = \int_0^{u_+} f(s) ds$. They also gave some important preliminary results about the mountain-pass solutions. In [13,14], Ni and Takagi investigated the asymptotic behavior of the least-energy solution u_ε as $\varepsilon \rightarrow 0^+$ and showed that the least-energy solution has a single-spike which is achieved at a point on $\partial\Omega$ and moreover for $\varepsilon > 0$ sufficiently small this point stays at a point where maximum of mean curvature of $\partial\Omega$ is attained. For the general case $1 < m < N$, it was shown [9,10] that as $\varepsilon \rightarrow 0^+$ the global maximum point of the least-energy solution u_ε approaches a point where maximum of mean curvature of $\partial\Omega$ is attained at a rate of $o(\varepsilon)$. Also see [3–7] for existence and properties of multiple-peaks solutions for the case $m = 2$.

In our former work [8] we gave some a-priori estimates of mountain-pass solutions and investigated uniform upper bound and Harnack inequalities etc for all positive solutions of (1)–(3) under assumption $1 < m < N$ and the following assumptions on f .

(f₁): $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $f: (0, \infty) \rightarrow (0, \infty)$ is C^1 .

(f₂): $f(t) \equiv 0$ for $t \leq 0$ and $\frac{f(t)}{t^{m-1}} \rightarrow 0$ as $t \rightarrow 0^+$ and $f(t) > 0$ for $t > 0$.

(f₃): $\lim_{t \rightarrow \infty} \frac{f(t)}{t^p} = B > 0$ with $m - 1 < p < v - 1$ and $v = \frac{Nm}{N - m}$.

(f₄): Let $F(t) = \int_0^t f(s) ds$. There exists a constant $\theta > m$ such that $0 \leq \theta F(t) \leq tf(t)$ for $t \geq 0$.

Especially we proved the following results (see [11,12] for the case $m = 2$).

Theorem A

([8], Theorem 1.3, P. 210) Under assumptions (f₁)–(f₄), for any positive solution $u_\varepsilon \in W^{1,m}(\Omega)$ to (1)–(3), there exists a constant C independent of $\varepsilon \in (0, +\infty)$ such that

$$\sup_{x \in \Omega} u_\varepsilon(x) \leq C.$$

Moreover if $1 < m \leq 2$, there exists a positive constant ε_1 sufficiently large such that for $\varepsilon > \varepsilon_1$, any positive solution to the problem (1)–(3) that belongs to $W^{1,m}(\Omega)$ must be a constant.

In this paper we continue to investigate asymptotic behaviors of all solutions of (1)–(3) under assumptions of (f₁)–(f₄) with $2 < m < N$, and least-energy solutions of (1)–(3) with $1 <$

$m < N$ and $f(u) = u_+^{\frac{N(m-1)+m}{N-m}}$ (i.e., the critical case) as $\varepsilon \rightarrow \infty$. For $1 < m < N$ and $f(u) = u_+^{\frac{N(m-1)+m}{N-m}}$ associated with (1)–(3) is the functional $J_\varepsilon: W^{1,m}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_\varepsilon(v) = \frac{1}{m} \int_\Omega (\varepsilon^m |\nabla v|^m + |v|^m) dx - \frac{N-m}{Nm} \int_\Omega v_+^{\frac{Nm}{N-m}} dx. \quad (4)$$

Let

$$\Gamma_{\text{set}} = \left\{ h \in C([0, 1]; W^{1,m}(\Omega)) \mid h(0) = 0, h(1) = e \right\}, \quad (5)$$

where $e \neq 0$ is a nonnegative function in $W^{1,m}(\Omega)$ with $J_\varepsilon(e) \leq 0$ (e.g., a sufficiently large constant). Later we will see

$$c_\varepsilon = \inf_{h \in \Gamma_{\text{set}}} \max_{0 \leq t \leq 1} J_\varepsilon(h(t)) \quad (6)$$

is a positive critical value of J_ε .

Moreover, since $t^{\frac{N(m-1)+m}{N-m}} / t^{m-1} = t^{\frac{m^2}{N-m}}$ is increasing for $t > 0$ it is known [9] that c_ε can also be characterized as

$$c_\varepsilon = \inf \left\{ M_*[v] \mid v \in W^{1,m}(\Omega), v \neq 0 \text{ and } v \geq 0 \text{ in } \Omega \right\} \quad (7)$$

with

$$M_*[v] = \sup_{t \geq 0} J_\varepsilon(tv).$$

Hence c_ε is the least among all positive critical values of J_ε . Therefore we call such a critical point u_ε of J_ε with $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ a least-energy solution. Now our results can be stated as follows:

Theorem 1

Under assumptions $2 < m < N$ and (f_1) – (f_4) on f , for any positive solution $u_\varepsilon \in W^{1,m}$ with $\varepsilon \in [1, \infty)$ there exist two positive constants C independent of ε or u_ε and $\alpha = \alpha(m, N, \Omega) \in (0, 1)$ such that

$$\|u_\varepsilon - \bar{u}_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} \leq C\varepsilon^{-\beta},$$

where $\beta = \min \left\{ \frac{m^2}{N(m-2)}, \frac{m}{m-1} \right\}$ and $\bar{u}_\varepsilon$ is a positive constant defined as the average of u_ε over Ω i.e., $\bar{u}_\varepsilon = \frac{1}{|\Omega|} \int_\Omega u_\varepsilon dx$. Especially we have $\|u_\varepsilon - \bar{u}_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} \rightarrow 0$ as $\varepsilon \rightarrow \infty$. Moreover, after passing by a sequence $\bar{u}_{\varepsilon_k}$ converges to a positive zero of $t^{m-1} - f(t)$ as $\varepsilon \rightarrow \infty$.

Next is our result about uniform upper bound of the least-energy solution u_ε for $\varepsilon \in [1, \infty)$,

$$1 < m < N \text{ and } f(u) = u_+^{\frac{N(m-1)+m}{N-m}}.$$

Theorem 2

Suppose $1 < m < N$ and $\bar{u}_\varepsilon$. For $\varepsilon \in [1, \infty)$ there exists a constant $C > 0$ independent of ε such that any least-energy solution u_ε to (1)–(3) satisfies

$$\sup_{x \in \Omega} u_\varepsilon(x) \leq C.$$

For asymptotic behaviors of the least-energy solutions u_ε as $\varepsilon \rightarrow \infty$, we have

Theorem 3

Let $1 < m < N$, $f(u) = u_+^{\frac{N(m-1)+m}{N-m}}$ and u_ε be a least energy solution. Then we have

(f₁): For $1 < m \leq 2$ there exists a positive constant ε_1 sufficiently large such that for $\varepsilon > \varepsilon_1$, $u_\varepsilon \equiv 1$.

(f₂): For $2 < m < N$, $\|u_\varepsilon - 1\|_{C^{1,\alpha}(\bar{\Omega})} \rightarrow 0$ as $\varepsilon \rightarrow \infty$, where α is stated as in Theorem 1.

The organization of this paper is as follows. Some a-priori estimates and proof of Theorem 1 will be presented in Sect. 2. In Sect. 3 we will discuss existence of least energy solutions and prove Theorem 2. The proof of Theorem 3 will be given in Sect. 4.

2 Some a-priori estimates and proof of Theorem 1

In this section we will give some a-priori estimates related to u_ε and use them to prove Theorem 1. From now on C, C_i ($i = 0, 1, \dots$) are generic positive constants, we will specify them whenever it is necessary.

Let $u_\varepsilon \in W^{1,m}(\Omega)$ be a positive solution to (1)–(3) with $2 < m < N$ and assumptions (f₁)–(f₄)

on f . Then we know $u_\varepsilon \in C^1(\bar{\Omega})$ and u_ε is uniformly bounded due to Theorem A, i.e.,

$$\sup_{x \in \Omega} u_\varepsilon(x) \leq C. \quad (8)$$

Decompose u_ε as $u_\varepsilon = \bar{u}_\varepsilon + v_\varepsilon$, where

$$\bar{u}_\varepsilon = \frac{1}{|\Omega|} \int_{\Omega} u_\varepsilon dx \quad \text{and} \quad \int_{\Omega} v_\varepsilon dx = 0. \quad (9)$$

Then it follows from (1)–(3) that v_ε satisfies the following equations

$$\begin{cases} \varepsilon^m \Delta_m v_\varepsilon - u_\varepsilon^{m-1} + f(u_\varepsilon) = 0 & \text{in } \Omega, \\ \frac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (10)$$

Note that

$$\begin{aligned}
 u_{\varepsilon}^{m-1} - f(u_{\varepsilon}) - u_{\varepsilon}^{-m-1} + f(\bar{u}_{\varepsilon}) &= \left(\bar{u}_{\varepsilon} + v_{\varepsilon} \right)^{m-1} - u_{\varepsilon}^{-m-1} - f(\bar{u}_{\varepsilon} + v_{\varepsilon}) + f(\bar{u}_{\varepsilon}) \\
 &= \left[(m-1) \int_0^1 \left(\bar{u}_{\varepsilon} + tv_{\varepsilon} \right)^{m-2} dt - \int_0^1 f'(\bar{u}_{\varepsilon} + tv_{\varepsilon}) dt \right] v_{\varepsilon}.
 \end{aligned}$$

Then the first equation in (10) can be expressed as

$$\varepsilon^m \Delta_m v_{\varepsilon} - \left[(m-1) \int_0^1 \left(\bar{u}_{\varepsilon} + tv_{\varepsilon} \right)^{m-2} dt - \int_0^1 f'(\bar{u}_{\varepsilon} + tv_{\varepsilon}) dt \right] v_{\varepsilon} = u_{\varepsilon}^{-m-1} - f(\bar{u}_{\varepsilon}). \quad (11)$$

Multiplying both sides of (11) by v_{ε} , integrating over Ω and using integration by parts we have

$$\varepsilon^m \int_{\Omega} |\nabla v_{\varepsilon}|^m dx + \int_{\Omega} \left[(m-1) \int_0^1 \left(\bar{u}_{\varepsilon} + tv_{\varepsilon} \right)^{m-2} dt - \int_0^1 f'(\bar{u}_{\varepsilon} + tv_{\varepsilon}) dt \right] v_{\varepsilon}^2 dx = 0.$$

Since $0 < \bar{u}_{\varepsilon} + tv_{\varepsilon} \leq 2 \max_{\Omega} u_{\varepsilon}(x) \leq C$ and $\lim_{t \rightarrow 0^+} \frac{f'(t)}{t^{m-2}} = 0$ for $m > 2$ it follows that

$$\left| \left((m-1) \left(\bar{u}_{\varepsilon} + tv_{\varepsilon} \right)^{m-2} - f'(\bar{u}_{\varepsilon} + tv_{\varepsilon}) \right) \right| \leq C \quad \text{for } (x, t) \in \bar{\Omega} \times [0, 1],$$

where $(v)_{-} = \min\{v, 0\}$. Thus we get

$$\varepsilon^m \int_{\Omega} |\nabla v_{\varepsilon}|^m dx \leq C \int_{\Omega} v_{\varepsilon}^2 dx. \quad (12)$$

On the other hand, (9) implies that the following Poincaré inequality holds for v_{ε} with some positive constant γ :

$$\int_{\Omega} |\nabla v_{\varepsilon}|^m dx \geq \gamma^m \int_{\Omega} |v_{\varepsilon}|^m dx.$$

. Thus by Hölder inequality we have

$$(\varepsilon \gamma)^m \int_{\Omega} |v_{\varepsilon}|^m dx \leq C \left(\int_{\Omega} |v_{\varepsilon}|^m dx \right)^{\frac{2}{m}},$$

which yields

$$\|v_{\varepsilon}\|_{L^m(\Omega)} \leq C \varepsilon^{-\frac{m}{m-2}}.$$

Therefore by interpolation and the fact that $\|v_{\varepsilon}\|_{L^{\infty}(\Omega)} \leq C$ it follows that for all $q \geq m$

$$\|v_{\varepsilon}\|_{L^q(\Omega)} \leq C \varepsilon^{-\frac{m^2}{q(m-2)}}. \quad (13)$$

Especially we have

$$\| |v_\varepsilon|^{m-2} v_\varepsilon \|_{L^{\frac{N}{m-1}}(\Omega)} \leq C \varepsilon^{-\frac{m^2(m-1)}{N(m-2)}}. \quad (14)$$

Lemma 1

The following estimate holds for v_ε :

$$\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C \varepsilon^{-\beta},$$

where $\beta = \min \left\{ \frac{m^2}{N(m-2)}, \frac{m}{m-1} \right\}$ and C is a positive constant depending only on N, m, Ω and the uniform upper bound as stated in Theorem A.

Proof We can rewrite (10) as follows.

$$\begin{cases} \Delta_m v_\varepsilon - |v_\varepsilon|^{m-2} v_\varepsilon = -|v_\varepsilon|^{m-2} v_\varepsilon + \frac{1}{\varepsilon^m} (u_\varepsilon^{m-1} - f(u_\varepsilon)) & \text{in } \Omega, \\ \frac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (15)$$

For convenience, let $g = -|v_\varepsilon|^{m-2} v_\varepsilon + \frac{1}{\varepsilon^m} (u_\varepsilon^{m-1} - f(u_\varepsilon))$. It is easy to check $g \in L^{\frac{N}{m-1}}(\Omega)$ due to (14) and uniform boundedness of u_ε as stated in Theorem A. Without loss of generality we

assume $\|g\|_{L^{\frac{N}{m-1}}(\Omega)} \neq 0$, or else the lemma holds trivially. Let $\tilde{v}_\varepsilon = v_\varepsilon / \|g\|_{L^{\frac{N}{m-1}}(\Omega)}^{\frac{1}{m-1}}$ and $\tilde{g} = g / \|g\|_{L^{\frac{N}{m-1}}(\Omega)}$. We can find that $\tilde{v}_\varepsilon$ satisfies the following equations

$$\begin{cases} \Delta_m \tilde{v}_\varepsilon - |\tilde{v}_\varepsilon|^{m-2} \tilde{v}_\varepsilon = \tilde{g} & \text{in } \Omega, \\ \frac{\partial \tilde{v}_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (16)$$

Next we use Moser iteration method to prove this lemma. For convenience, write v instead of $\tilde{v}_\varepsilon$.

If we multiply both sides of the first equation in (16) by $|v|^{m(s-1)} v$ ($s \geq 1$) and integrate over Ω , then by virtue of the homogeneous Neumann boundary condition we have

$$s^{-m} [m(s-1)+1] \int_\Omega |\nabla |v|^s|^m dx + \int_\Omega |v|^{ms} dx = - \int_\Omega \tilde{g} |v|^{m(s-1)} v dx. \quad (17)$$

Since $s^{-m} [m(s-1)+1] \leq m+1$ for $s \geq 1$ and $\|\tilde{g}\|_{L^{\frac{N}{m-1}}(\Omega)} = 1$ and we obtain from (17) that

$$s^{-m} [m(s-1)+1] \int_\Omega (|\nabla |v|^s|^m + |v|^{ms}) dx \leq (m+1) \left(\int_\Omega |v|^{\frac{[m(s-1)+1]N}{N-m+1}} dx \right)^{\frac{N-m+1}{N}}. \quad (18)$$

Recall the Sobolev embedding theorem

$$\left(\int_{\Omega}|w|^v dx\right)^{\frac{m}{v}} \leq \Gamma^m \int_{\Omega} (|\nabla w|^m + |w|^m) dx \quad (19)$$

for $w \in W^{1,m}(\Omega)$, where $\Gamma > 0$ is a constant depending on Ω .

It follows from (18) that

$$s^{-m} [m(s-1)+1] \Gamma^{-m} \left(\int_{\Omega}|v|^{vs} dx\right)^{\frac{m}{v}} \leq (m+1) \left(\int_{\Omega}|v|^{\frac{m(s-1)+1}{N-m+1}} dx\right)^{\frac{N-m+1}{N}}.$$

Noting that $s^{-m}[m(s-1)+1] \geq s^{-(m-1)}$ for $s \geq 1$ thus we have

$$\left(\int_{\Omega}|v|^{vs} dx\right)^{\frac{mN}{v(N-m+1)}} \leq ((m+1)) \Gamma^m s^{m-1} \int_{\Omega}|v|^{\frac{m(s-1)+1}{N-m+1}} dx \quad (20)$$

for $s \geq 1$. Here we remark that if we take $s = 1$ in (17) and take care of $\|\tilde{g}\|_{L^{\frac{N}{N-m+1}}(\Omega)} = 1$ we have

$$\int_{\Omega} (|\nabla v|^m + |v|^m) dx \leq |\Omega|^{\frac{(m-1)(N-m)+m}{Nm}} \|v\|_{L^v(\Omega)},$$

which yields from the Sobolev embedding theorem that

$$\int_{\Omega}|v|^v dx \leq C_0 \quad (21)$$

$$\text{with } C_0 = \left(\Gamma^m |\Omega|^{\frac{(m-1)(N-m)+m}{Nm}} \right)^{\frac{v}{m-1}}.$$

Next we define two sequences $\{s_j\}$ and $\{M_j\}$ by

$$\begin{aligned} [m(s_0-1)+1]N/(N-m+1) &= v, \\ [m(s_{j+1}-1)+1]N/(N-m+1) &= vs_j \quad \text{for } j=0, 1, 2, \dots, \end{aligned} \quad (22)$$

$$\begin{aligned} M_0 &= C_0, \\ M_{j+1} &= \left([(m+1)\Gamma^m s_j^{m-1}]^{\frac{N}{N-m+1}} M_j \right)^{\frac{v(N-m+1)}{mN}} \quad \text{for } j=0, 1, 2, \dots, \end{aligned} \quad (23)$$

We note that s_j can be explicitly given by

$$s_j = \left(\frac{1}{N-m} + \frac{(m-1)(N-m)+m}{m} \right) \left[\left(\frac{N-m+1}{N-m} \right)^j - 1 \right] + \frac{N-m+1}{N-m} \quad (24)$$

for $j = 0, 1, 2, \dots$, and it follows that $s_j > 1$ for all $j \geq 0$ and $s_j \rightarrow \infty$ as $j \rightarrow \infty$.

We shall show that

$$\int_{\Omega} |v|^{\frac{[m(s_{j-1})+1]N}{N-m+1}} dx \leq M_j \quad \text{for } j \geq 0 \quad (25)$$

and

$$M_j \leq e^{Ks_{j-1}} \quad (26)$$

for some constant $K > 0$.

Clearly (25) holds for $j = 0$ due to (21). Suppose that we have proved (25) for $j \geq 0$. Then by (20) we have

$$\begin{aligned} \int_{\Omega} |v|^{\frac{[m(s_{j+1})+1]N}{N-m+1}} dx &\leq \left([(m+1)\Gamma^m s_j^{m-1}]^{\frac{N}{N-m+1}} \int_{\Omega} |v|^{\frac{[m(s_{j-1})+1]N}{N-m+1}} dx \right)^{\frac{v(N-m+1)}{mN}} \\ &\leq \left([(m+1)\Gamma^m s_j^{m-1}]^{\frac{N}{N-m+1}} M_j \right)^{\frac{v(N-m+1)}{mN}} \\ &= M_{j+1}, \end{aligned}$$

so that (25) is true also for $j + 1$. Therefore it remains to show (26).

Put $\sigma_j = \frac{v}{m} \log((m+1)\Gamma^m s_j^{m-1})$ and $\mu_j = \log M_j$. Hence

$$\mu_{j+1} = \frac{N-m+1}{N-m} \mu_j + \sigma_j.$$

Then from (24) it follows that

$$\begin{aligned} \sigma_j &= \frac{v}{m} \log((m+1)\Gamma^m) \\ &\quad + \frac{v(m-1)}{m} \log\left(\left(\frac{1}{N-m} + \frac{(m-1)(N-m)+m}{m}\right)\right. \\ &\quad \times \left.\left[\left(\frac{N-m+1}{N-m}\right)^j - 1\right] + \frac{N-m+1}{N-m}\right) \end{aligned}$$

and therefore we can find a constant C^* such that

$$\sigma_j \leq C^*(j+1)$$

for all $j \geq 0$. We now define $\{\tau_j\}$ by $\tau_0 = \mu_0$ and

$$\tau_{j+1} = \frac{N-m+1}{N-m} \tau_j + C^*(j+1) \quad \text{for } j \geq 0.$$

Clearly $\mu_j \leq \tau_j$ for all $j \geq 0$. Moreover

$$\tau_j = \left(\frac{N-m+1}{N-m} \right)^j [\mu_0 + C^* (N-m)(N-m+1)] - C^* (j+N-m+1)(N-m),$$

we know there is a constant $K > 0$ such that

$$\tau_j \leq K s_{j-1}$$

and then we obtain (26). Here we note that K depends only on μ_0, m, N and C^* , whereas μ_0 depends only on Γ, m, N and $|\Omega|$, and C^* depends only on m, N and Γ . Thus (25) and (26) are true. So (22), (25) and (26) tell us that

$$\|v\|_{L^{s_{j-1}}(\Omega)}^{s_{j-1}} \leq e^{K s_{j-1}},$$

and hence letting $j \rightarrow \infty$ we obtain

$$\|v\|_{L^\infty(\Omega)} \leq e^{K/v}.$$

Note that v stands for $\bar{v}_\varepsilon$. Therefore till now we have shown that there is a constant C depends only on m, N , and Ω such that

$$\|\bar{v}_\varepsilon\|_{L^\infty(\Omega)} \leq C.$$

From the relation $\bar{v}_\varepsilon = v_\varepsilon / \|g\|_{L^{\frac{N}{m-1}}(\Omega)}^{\frac{1}{m-1}}$ it follows that

$$\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C \|g\|_{L^{\frac{N}{m-1}}(\Omega)}^{\frac{1}{m-1}}. \quad (27)$$

Note that (14) and Theorem A tell us that

$$\begin{aligned} \|g\|_{L^{\frac{N}{m-1}}(\Omega)} &\leq \| |v_\varepsilon|^{m-2} \|_{L^{\frac{N}{m-1}}(\Omega)} + \left\| \frac{1}{\varepsilon^m} (u_\varepsilon^{m-1} - f(u_\varepsilon)) \right\|_{L^{\frac{N}{m-1}}(\Omega)} \\ &\leq C \varepsilon^{-\frac{m^2(m-1)}{N(m-2)}} + C \varepsilon^{-m} |\Omega|^{\frac{m-1}{N}} \\ &\leq C \varepsilon^{-\tilde{\beta}} \end{aligned}$$

with $\tilde{\beta} = \min \left\{ \frac{m^2}{N(m-2)}, m \right\}$. Thus it follows from (27) that

$$\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C \varepsilon^{-\beta}$$

with $\beta = \frac{\tilde{\beta}}{m-1} = \min \left\{ \frac{m^2}{N(m-2)}, \frac{m}{m-1} \right\}$. Proof of Lemma 1 is complete.

Next is a lemma related to $C^{1,\alpha}$ -estimate of v_ε .

Lemma 2

Let $w \in W^{1,m}(\Omega) \cap L^\infty(\Omega)$ ($1 < m < \infty$) be a weak solution to the following problem:

$$\begin{cases} \Delta_m w = h & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (28)$$

Then there exist two constants $C > 0$ and $\alpha \in (0, 1)$ depending only on m, N and Ω such that

$$\|w\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \left(\|w\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)}^{\frac{1}{m-1}} \right).$$

Proof In [16] the author proved the follows: Suppose $w \in W^{1,m}(B_1(0)) \cap L^\infty(B_1(0))$ (hereafter $B_R(x)$ denotes the ball centered at x with radius R) is a solution of the differential equation

$$\int_{B_1(0)} \sum_{j=1}^N \{a_j(x, w, \nabla w) \cdot \psi_{x_j}\} - h \psi dx = 0, \quad \forall \psi \in C_c^\infty(B_1(0))$$

with $\|w\|_{L^\infty(B_1(0))}, \|h\|_{L^\infty(B_1(0))} \leq 1$ and for $j = 1, \dots, N$,

$$a_j \in C^0(B_1(0) \times \mathbb{R} \times \mathbb{R}^N) \cap C^1(B_1(0) \times \mathbb{R} \times (\mathbb{R}^N - \{0\}))$$

and satisfies the following ellipticity and growth conditions

$$\begin{aligned} a_j(x, \mu, 0) &= 0, \\ \sum_{i,j=1}^N \frac{\partial a_j}{\partial \eta_i}(x, \mu, \eta) \cdot \xi_i \xi_j &\geq \gamma_1 |\eta|^{m-2} \cdot |\xi|^2, \\ \sum_{i,j=1}^N \left| \frac{\partial a_j}{\partial \eta_i}(x, \mu, \eta) \right| &\leq \Gamma_1 |\eta|^{m-2}, \\ \sum_{i,j=1}^N \left| \frac{\partial a_j}{\partial x_i}(x, \mu, \eta) \right| + \left| \frac{\partial a_j}{\partial \mu}(x, \mu, \eta) \right| &\leq \Gamma_1 |\eta|^{m-1}, \end{aligned} \quad (29)$$

for some positive constants γ_1, Γ_1 and $x \in B_1(0), \mu \in \mathbb{R}, \eta \in \mathbb{R}^N - \{0\}$ and $\xi \in \mathbb{R}^N$. Then there exist constants $\alpha \in (0, 1), C$ depending only on m, N, γ_1 and Γ_1 such that

$$\|w\|_{C^{1,\alpha}(\bar{B}_{\frac{1}{2}}(0))} \leq C.$$

Here we remark that the proof also works well if $C^1(B_1(0) \times \mathbb{R} \times (\mathbb{R}^N - \{0\}))$ is replaced by locally Lipschitz for $(x, \mu, \eta) \in B_1(0) \times \mathbb{R} \times (\mathbb{R}^N - \{0\})$.

For any compact set $K_\Omega \subset \subset \Omega$ and $x_0 \in K_\Omega$ let $\rho = \min \{\text{dist}(K_\Omega, \partial\Omega), 1\}$. Applying above interior estimate to

$$\tilde{w}(x) = \frac{w(x_0 + \rho x)}{\left(\|w\|_{L^\infty(\Omega)} + \|\rho^m h\|_{L^\infty(\Omega)}^{\frac{1}{(m-1)}} \right)}$$

and using standard covering argument we have a constant $\alpha_* \in (0, 1)$ depending only on m and N such that

$$\|\tilde{w}\|_{C^{1,\alpha_*}(K_\Omega)} \leq C$$

with C depending only on m, N, K_Ω and ρ , which tells us that

$$\|\tilde{w}\|_{C^{1,\alpha_*}(K_\Omega)} \leq C(m, N, K_\Omega, \rho) \left(\|w\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)}^{\frac{1}{(m-1)}} \right). \quad (30)$$

For the boundary regularity we can use the local reflection method. For $P \in \partial\Omega$, without loss of generality, we may assume that P is the coordinate origin and the x_N -axis is normal to $\partial\Omega$ at P , hence there exists a smooth function $h_*(x')$, $x' = (x_1, \dots, x_{N-1})$ defined for $|x'| < \delta$ satisfying

$$\begin{aligned} (f_1): \quad & h_*(0) = 0 \text{ and } \frac{\partial h_*}{\partial x_j}(0) = 0 \text{ for } j = 1, \dots, N-1 \text{ and} \\ (f_2): \quad & \Omega \cap \mathcal{B} = \left\{ (x', x_N) \mid x_N > h_*(x') \right\} \text{ and } \partial\Omega \cap \mathcal{B} = \left\{ (x', x_N) \mid x_N = h_*(x') \right\} \text{ in a neighborhood } \mathcal{B} \text{ of } P. \end{aligned}$$

For $y \in \mathbb{R}^N$ and $|y|$ sufficiently small, we define a mapping straightening the boundary portion around P as in [11]: $x = \Phi(y) = (\phi_1(y), \dots, \phi_N(y))$ by

$$\begin{aligned} \phi_j(y) &= y_j - y_N \left(\frac{\partial h_*}{\partial y_j}(y') \right) \quad \text{for } j = 1, \dots, N-1, \\ \phi_N(y) &= y_N + h_*(y'). \end{aligned}$$

Since in virtue of property f_1 the differential map $D\Phi$ of Φ satisfies $D\Phi(0) = I_{N \times N}$, the identity matrix, we can assume Φ is defined on $B_\kappa(0)$ for some sufficiently small constant $\kappa > 0$ such that

$$\|D\Phi(y) - I_{N \times N}\| \leq \frac{1}{2} \quad (31)$$

for all $y \in B_\kappa(0)$ where $\|A\| = \max_{\|x\|=1} \|Ax\|$ for a $N \times N$ matrix A . Then Φ has an inverse mapping $y = \Psi(x) = \Phi^{-1}(x)$ in the neighborhood $(B_\kappa(0))$ of $x = 0$. We write $\Psi(x) = (\psi(x), \dots, \psi_N(x))$. Set $\widehat{w}(y) = w(\Phi(y))$ and

$$\bar{w}(y', y_N) = \begin{cases} \bar{w}(y', y_N) & \text{for } (y', y_N) \in B_k^+(0), \\ \bar{w}(y', -y_N) & \text{for } (y', y_N) \in B_k^-(0). \end{cases} \quad (32)$$

Let

$$g_{ij} = \sum_{k=1}^N \frac{\partial \phi_k}{\partial y_i} \frac{\partial \phi_k}{\partial y_j}(y), \quad g^{ij} = \sum_{k=1}^N \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_j}{\partial x_k}(\Phi(y)).$$

Here we note that $g_{iN|B_k(0) \cap \{y_N=0\}} = 0$ for $i = 1, \dots, N-1$. Next denote

$$\bar{g}_{ij}(y', y_N) = \begin{cases} g_{ij}(y', y_N) & \text{for } (y', y_N) \in B_k^+(0), \\ (-1)^{\delta_{iN} + \delta_{jN}} g_{ij}(y', -y_N) & \text{for } (y', y_N) \in B_k^-(0), \end{cases}$$

$$\bar{g}^{ij}(y', y_N) = \begin{cases} g^{ij}(y', y_N) & \text{for } (y', y_N) \in B_k^+(0), \\ (-1)^{\delta_{iN} + \delta_{jN}} g^{ij}(y', -y_N) & \text{for } (y', y_N) \in B_k^-(0), \end{cases}$$

where δ_{ij} is the Kronecker symbol. Denote $g_*(y) = \det(\bar{g}_{ij}(y))$. Then from (31) we know

$$\sqrt{g_*(y)} \leq C(N), \quad \forall y \in B_k(0) \quad (33)$$

for some absolute constant $C(N) > 0$. Then $\tilde{w} \in W^{1,m}(B_k(0)) \cap L^\infty(B_k(0))$ satisfies

$$\begin{cases} \sum_{j=1}^N \frac{\partial}{\partial y_j} a_j(y, \nabla \tilde{w}) = \sqrt{g_*}(h \circ \Phi) & \text{in } B_k(0), \\ \frac{\partial \tilde{w}}{\partial y_N} = 0 & \text{on } B_k(0) \cap \{y_N=0\} \end{cases} \quad (34)$$

with

$$a_j(y, \nabla \tilde{w}) = \sum_{i=1}^N \left[\sum_{s,l} g^{-sl} \frac{\partial \tilde{w}}{\partial y_s} \frac{\partial \tilde{w}}{\partial y_l} \right]^{\frac{m-2}{2}} \sqrt{g_*} g^{-ji} \frac{\partial \tilde{w}}{\partial y_i}, \quad \text{for } j=1, \dots, N. \quad (35)$$

It is not hard to check that $a_j(y, \eta)$ ($j = 1, \dots, N$) satisfies all inequalities in (29) with some positive constants γ_1 depending only on m and N due to (31) and (33) and Γ_1 depending only

on $m, N, D^2\Phi_i$ and $D^2\psi_i$ ($i = 1, \dots, N$) via $\frac{\partial g_{sl}}{\partial y_j}$ and $\frac{\partial g^{-sl}}{\partial y_j}(s, l, j = 1, \dots, N)$ (hence Γ_1 depends only on m, N and Ω).

Applying the interior estimate to

$$\tilde{w}(y) = \frac{\bar{w}(\kappa y)}{\|\bar{w}\|_{L^\infty(B_\kappa(0))} + \|\kappa^m \sqrt{g_*}(h \circ \Phi)\|_{L^\infty(B_\kappa(0))}^{\frac{1}{m-1}}}$$

we know there exist two constants $\alpha_0 \in (0, 1)$ and C depending only on m, N and Ω such that

$$\|\tilde{w}\|_{C^{1,\alpha_0}\left(B_{\frac{1}{2}}(0)\right)} \leq C,$$

which tells us that

$$\|\bar{w}\|_{C^{1,\alpha_0}\left(\Phi\left(B_{\frac{\kappa}{2}}(0)\right)\right)} \leq C(m, N, \kappa, \Omega) \left(\|w\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}^{\frac{1}{m-1}} \right). \quad (36)$$

Especially, we have

$$\|w\|_{C^{1,\alpha_0}\left(\Phi\left(B_{\frac{\kappa}{2}}(0)\right)\right)} \leq C(m, N, \kappa, \Omega) \left(\|w\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}^{\frac{1}{m-1}} \right). \quad (37)$$

Since $\partial\Omega$ is compact Lemma 2 follows from standard covering argument to patch the interior estimate (30) and the boundary estimate (37) together. Proof of Lemma 2 is complete.

With helps of Lemma 1 and Lemma 2 we can give proof of Theorem 1 as follows.

Proof of Theorem 1 It follows from (10), Lemma 1 and Lemma 2 that there exist a positive constant $\alpha \in (0, 1)$ depending only on m, N, Ω such that

$$\begin{aligned} \|v_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} &\leq C \left(\|v_\varepsilon\|_{L^\infty(\Omega)} + \left\| \frac{1}{\varepsilon^m} (u_\varepsilon^{m-1} - f(u_\varepsilon)) \right\|_{L^\infty(\Omega)}^{\frac{1}{m-1}} \right) \\ &\leq C \left(\varepsilon^{-\beta} + \varepsilon^{-\frac{m}{m-1}} \right) \leq C \varepsilon^{-\beta}, \end{aligned} \quad (38)$$

with $\beta = \min \left\{ \frac{m^2}{N(m-2)}, \frac{m}{m-1} \right\}$ as defined in Lemma 1. that we have used uniform boundedness of u_ε as stated in Theorem A. Therefore the constant C depends only on m, N, Ω, f and the uniform bound of u_ε as stated in Theorem A.

Let $Z_f = \min\{t > 0 \mid t^{m-1} = f(t)\}$. It follows from assumption (f₂) that $Z_f > 0$. The maximum principle tells us that $\sup_\Omega u_\varepsilon \geq Z_f$ ([8]). Thus it follows from (38) and Theorem A that for sufficiently large ε ,

$$\frac{1}{2} Z_f \leq \bar{u}_\varepsilon \leq C, \quad (39)$$

which implies there exist a sequence $\{\varepsilon_j\}$ with $\lim_{j \rightarrow \infty} \varepsilon_j = \infty$ and a constant $u_c > 0$ such that $\lim_{j \rightarrow \infty} \bar{u}_{\varepsilon_j} = u_c$. Note that from (1)–(3) it follows that

$$\varepsilon_j^m \int_{\Omega} |\nabla u_{\varepsilon_j}|^m dx = \int_{\Omega} u_{\varepsilon_j} \left(f(u_{\varepsilon_j}) - u_{\varepsilon_j}^{m-1} \right) dx. \quad (40)$$

Note that (12) and (38) imply

$$\varepsilon_j^m \int_{\Omega} |\nabla u_{\varepsilon_j}|^m dx = \varepsilon_j^m \int_{\Omega} |\nabla v_{\varepsilon_j}|^m dx \leq C \int_{\Omega} v_{\varepsilon_j}^2 dx \leq C \varepsilon_j^{-2\beta} |\Omega|. \quad (41)$$

Taking limit at both sides of (40) as $j \rightarrow \infty$ and using (38) we obtain

$$\int_{\Omega} u_c \left(f(u_c) - u_c^{m-1} \right) dx = 0$$

which implies $f(u_c) - u_c^{m-1} = 0$, i.e., the constant u_c is a positive zero of $t^{m-1} - f(t)$. Proof of Theorem 1 is complete.

3 Uniform upper bound for least energy solutions of the critical case

In this section we first discuss the existence of least energy solutions of the critical case

$f(u) = u_+^{\frac{N(m-1)+m}{N-m}}$ ($1 < m < N$) and then show that for $\varepsilon \in [1, \infty)$ there is an upper bound independent of ε (hence uniform) for the least energy solutions.

Note that for this critical case the problem (1)–(3) is as follows.

$$\begin{cases} \varepsilon^m \lambda_m u - u^{m-1} + u^{\frac{N(m-1)+m}{N-m}} = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (42) \quad (43) \quad (44)$$

Let $u = \varepsilon^{\frac{N-m}{m}} v$. Then v solves the following problem:

$$\begin{cases} \Delta_m v - \frac{1}{\varepsilon^m} v^{m-1} + v^{\frac{N(m-1)+m}{N-m}} = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (45) \quad (46) \quad (47)$$

Associated with (45)–(47) is functional $J^\varepsilon: W^{1,m}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J^\varepsilon(v) = \frac{1}{m} \int_{\Omega} \left(|\nabla v|^m + \frac{1}{\varepsilon^m} |v|^m \right) dx - \frac{N-m}{Nm} \int_{\Omega} v_+^N dx. \quad (48)$$

Recall that $J_\varepsilon: W^{1,m}(\Omega) \rightarrow \mathbb{R}$ was defined by (4) and also note that for $u = \varepsilon^{\frac{N-m}{m}} v \in W^{1,m}(\Omega)$ we have

$$J^\varepsilon(u) = \varepsilon^N J^\varepsilon(v). \quad (49)$$

Thus existence of least energy solutions of (45)–(47) implies the existence of least energy solution of (42)–(44).

Recall Γ_{set} defined by (5) and let

$$\tilde{c}_\varepsilon = \inf_{h \in \Gamma_{set}} \max_{0 \leq t \leq 1} J^\varepsilon(h(t)). \quad (50)$$

The following result was proved in [18] (see also [15]) for $1 < m < N$ and also see [17] for $m = 2$.

Lemma 3 $0 < \tilde{c}_\varepsilon \leq \frac{1}{2N} S_m^{\frac{N}{m}}$ is a critical value of the functional J^ε defined by (48), where

$$S_m = \inf_{w \in C_c^\infty(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla w|^m dx : \int_{\mathbb{R}^N} |w|^v = 1 \right\}$$

is the best Sobolev constant for the embedding $\mathbb{D}^{1,m}(\mathbb{R}^N) \hookrightarrow L^v(\mathbb{R}^N)$, and

$$\mathbb{D}^{1,m}(\mathbb{R}^N) = \left\{ \nabla w \in L^m(\mathbb{R}^N) : w \in L^v(\mathbb{R}^N) \right\} \text{ with norm } \left(\int_{\mathbb{R}^N} |\nabla w|^m dx \right)^{\frac{1}{m}}.$$

From above discussions we know there is a least energy solution u_ε to the problem (42)–(44) with

$$J_\varepsilon(u_\varepsilon) \leq \frac{1}{2N} \varepsilon^N S_m^{\frac{N}{m}}. \quad (51)$$

Note that due to the characterization (7) we also know

$$J_\varepsilon(u_\varepsilon) \leq M_*[1] = \frac{1}{N} |\Omega|. \quad (52)$$

Therefore (51) and (52) together imply

$$J_\varepsilon(u_\varepsilon) = \frac{1}{N} \int_{\Omega} u_\varepsilon^v dx \leq C_1. \quad (53)$$

with $C_1 = \min \left\{ \frac{1}{2N} \varepsilon^N S_m^{\frac{N}{m}}, \frac{1}{N} |\Omega| \right\}$. Also note that

$$\int_{\Omega} (\varepsilon^m |\nabla u_\varepsilon|^m + u_\varepsilon^m) dx = \int_{\Omega} u_\varepsilon^v dx \leq C_1 N. \quad (54)$$

Next we give proof of Theorem 2.

Proof of Theorem 2 We need several steps to prove this theorem.

Claim 1

There exists a constant $C_* > 0$ such that

$$\sup_{\Omega} u_{\varepsilon} \leq C_* \varepsilon^{\frac{N-m}{m}}. \quad (55)$$

Suppose not. Then there exist a sequence $\{\varepsilon_k\}$ ($k \geq 1$), a sequence of constants $\{c_k\}$ and a sequence of points $\{P_k\} \subset \bar{\Omega}$ with $\varepsilon_k, C_k \rightarrow \infty$ such that

$$\tilde{M}_k = \sup_{\Omega} u_{\varepsilon_k} = u_{\varepsilon_k}(P_k) \leq c_k \varepsilon_k^{\frac{N-m}{m}}. \quad (56)$$

Since $\bar{\Omega}$ is compact, passing by a subsequence if it is necessary we can assume $P_k \rightarrow P_0 \in \bar{\Omega}$. For convenience, we write u_k instead of u_{ε_k} .

Case 1

$P_0 \in \partial\Omega$. Without loss of generality we may assume that P_0 is the coordinate origin and the x_N -axis is normal to $\partial\Omega$ at P_0 . As in the proof of Lemma 2 we introduce a mapping Φ straightening the boundary portion around P_0 which is defined on $B_\kappa(0)$ for some constant $\kappa > 0$. Let $\bar{u}_k$ be defined as in the proof of Lemma 2 with w replaced by u_k . We know $\bar{u}_k \in W^{1,m}(B_\kappa(0))$ satisfies

$$\begin{cases} \varepsilon^m \sum_{j=1}^N \frac{\partial}{\partial y_j} a_j(y, \nabla \bar{u}_k) = \sqrt{g_*} \left(u_k^{m-1} - \bar{u}_k^{\frac{N(m-1)+m}{N-m}} \right) & \text{in } B_\kappa(0) \\ \frac{\partial \bar{u}_k}{\partial y_N} = 0 & \text{on } B_\kappa(0) \cap \{y_N = 0\} \end{cases} \quad (57)$$

with

$$a_j(y, \nabla \bar{u}_k) = \sum_{i=1}^N \left[\sum_{s,l} g^{sl} \frac{\partial \bar{u}_k}{\partial y_s} \frac{\partial \bar{u}_k}{\partial y_l} \right]^{\frac{m-2}{2}} \sqrt{g_*} g^{-ji} \frac{\partial \bar{u}_k}{\partial y_i}, \quad (58)$$

for $j = 1, \dots, N$, where g^{sl} is defined as in the proof of Lemma 2.

Put $Q_k = \Psi(P_k)$ with Ψ being the inverse of Φ as defined in the proof of Lemma 2 and write $Q_k = (q'_k, \alpha_k)$, $\alpha_k \geq 0$. Since $Q_k \rightarrow 0$ as $k \rightarrow \infty$ we may assume that $|Q_k| \leq \frac{\kappa}{2}$ for all k with κ being chosen as in the proof of Lemma 2.

Let $\lambda_k = \varepsilon_k / \tilde{M}_k^{\frac{m}{N-m}}$. From (56) it follows that $\lambda_k \leq 1 / c_k^{\frac{m}{N-m}} \rightarrow 0$ as $k \rightarrow \infty$. Then the proof can be divided to treat two cases according to the behavior of $\frac{\alpha_k}{\lambda_k}$ as $k \rightarrow \infty$.

Subcase (i). The sequence $\left\{ \frac{\alpha_k}{\lambda_k} \right\}$ remain bounded. By passing to a subsequence if it is necessary we may assume

$$\frac{\alpha_k}{\lambda_k} \rightarrow \beta_* \geq 0 \quad \text{as } k \rightarrow \infty.$$

Define a scaled function by

$$v_k(z) = \tilde{M}_k^{-1} u_k(\lambda_k z' + q'_k, \lambda_k z_N).$$

Note that v_k is well-defined in the ball $B_{\frac{\kappa}{\lambda_k}}$ and that $0 < v_k(z) \leq 1$ for all k . From (57) we see that v_k satisfies

$$\begin{cases} \sum_{j=1}^N \frac{\partial}{\partial z_j} \tilde{a}_j(z, \nabla v_k) = \sqrt{g_*} \left(\tilde{M}_k^{m-\nu} v_k^{m-1} - v_k^{\frac{N(m-1)+m}{N-m}} \right) & \text{in } B_{\frac{\kappa}{\lambda_k}}(0), \\ \frac{\partial v_k}{\partial z_N} = 0 & \text{on } B_{\frac{\kappa}{\lambda_k}}(0) \cap \{z_N = 0\} \end{cases} \quad (59)$$

with

$$\tilde{a}_j(z, \nabla v_k) = \sum_{i=1}^N \left[\sum_{s,l} \tilde{g}^{sl}(z) \frac{\partial v_k}{\partial z_s} \frac{\partial v_k}{\partial z_l} \right]^{\frac{m-2}{2}} \sqrt{\tilde{g}_*(z)} \tilde{g}^{ji}(z) \frac{\partial v_k}{\partial z_i}, \quad (60)$$

for $j = 1, \dots, N$, where $\tilde{g}^{sl}(z) = \tilde{g}^{-sl}(\lambda_k z' + q'_k, \lambda_k z_N)$, $\tilde{g}_*(z) = g_*(\lambda_k z' + q'_k, \lambda_k z_N)$ for $s, l = 1, \dots, N$.

Note that $\tilde{a}_j(z, \nabla v_k)$ satisfies the structure conditions (29) with two positive constants $\gamma_1 = \gamma_1(m, N)$ and $\Gamma_1 = \Gamma_1(m, N, \Omega)$ (see the proof of Lemma 2). Next choose a sequence $\{R_l\}$ such that $R_l \rightarrow \infty$ as $l \rightarrow \infty$. For fixed l , $B_{4R_l}(0) \subset B_{\frac{\kappa}{\lambda_k}}(0)$ provided k is sufficiently large. Thus by the interior estimate we mentioned at the beginning of the proof of Lemma 2 it follows that $\{v_k\}$ is uniformly bounded in $C^{1,\alpha}(B_{2R_l}(0))$ for some $\alpha \in (0, 1)$ which is independent of k . By standard arguments using a diagonal process, we can find a subsequence $\{v_{k_j}\}$ which converge uniformly to $\tilde{v} \in C^{1,\alpha^\#}(\mathbb{R}^N) (0 < \alpha^\# < \alpha)$ on any compact subset of $\mathbb{R}^N$. Note that

$\tilde{g}^{ij}(z) \rightarrow \delta_{ij}$ as $k \rightarrow \infty$ due to the fact that $\lambda_k \rightarrow 0$ and $D\Psi(x) \rightarrow D\Psi(0) = I_{N \times N}$ as $x \rightarrow 0$.

Also note that $\tilde{M}_k \rightarrow \infty$ as $k \rightarrow \infty$. Thus we conclude that $\tilde{v}$ satisfies

$$\begin{cases} \Delta_m \tilde{v} + \tilde{v}^{\frac{N(m-1)+m}{N-m}} = 0 & \text{in } \mathbb{R}^N, \\ 0 \leq \tilde{v} \leq 1, \quad \tilde{v}(0, \dots, 0, \beta_*) = 1. \end{cases} \quad (61)$$

On the other hand, for any fixed $R > \beta_* + 1$ we have

$$\begin{aligned}
\int_{B_R(0)} |\nabla \bar{v}|^m dz &= \lim_{j \rightarrow \infty} \int_{B_R(0)} |\nabla v_{k_j}|^m dz \leq \lim_{j \rightarrow \infty} \int_{B_{\frac{k}{\lambda_{k_j}}}(0)} |\nabla v_{k_j}|^m dz \\
&= \lim_{j \rightarrow \infty} \varepsilon_{k_j}^{m-N} \int_{B_k(0)} |\nabla \bar{u}_{k_j}|^m dy \\
&= \lim_{j \rightarrow \infty} 2 \varepsilon_{k_j}^{m-N} \int_{B_k^+(0)} |\nabla \bar{u}_{k_j}|^m dy \\
&= \lim_{j \rightarrow \infty} 2 \varepsilon_{k_j}^{m-N} \int_{\Phi(B_k^+(0))} |\nabla u_{k_j}|^m |\det(D\Psi)| dx \\
&\leq \lim_{j \rightarrow \infty} \varepsilon_{k_j}^{-N} \left(C(N) \int_{\Omega} \varepsilon_{k_j}^m |\nabla u_{k_j}|^m dx \right) = 0 \quad (\text{by (54)}).
\end{aligned}$$

Hence $\nabla \bar{v} \equiv 0$ in $B_R(0)$, which implies $\bar{v} \neq 1$ in $B_R(0)$ since $\bar{v}(0, \dots, 0, \beta_*) = 1$, and this contradicts, (61).

Subcase (ii). The sequence $\left\{ \frac{\alpha_k}{\lambda_k} \right\}$ is unbounded. We may assume that $\frac{\alpha_k}{\lambda_k} \rightarrow \infty$ as $k \rightarrow \infty$. In this case we define

$$v_k(z) = \bar{M}_k^{-1} \bar{u}_k(\lambda_k z + Q_k).$$

Then v_k is well defined in the ball $B_{\frac{k}{\lambda_k}}(0)$ and $v_k(0) = 1$ for all $k = 1, 2, \dots$. From (58) we see that v_k satisfies

$$\begin{cases} \sum_{j=1}^N \frac{\partial}{\partial z_j} \tilde{a}_j(z, \nabla v_k) = \sqrt{g_*} \left(\bar{M}_k^{m-\nu} v_k^{m-1} - v_k^{\frac{N(m-1)+m}{N-m}} \right) & \text{in } B_{\frac{k}{\lambda_k}}(0), \\ \frac{\partial v_k}{\partial z_N} = 0 & \text{on } B_{\frac{k}{\lambda_k}}(0) \cap \left\{ z_N = -\frac{\alpha_k}{\lambda_k} \right\} \end{cases} \quad (62)$$

with

$$\tilde{a}_j(z, \nabla v_k) = \sum_{i=1}^N \left[\sum_{s,l} \bar{g}^{sl}(z) \frac{\partial v_k}{\partial z_s} \frac{\partial v_k}{\partial z_l} \right]^{\frac{m-2}{2}} \sqrt{g_*} \bar{g}^{ji}(z) \frac{\partial v_k}{\partial z_i}, \quad (63)$$

for $j = 1, \dots, N$, where $\bar{g}^{sl}(z) = \bar{g}^{sl}(\lambda_k z + Q_k)$, $\bar{g}_*(z) = g_*(\lambda_k z + Q_k)$ in (63) for $s, l = 1, \dots, N$.

For any $R > 0$ we have $\frac{\alpha_k}{\lambda_k} > 2R$ if k is sufficiently large, so that the entire ball $\bar{B}_{2R}(0)$ is

contained in $B_{\frac{k}{\lambda_k}}(0) \cap \left\{ z_N > -\frac{\alpha_k}{\lambda_k} \right\}$. Repeating the argument as in the subcase (i), we obtain a subsequence v_k which converges uniformly to $C^{1,\alpha^\#}$ function $\bar{v}$ on any compact set of $\mathbb{R}^N$, and $\bar{v}$ satisfies

$$\begin{cases} \Delta_m \bar{v} + \bar{v}^{\frac{N(m-1)+m}{N-m}} = 0 & \text{in } \mathbb{R}^N, \\ 0 \leq \bar{v} \leq 1, \quad \bar{v}(0) = 1. \end{cases} \quad (64)$$

Nevertheless, for any fixed $R > 0$ we know

$$\begin{aligned}
\int_{B_R(0)} |\nabla \tilde{v}|^m dz &= \lim_{j \rightarrow \infty} \int_{B_R(0)} |\nabla v_{k_j}|^m dz \leq \lim_{j \rightarrow \infty} \int_{B_{\frac{k}{\lambda_{k_j}}}(0) \cap \{z_N > -\frac{\alpha_k}{\lambda_k}\}} |\nabla v_{k_j}|^m dz \\
&= \lim_{j \rightarrow \infty} \varepsilon_{k_j}^{m-N} \int_{B_k^+(0)} |\nabla u_{k_j}|^m dy \\
&= \lim_{j \rightarrow \infty} \varepsilon_{k_j}^{m-N} \int_{\Phi(B_k^+(0))} |\nabla u_{k_j}|^m |\det(D\Psi)| dx \\
&\leq \lim_{j \rightarrow \infty} \varepsilon_{k_j}^{-N} \left(C(N) \int_{\Omega} \varepsilon_{k_j}^m |\nabla u_{k_j}|^m dx \right) = 0 \quad (\text{by (54)}).
\end{aligned}$$

Hence $\nabla \tilde{v} \equiv 0$ in $B_R(0)$, which implies $\tilde{v} \equiv 1$ in $B_R(0)$ since $\tilde{v}(0, \dots, 0, \beta_*) = 1$, and this contradicts (64).

Case 2

$P_0 \in \Omega$. In this case we define

$$v_k(z) = \tilde{M}_k^{-1} u_k(\lambda_k z + P_k).$$

Then $v_k(0) = 1$ for all $k = 1, 2, \dots$; v_k satisfies

$$\begin{cases} \Delta_m v_k - \tilde{M}_k^{-m-\gamma} v_k^{m-1} + v_k^{\frac{N(m-1)+m}{N-m}} = 0 & \text{in } \Omega_k, \\ \frac{\partial v_k}{\partial n_k} = 0 & \text{on } \partial\Omega_k, \end{cases} \quad (65)$$

where n_k is the unit outer normal of $\Omega_k = \{z | \lambda_k z + P_k \in \Omega\}$. For any $R > 0$ we have $\frac{\alpha_k}{\lambda_k} > 2R$ if k is sufficiently large, so that the entire ball $B_{2R}(0)$ is contained in Ω_k . Repeating the argument as in the subcase (i), we obtain a subsequence such that $\{v_{k_j}\}$ is uniformly convergent to a $C^{1,\alpha^\#}$ function $\widehat{v}$ on any compact set of $\mathbb{R}^N$ and moreover it follows from (65) that $\widehat{v}$ satisfies (64) with $\tilde{v}$ replaced by $\widehat{v}$. On the other hand, for any fixed $R > 0$ we have

$$\begin{aligned}
\int_{B_R(0)} |\nabla \widehat{v}|^m dz &= \lim_{j \rightarrow \infty} \int_{B_R(0)} |\nabla v_{k_j}|^m dz \leq \lim_{j \rightarrow \infty} \int_{\Omega_{k_j}} |\nabla v_{k_j}|^m dz \\
&= \lim_{j \rightarrow \infty} \varepsilon_{k_j}^{m-N} \int_{\Omega} |\nabla u_{k_j}|^m dx \\
&= \lim_{j \rightarrow \infty} \varepsilon_{k_j}^{-N} \left(\int_{\Omega} \varepsilon_{k_j}^m |\nabla u_{k_j}|^m dx \right) = 0 \quad (\text{by (54)}).
\end{aligned}$$

Hence $\nabla \widehat{v} \equiv 0$ in $B_R(0)$, which implies $\widehat{v} \equiv 1$ in $B_R(0)$ since $\widehat{v}(0) = 1$, and this contradicts (64). Hence Claim 1 is proved by combining the above cases together.

Claim 2

There exists a constant $0 < C < \infty$ independent of ε such that $u_\varepsilon \leq C$.

If we multiply both sides of (42) by $u_\varepsilon^{m(s-1)+1}$ ($s \geq 1$) and integrate over Ω , then by virtue of the homogeneous Neumann boundary condition (44) it follows that

$$s^{-m} [m(s-1)+1] \varepsilon^m \int_{\Omega} |\nabla u_\varepsilon|^m dx + \int_{\Omega} u_\varepsilon^{ms} dx = \int_{\Omega} u_\varepsilon^{m(s-1)+\gamma} dx. \quad (66)$$

Note that Claim 1 tells us that

$$\int_{\Omega} u_{\varepsilon}^{m(s-1)+\nu} dx \leq \sup_{\Omega} u_{\varepsilon}^{\nu-m} \int_{\Omega} u_{\varepsilon}^{ms} dx \leq C\varepsilon^m \int_{\Omega} u_{\varepsilon}^{ms} dx. \quad (67)$$

Since $s^{-m} [m(s-1)+1] \leq m+1$ for $s \geq 1$ we obtain from (66) and (67) that

$$s^{-m} [m(s-1)+1] \int_{\Omega} (|\nabla u_{\varepsilon}^s|^m + u_{\varepsilon}^{ms}) dx \leq C \int_{\Omega} u_{\varepsilon}^{ms} dx. \quad (68)$$

Note that $\frac{s^m}{m(s-1)+1} \leq \frac{N}{m^2+(N-m)} s^{m-1}$ for $s \geq \frac{\nu}{m}$. Thus it follows from the Sobolev embedding theorem (19) and (68) that for $s \geq \frac{\nu}{m}$,

$$\left(\int_{\Omega} u_{\varepsilon}^{\nu s} dx \right)^{\frac{m}{\nu}} \leq (Cs)^{m-1} \int_{\Omega} u_{\varepsilon}^{ms} dx. \quad (69)$$

Next we define two sequences $\{\widehat{s}_j\}$ and $\{\widehat{M}_j\}$ by

$$\widehat{s}_j = \left(\frac{\nu}{m} \right)^{j-1} \nu, \quad \widehat{M}_j = \int_{\Omega} u_{\varepsilon}^{\widehat{s}_j} dx \quad \text{for } j=1, 2, \dots \quad (70)$$

Note that $\widehat{s}_j \geq \nu$ for all $j \geq 0$ and $\widehat{s}_j \rightarrow \infty$ as $j \rightarrow \infty$. The inequality (69) yields that

$$\left(\widehat{M}_{j+1} \right)^{\frac{1}{\widehat{s}_{j+1}}} \leq \left(C\widehat{s}_j \right)^{\frac{m-1}{m} \nu \cdot \frac{1}{\widehat{s}_{j+1}}} \widehat{M}_j^{\frac{1}{\widehat{s}_j}}.$$

If we let $\widehat{\mu}_j = \frac{1}{\widehat{s}_j} \log \widehat{M}_j$ we get

$$\widehat{\mu}_{j+1} \leq \widehat{\mu}_j + \frac{m-1}{\nu} \left(\frac{\nu}{m} \right)^{-(j-1)} \left[\log(C\nu) + (j-1) \log \frac{\nu}{m} \right].$$

Thus we have

$$\widehat{\mu}_{j+1} \leq \widehat{\mu}_1 + \sum_{j=1}^{\infty} \frac{m-1}{\nu} \left(\frac{\nu}{m} \right)^{-(j-1)} \left[\log(C\nu) + (j-1) \log \frac{\nu}{m} \right]$$

Note that

$$\sum_{j=1}^{\infty} \frac{m-1}{\nu} \left(\frac{\nu}{m} \right)^{-(j-1)} \left[\log(C\nu) + (j-1) \log \frac{\nu}{m} \right] \leq C_2$$

for some constant C_2 independent of ε . Therefore we get

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq \lim_{j \rightarrow \infty} e^{\widehat{\mu}_j} \leq e^{C_2} e^{\widehat{\mu}_1} = C_3 \left(\int_\Omega u_\varepsilon^\nu \right)^{\frac{1}{\nu}}. \quad (71)$$

(54) tells us that

$$\left(\int_\Omega u_\varepsilon^\nu \right)^{\frac{1}{\nu}} \leq |\Omega|^{\frac{1}{\nu}}.$$

Hence we have

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C, \quad (72)$$

for some positive constant independent of ε . Proof of Theorem 2 is complete.

By applying Lemma 2 to u_ε we have the following uniform $C^{1,\alpha}$ -estimate of u_ε as a corollary of Theorem 2.

Corollary 1

Suppose $1 < m < N$. For $\varepsilon \in [1, \infty)$ there exist two positive constants $\alpha \in (0, 1)$ and C depending only on m, N and Ω such that for any least-energy solution u_ε to (42)–(44) we have

$$\|u_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} \leq C.$$

4 Asymptotic behaviors of least energy solutions of the critical case as $\varepsilon \rightarrow \infty$

In this section we study asymptotic behaviors of least energy solution u_ε of the critical case as $\varepsilon \rightarrow \infty$ and give Proof of Theorem 3. We decompose u_ε as we did before in (9), i.e., $u_\varepsilon = \bar{u}_\varepsilon + v_\varepsilon$, where

$$\bar{u}_\varepsilon = \frac{1}{|\Omega|} \int_\Omega u_\varepsilon dx \quad \text{and} \quad \int_\Omega v_\varepsilon dx = 0. \quad (73)$$

Then (42)–(44) tell us that v_ε satisfies the following equations

$$\begin{cases} \varepsilon^m \Delta_m v_\varepsilon - u_\varepsilon^{m-1} + u_\varepsilon^{\frac{N(m-1)+m}{N-m}} = 0 & \text{in } \Omega, \\ \frac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (74)$$

Since

$$u_\varepsilon^{m-1} - u_\varepsilon^{\frac{N(m-1)+m}{N-m}} - u_\varepsilon^{m-1} + u_\varepsilon^{\frac{N(m-1)+m}{N-m}} = \left[(m-1) \int_0^1 \left(\bar{u}_\varepsilon + tv_\varepsilon \right)^{m-2} dt \right] - \frac{N(m-1)+m}{N-m} \int_0^1 \left(\bar{u}_\varepsilon + tv_\varepsilon \right)^{\frac{N(m-2)+2m}{N-m}} dt \Big] v_\varepsilon$$

we have

$$\varepsilon^m \Delta_m v_\varepsilon - \left[(m-1) \int_0^1 \left(\bar{u}_\varepsilon + t v_\varepsilon \right)^{m-2} dt - \frac{N(m-1)+m}{N-m} \int_0^1 \left(\bar{u}_\varepsilon + t v_\varepsilon \right)^{\frac{N(m-2)+2m}{N-m}} dt \right] v_\varepsilon = \bar{u}_\varepsilon^{m-1} - \bar{u}_\varepsilon^{\frac{N(m-1)+m}{N-m}}. \quad (75)$$

Multiplying both sides of (75) by v_ε , integrating over Ω and using integration by parts we have

$$\varepsilon^m \int_\Omega |\nabla v_\varepsilon|^m dx + \int_\Omega \left[(m-1) \int_0^1 \left(\bar{u}_\varepsilon + t v_\varepsilon \right)^{m-2} dt - \frac{N(m-1)+m}{N-m} \int_0^1 \left(\bar{u}_\varepsilon + t v_\varepsilon \right)^{\frac{N(m-2)+2m}{N-m}} dt \right] v_\varepsilon^2 dx = 0. \quad (76)$$

Theorem 2 tells us

$$0 < \bar{u}_\varepsilon + t v_\varepsilon \leq 2 \max_{\bar{\Omega}} u_\varepsilon(x) \leq C.$$

Since

$$\lim_{t \rightarrow 0^+} \frac{t^{\frac{N(m-2)+2m}{N-m}}}{t^{m-2}} = 0$$

it follows from (76) that

$$\left| \left((m-1) \left(\bar{u}_\varepsilon + t v_\varepsilon \right)^{m-2} - \frac{N(m-1)+m}{N-m} \left(\bar{u}_\varepsilon + t v_\varepsilon \right)^{\frac{N(m-2)+2m}{N-m}} \right) \right| \leq C$$

for $(x, t) \in \bar{\Omega} \times [0, 1]$. Thus we get by the Poincaré inequality

$$\varepsilon^m \gamma^m \int_\Omega |v_\varepsilon|^m dx \leq \varepsilon^m \int_\Omega |\nabla v_\varepsilon|^m dx \leq C \int_\Omega v_\varepsilon^2 dx. \quad (77)$$

Proof of Theorem 3 Case 1. $1 < m \leq 2$. It follows from (77) that

$$\varepsilon^m \gamma^m \int_\Omega |v_\varepsilon|^m \leq C \int_\Omega v_\varepsilon^2 dx \leq C \sup_\Omega |v_\varepsilon|^{2-m} \int_\Omega |v_\varepsilon|^m dx \leq C \int_\Omega |v_\varepsilon|^m dx \quad (\text{by Theorem 2}).$$

If we take $\varepsilon > \varepsilon_1 = \frac{\sqrt[m]{C}}{\gamma}$, we know $\int_\Omega |v_\varepsilon|^m dx = 0$, which yields that $v_\varepsilon \equiv 0$. Hence

$$u_\varepsilon \equiv \frac{1}{|\Omega|} \int_\Omega u_\varepsilon dx \quad \text{on} \quad \bar{\Omega}.$$

Maximum Principle tells us that $\sup_\Omega u_\varepsilon \geq 1$ ([8]). Therefore $u_\varepsilon \not\equiv 1$ for $\varepsilon > \varepsilon_1$.

Case 2. $2 < m < N$. For this case, by using Hölder inequality, interpolation and the fact that $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C$ as we did in Sect. 2 we obtain that for all $q \geq m$

$$\|v_\varepsilon\|_{L^q(\Omega)} \leq C\varepsilon^{-\frac{m^2}{q(m-2)}}.$$

By repeating the proof of Lemma 1 we have

$$\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^{-\beta} \quad (78)$$

with β as defined in Lemma 1. Applying Lemma 2 to v_ε and taking care of (78) yield

$$\|v_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} \leq C\varepsilon^{-\beta} \quad (79)$$

for some constants $\alpha \in (0, 1)$ depending only on m, N and Ω , and C depending only on m, N, Ω and the uniform upper bound in Theorem 2. Note that $t^{m-1} - t^{\frac{N(m-1)+m}{N-m}}$ has only 1 as its positive zero. Next we show that $\bar{u}_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow \infty$. Since maximum principle tells us that $\sup_\Omega \geq 1$ ([8]) it follows from (78) and Theorem 2 that for sufficiently large ε ,

$$\frac{1}{2} \leq \bar{u}_\varepsilon \leq C. \quad (80)$$

To show $\bar{u}_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow \infty$ it is equivalent to show that for any sequence $\{\varepsilon_j\}$ with $\lim_{j \rightarrow \infty} \varepsilon_j = \infty$, $\lim_{j \rightarrow \infty} \lim_{j \rightarrow \infty} \bar{u}_{\varepsilon_j} = 1$.

Suppose not. Then it follows from (80) that there exist a sequence $\{\varepsilon_j\}$ with $\lim_{j \rightarrow \infty} \varepsilon_j = \infty$ and a constant $\frac{1}{2} \leq u_c \leq C$ such that $\lim_{j \rightarrow \infty} \bar{u}_{\varepsilon_j} = u_c \neq 1$. Note that from (42)–(44) it follows that

$$\varepsilon_j^m \int_\Omega |\nabla u_{\varepsilon_j}|^m dx = \int_\Omega (u_{\varepsilon_j}^v - u_{\varepsilon_j}^m) dx. \quad (81)$$

Note that (77) and (78) imply

$$\varepsilon_j^m \int_\Omega |\nabla u_{\varepsilon_j}|^m dx = \varepsilon_j^m \int_\Omega |\nabla v_{\varepsilon_j}|^m dx \leq C \int_\Omega v_{\varepsilon_j}^2 dx \leq C \varepsilon_j^{-2\beta} |\Omega| \quad (82)$$

Taking limit at both sides of (81) as $j \rightarrow \infty$ and using (79) we obtain

$$\int_\Omega (u_c^v - u_c^m) dx = 0$$

which implies $u_c^v - u_c^m = 0$, i.e., the constant u_c must be 1, contradiction. Therefore we have

$$\|u_\varepsilon - 1\|_{C^{1,\alpha}(\bar{\Omega})} \leq \|v_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} + \|\bar{u}_\varepsilon - 1\|_{C^{1,\alpha}(\bar{\Omega})} \rightarrow 0$$

as $\varepsilon \rightarrow \infty$. Proof of Theorem 3 is complete.

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