Simultaneous segmentation and reconstruction: A level set method approach for limited view computed tomography

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Purpose: An iterative tomographic reconstruction algorithm that simultaneously segments and reconstructs the reconstruction domain is proposed and applied to tomographic reconstructions from a sparse number of projection images.

Methods: The proposed algorithm uses a two-phase level set method segmentation in conjunction with an iterative tomographic reconstruction to achieve simultaneous segmentation and reconstruction. The simultaneous segmentation and reconstruction is achieved by alternating between level set function evolutions and per-region intensity value updates. To deal with the limited number of projections, *a priori* information about the reconstruction is enforced via penalized likelihood function. Specifically, smooth function within each region (piecewise smooth function) and bounded function intensity values for each region are assumed. Such *a priori* information is formulated into a quadratic objective function with linear bound constraints. The level set function evolutions are achieved by artificially time evolving the level set function in the negative gradient direction; the intensity value updates are achieved by using the gradient projection conjugate gradient algorithm.

Results: The proposed simultaneous segmentation and reconstruction results were compared to “conventional” iterative reconstruction (with no segmentation), iterative reconstruction followed by segmentation, and filtered backprojection. Improvements of 6%–13% in the normalized root mean square error were observed when the proposed algorithm was applied to simulated projections of a numerical phantom and to real fan-beam projections of the Catphan phantom, both of which did not satisfy the *a priori* assumptions.

Conclusions: The proposed simultaneous segmentation and reconstruction resulted in improved reconstruction image quality. The algorithm correctly segments the reconstruction space into regions, preserves sharp edges between different regions, and smoothes the noise within each region. The proposed algorithm framework has the flexibility to be adapted to different *a priori* constraints while maintaining the benefits achieved by the simultaneous segmentation and reconstruction. © 2010 American Association of Physicists in Medicine.

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I. INTRODUCTION

Tomographic reconstruction from projection images is an inverse problem with a wide range of applications in the medical imaging field, e.g., CT, MRI, SPECT, and PET. With sufficiently large number of projections over the required angular sampling range, which is equivalent to sufficient sampling of the Fourier space, fast and accurate reconstructions are achievable using the well-known filtered backprojection (FBP) algorithms. However, situations exist where the number of projections is inherently limited, such as in cardiac imaging using C-arm CT^{1,2} and in imaging in the presence of respiratory motion. To minimize motion artifacts, a limited number of projection images, where the motion is

minimal or corresponds to the same phase of the motion cycle, can be used for reconstruction. Even with multiple acquisitions with gating information to sample the required angular sampling range, the number of acquired projections is inherently limited due to time and dose issues. As the angular sampling intervals increase (*limited view* problem) and/or the angular sampling range decreases (*limited angle* problem), the inverse problem becomes progressively more ill-conditioned, and the reconstructions using FBP display significant artifacts.^{3–6}

An alternate reconstruction approach that suppresses the artifacts from the limited number of projections is to find the maximum likelihood (ML) solution of the tomographic in-

verse problem.^{7–9} Typically, an iterative algorithm is used, which computes an estimate of the unknown object (for reconstruction) that exhibits maximum likelihood given the observed (projection) data. Equivalently, the log-likelihood function $L(u;g)$ is maximized, where u is the object to reconstruct and g is the observed data. In the presence of noise, ML algorithms typically amplify noise as the number of iterations increases. By using a Bayesian formulation⁵ and incorporating *a priori* information about the object u as regularizations in the ML reconstruction algorithm, maximum *a posteriori* (MAP) or regularized likelihood solutions can be computed. In other words, the MAP solution comes from

$$\arg \min_u -L(u;g) + \beta R(u), \quad (1)$$

where $R(u)$ is a regularization function of the object u . The parameter β controls the trade-off between the negative likelihood $-L(u;g)$ and *a priori* constraints $R(u)$. Regularized likelihood reconstructions have been reported, in Refs. 5 and 10–25 and references therein, to significantly reduce artifacts. A wide range of priors, such as smoothness, shape, and location priors, that characterize the object u has been investigated in literature. For example, a commonly used regularization regarding the intensity values is the non-negativity constraint $u \geq 0$, which reflects the *a priori* information that attenuation coefficients are non-negative.

For the limited view problem, regularizations based on *a priori* information can improve the reconstruction quality, whereas the reconstruction quality will be unacceptable without such prior information. For example, discrete tomography (DT) (Ref. 26) has been applied to reconstruct u with the *a priori* information that u consists of a finite range of pixel values. Prior knowledge of the finite range of pixel values with additional *a priori* information regarding u can significantly reduce the necessary number of projections for an accurate reconstruction. Specifically, DT has been applied to angiography applications with the assumption that u is comprised of two homogeneous materials (regions): contrast material and the “background.”

Instead of having a regularization function that is applied globally on the object u , the regularization constraints can be enforced locally inside a known support region if the support region is known *a priori*. It is often the case that the support region is not well-known *a priori*, but numerous works have focused on estimating such support region boundaries based on the level set method. The level set method was introduced by Osher and Sethian²⁷ for capturing the moving fronts of curves and surfaces (see Refs. 28 and 29 for an overview of this topic). The level set method has been used to reconstruct shapes (support regions) posed as an inverse problem.^{30–33}

In imaging science, level set based shape reconstruction algorithms have been applied to simultaneously estimate values within the reconstructed shapes.^{34–43} Such simultaneous shape and value estimation algorithms have been applied to segmentation of medical images. Additionally, simultaneous shape and value estimation algorithms for the tomographic reconstruction problem, using the level set method, have shown promising results.^{32,44–57} Dorn *et al.*³² applied the

level set method to reconstruct the shape for noisy limited view electromagnetic tomography. Chan and Vese^{34,35} demonstrated simultaneous shape and value estimation algorithms using a piecewise constant model (PCM). A two-phase PCM, which divides the domain into two regions (phases) and assumes constant values for each phase, has been used in conjunction with the level set method to reconstruct electrical impedance tomography,⁵⁰ diffusion optical tomography,⁵¹ and limited view tomography.⁴⁴ Furthermore, four-phase PCM for the limited view problem has been used to reconstruct from tomographic data.^{47,52,53,55}

A more flexible object representation, as compared to the PCM, is to use a small set of basis functions in place of constant values for each region. Feng *et al.*⁴⁶ used the level set method with a small number of “texture” coefficients to reconstruct from limited view noisy tomographic data. Shi and Karl⁴⁹ used the level set method with an object-based model to reconstruct dynamic objects from limited view noisy tomographic data. Another improvement from the PCM object representation is the piecewise smooth model (PSM). The PSM assumes that the object u is divided into regions and that the values within each region vary slowly, but allows the values from region to region to have large discontinuities. We have observed that such relaxed constraints using the PSM allow for better convergence, when compared to PCM, in the presence of data inconsistencies arising from noisy projections. Yu and Fessler⁴⁵ used the PSM with the level set method to incorporate nonlocal boundary information and applied their approach to emission computed tomography. Alvino and Yezzi⁴⁸ used the PSM to generalize the work by Whitaker and Elangovan⁴⁴ and applied it to CT reconstruction from noiseless projections. Kolehmainen *et al.*⁵⁶ used a slightly modified PSM, which divides the reconstruction domain into “valid” and “invalid” regions, to reconstruct limited angle dental data.

In this work, we propose an iterative tomographic reconstruction method for the limited view problem that simultaneously segments and reconstructs the unknown function u using the following *a priori* information:

- (a) u is divided (segmented) into nonoverlapping regions.
- (b) Intensity values within each region are slowly varying.
- (c) Intensity values within each region are bounded.

Priors (a) and (b) assume that the object u to reconstruct is a piecewise smooth function, similar to that in Refs. 45, 48, and 56. In addition, we encode each region by bounding the intensity values [prior (c)]. For example, in diagnostic CT reconstructions, human anatomy is often represented by a finite number of intensity value ranges corresponding to different material compositions, e.g., air (≈ -1000 HU), soft tissue, and bone (≈ 1000 HU). The bounds on the intensity values constrain each segmented region to represent one (or possibly more) material.

Prior (a) is incorporated into the object model using the level set method segmentation framework; prior (b) is incorporated into the objective function as a smoothness penalty terms; and prior (c) is incorporated into the search algorithm

of the objective function minimization. We achieve simultaneous segmentation and reconstruction by iteratively alternating between the level set based segmentation and the iterative tomographic reconstruction algorithms. We show that the simultaneous segmentation and reconstruction leads to improved results when compared to iterative reconstruction (IR), iterative reconstruction followed by segmentation, and FBP reconstructions. Improvements were observed when the proposed algorithm was applied to a numerical phantom as well as to the Catphan phantom that did not satisfy the *a priori* assumptions.

Section II presents the details of the algorithm used. Section III presents the numerical results for the proposed algorithm and compares the results to traditional reconstruction methods using a numerical phantom and using a real fan-beam data. Section IV discusses current implementation issues. Section V summarizes our work.

II. MATERIAL AND METHODS

II.A. Algorithm

We adopt the multiphase level set formulation and framework^{34,35,51} for the segmentation problem using the Mumford and Shah model.⁵⁸ We use a two-phase piecewise smooth approximation using a single level set function as our model to represent the *a priori* information of the underlying function. A two-phase function $u(\mathbf{x})$ can be expressed as

$$u(\mathbf{x}) = u_1(\mathbf{x})H(\phi(\mathbf{x})) + u_2(\mathbf{x})(1 - H(\phi(\mathbf{x}))), \quad (2)$$

where u_1 and u_2 describe the object in each region, ϕ is the implicit level set function, and point $\mathbf{x}$ is in the reconstruction domain. The Heaviside function, defined as $H(z)=1$ for $z \geq 0$ and $H(z)=0$ for $z < 0$, indicates the two different regions, determined by the sign of ϕ . In other words, the boundaries between the two regions are determined by $\mathbf{C}=\{\mathbf{x} | \phi(\mathbf{x})=0\}$. Segmentation of u into regions by the implicit level set function ϕ allows independent characterizations of u_1 and u_2 .

Assuming a PSM for u , we formulate additional *a priori* smoothness constraints for u_1 and u_2 by regularizing the gradients in each region. These regularizations encourage smoothly varying intensity values within each region. They bias the object to be a constant, and the strength of the bias is determined by the regularization parameters. The regularization is applied independently to each region because the boundary information is separately available through ϕ . Therefore, the objective function to minimize for the proposed reconstruction algorithm is expressed as

$$F(u_1, u_2, \phi; g, \beta_1, \beta_2, \nu) = -L(u, \phi; g) + R_u(u_1, u_2, \phi; \beta_1, \beta_2) + R_\phi(\phi; \nu), \quad (3)$$

where

$$-L(u, \phi; g) = \frac{1}{2} \int_{\theta} \int_r ([\mathcal{P}u](r, \theta) - g(r, \theta))^2 dr d\theta, \quad (4)$$

$$R_u(u_1, u_2, \phi; \beta_1, \beta_2) = \frac{\beta_1}{2} \int_{\Omega} |\nabla u_1|^2 H(\phi) d\mathbf{x} + \frac{\beta_2}{2} \int_{\Omega} |\nabla u_2|^2 (1 - H(\phi)) d\mathbf{x}, \quad (5)$$

$$R_\phi(\phi; \nu) = \nu \int_{\Omega} |\nabla H(\phi)| d\mathbf{x}, \quad (6)$$

where the dependence of u_j and ϕ on $\mathbf{x}$ has been omitted for simplicity. The likelihood function $L(u, \phi; g)$ is a quadratic function, which comes from the Gaussian approximation of the Poisson distribution for a large mean. The term $[\mathcal{P}u](r, \theta) = \int u(\mathbf{x}) \delta(\mathbf{x} \cdot \mathbf{d}(\theta) - r) d\mathbf{x}$ is the projection of $u(\mathbf{x})$, where $\mathbf{x} \cdot \mathbf{d}(\theta)$ denotes the inner product of $\mathbf{x}$ and the unit ray directional vector $\mathbf{d}(\theta)$. The term $g(r, \theta)$ is the measured projection with its sinogram space parametrized by (r, θ) , ∇ is the gradient operator, β_j are the smoothness regularization parameters associated with u_j , and Ω is the reconstruction domain. The variable ν is a commonly used regularization parameter in level set framework that controls the length of the curve $\mathbf{C}$.

Finally, we restrict u_1 and u_2 intensity values to be in nonoverlapping ranges $[l_1, h_1]$ and $[l_2, h_2]$, respectively. Therefore, our overall reconstruction algorithm is equivalent to finding the solution to the optimization problem,

$$\begin{aligned} &\arg \min_{u_1, u_2, \phi} F(u_1, u_2, \phi; g, \beta_1, \beta_2, \nu) \\ &\text{s.t. } l_1 \leq u_1 \leq h_1, \quad l_2 \leq u_2 \leq h_2. \end{aligned} \quad (7)$$

The optimization problem (7) is solved by iteratively updating the unknown functions u_j , while holding ϕ fixed and then evolving the level set function ϕ while holding u_j fixed. Updating u_j corresponds to a tomographic reconstruction algorithm, where for any reasonably sized data, iterative algorithms [e.g., conjugate gradient (CG) method and expectation maximization method] are better choices over direct inversion algorithms. To deal with the value bound constraints in updating u_j , the barrier/penalty function method^{17,59} or the active set/gradient projection (GP) method^{60,61} needs be incorporated. In our proposed algorithm, an active set/gradient projection method is used. On the other hand, since the intensity value bound constraints have no role in the updating of the level set function ϕ , a typical PDE-based level set function evolution method is used to artificially time evolve the level set function ϕ .

Given an initial u_0 as input, the overall algorithm is summarized in Fig. 1. We will now present the details for each part of the algorithm and its associated parameters that will determine the reconstructions in Sec. III.

The iterative tomographic reconstruction segment updates u_1 and u_2 simultaneously, where the update direction at each pixel is proportional to the negative gradient

FIG. 1. Proposed reconstruction algorithm flowchart.

$$-\nabla F = - \begin{bmatrix} \partial F / \partial u_1 \\ \partial F / \partial u_2 \end{bmatrix}. \quad (8)$$

The gradients of F with respect to u_j are computed under the assumption of fixed ϕ and are equal to

$$\frac{\partial F}{\partial u_j} = \frac{\partial F}{\partial u} \frac{\partial u}{\partial u_j} = [\mathcal{P}^*(\mathcal{P}u - g) + \beta_j \Delta u_j] I_j(\phi)$$

and

$$\frac{\partial u_j}{\partial \vec{n}} = 0 \quad \text{on } \mathbf{C}. \quad (9)$$

Here, $\mathcal{P}^*$ is the backprojection operator ($\mathcal{P}$ is the projection operator), Δ is the Laplacian operator, $I_1(\phi) = H(\phi)$, $I_2(\phi) = 1 - H(\phi)$, and $\partial / \partial \vec{n}$ denotes the partial derivative in the normal direction $\vec{n}$ at the boundaries $\mathbf{C}$. $\mathcal{P}$ is implemented as a linearly interpolated ray-driven projector, Δ is implemented as a second-order accurate finite difference, and $H(\phi)$ is implemented by using an approximate function³⁵

$$H_\epsilon(\phi) = \frac{1}{2} \left[1 + \frac{2}{\pi} \arctan \left(\frac{\phi}{\epsilon} \right) \right]. \quad (10)$$

Because of bound constraints $l_1 \leq u_1 \leq h_1$ and $l_2 \leq u_2 \leq h_2$, updates of u_j must be modified to enforce feasible updates. We have adapted an algorithm for solving large scale quadratic programming problems with bound constraints, namely, the gradient projection conjugate gradient (GPCG) method.^{60,61} The GPCG algorithm uses the GP method⁶² to reach a point where the algorithm fails to make reasonable progress or the *active set* does not change. The active set is defined as $\mathcal{A}(\mathbf{u}_1, \mathbf{u}_2) = \{i: u_1(i) = l_1 \text{ or } u_1(i) = h_1\} \cup \{i: u_2(i) = l_2 \text{ or } u_2(i) = h_2\}$, where $\mathbf{u}_j \in \mathbb{R}^n$ is the discretized u_j and $\mathbf{u}_j(i)$ is the i th component of $\mathbf{u}_j$. After reaching a solution point using the GP method, the GPCG algorithm uses the CG method to find an approximate solution to the subproblem of Eq. (7), where the components of $\mathbf{u}_j$ in the active set are

fixed to their bound values. The CG method searches for an approximate solution until the algorithm fails to make a reasonable progress or the new search direction indicates a change in the active set. Once a solution point for the CG method is found, the GP method is repeated.

Level set boundary evolution under the assumption of fixed u_j is computed by artificially time evolving ϕ in the negative direction of the gradient of F with respect to ϕ ,

$$\frac{\partial \phi}{\partial t} = - \frac{\partial F}{\partial \phi}. \quad (11)$$

The gradient of F with respect to ϕ is computed as

$$\begin{aligned} \frac{\partial F}{\partial \phi} &= \frac{\partial F}{\partial u} \frac{\partial u}{\partial \phi} \\ &= \left[\mathcal{P}^*(\mathcal{P}u - g)(u_1 - u_2) + \frac{\beta_1}{2} |\nabla u_1|^2 - \frac{\beta_2}{2} |\nabla u_2|^2 \right. \\ &\quad \left. - \nu \operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) \right] \delta(\phi), \end{aligned} \quad (12)$$

where $\operatorname{div}()$ is the divergence operator and $\delta(\phi) = (\partial / \partial \phi) H(\phi)$. We followed the implementation details of Eq. (12) presented in Ref. 35.

The choice of signed distance function for the implicit function ϕ results in more accurate numerical solutions⁶³ but requires a periodic *reinitialization*.^{29,34,64,65} Periodic reinitialization reduces the accumulation of numerical errors during the level set curve evolution iterations. We adopted the periodic reinitialization algorithm presented in Ref. 65.

The number of iterations for the level set method segmentation varies considerably, depending on the initial configuration of the level set function. For example, the number of iterations starting from a random initial configuration is large because the reconfiguration of the boundaries takes a long time to converge. In the proposed reconstruction method, the initial u_0 is computed using an iterative reconstruction algorithm (i.e., GPCG algorithm) with a global smoothness constraint. The initial u_0 is computed in a limited number of iterations by relaxing the iteration stopping criterion. After computing u_0 , it is segmented by simple thresholding, depending on the ranges $[l_1, h_1]$ and $[l_2, h_2]$. The initial ϕ is determined by the segmentation result such that the initial boundary curve $\mathbf{C}$ matches the initial segmentation boundaries. The initial u_j is assigned arbitrary values that satisfy $l_1 \leq u_1 \leq h_1$ and $l_2 \leq u_2 \leq h_2$.

II.B. Experiments

A numerical phantom representing a simplified heart with intravenous contrast agent was created and used to investigate the reconstruction properties. A 128×128 2D phantom, which has constant values for the “air” and the “body” tissue region, and linearly varying values in three other regions, is shown in Fig. 2(a). The “heart” region CT numbers are in [542, 1290] HU, the “spine” region values are in [686, 1180] HU, the “lung” region values are in [−933, 404] HU, the body tissue region value is equal to 25 HU, and the air region

FIG. 2. Data used for the reconstructions: (a) Numerical phantom and (b) FDK reconstruction of a Catphan phantom slice.

value is equal to -1000 HU. To simulate noisy projections used for the reconstructions, only the photon noise was considered, i.e., each projection was corrupted by Poisson noise under the following assumptions: 10^5 incident photons at 60 keV having passed through a 16 cm thick material.

The proposed reconstruction algorithm is applied to reconstruct from parallel-beam simulated projections of the numerical phantom, uniformly distributed over the 180° range, and the results are compared to iterative reconstructions using the GPCG algorithm (IR), iterative reconstructions using the GPCG algorithm followed by a level set method segmentation (IRPS), and conventional FBP reconstructions. The segmented regions from the proposed reconstruction algorithm are also compared to those from IRPS. The number of iterations to reach convergence is compared for different algorithms. Gradient computation, which requires a projection and backprojection operation, accounts for each iteration increment. Therefore, gradient projection, conjugate gradient direction, and projected search which require gradient computations, each adds a count to the total number of iterations. Because the level set curve ϕ evolution also requires a projection and backprojection operation, the number of iterations for ϕ evolution is computed separately. The normalized RMSE, defined as

$$\left[\frac{\sum_{x,y} |\hat{u} - u^*|^2}{\sum_{x,y} |u^*|^2} \right]^{1/2},$$

where $\hat{u}$ denotes the reconstruction and u^* denotes the reference object, is computed inside the reference object region of support. The normalized RMSE for reconstructions from the proposed algorithm and three other algorithms (IR, IRPS, and FBP) are compared. In addition to the RMSE computations, histograms of the absolute error $|\hat{u} - u^*|$ for the different reconstruction algorithms are plotted for comparison.

The proposed reconstruction algorithm is used to reconstruct from fan-beam projections of the Catphan phantom (The Phantom Laboratory, Salem, NY) using a table-top acquisition system. The table-top system is equipped with a PaxScan 4030CB (Varian Imaging Products, Palo Alto, CA) detector, and projections uniformly distributed over a 360° range were acquired (125 kVp, 80 mA tube parameters, bow-tie filtration, and antiscatter grid). A narrow collimator was

TABLE I. Algorithm parameters.

Parameter	Description
β_j	Smoothness regularization
ν	Curve length regularization
$[l_1, h_1], [l_2, h_2]$	Intensity value bounds
dt_ϕ	Time-step size for ϕ evolution
N_ϕ	No. of time steps for ϕ evolution
ϵ	Parameter for $H_\epsilon(\phi)$

used to acquire fan-beam projections of a Catphan phantom slice. Similar to the numerical phantom, the Catphan phantom slice has multiple high contrast and low contrast materials, which do not satisfy the *a priori* information. A phantom that does not satisfy the *a priori* assumption of two-phase piecewise smooth function was chosen to investigate the merits as well as the limitations of the proposed reconstruction algorithm. In addition, real noise in the projections adds to the inconsistencies that must be handled by the reconstruction algorithms.

Similar to the numerical phantom experiment, 2D fan-beam reconstructions (512×512) were performed and were compared to three other reconstruction algorithms: IR, IRPS, and FBP. A FBP reconstruction of the slice using the FDK (Ref. 66) reconstruction algorithm (Hamming windowed) from 625 projections uniformly distributed over a 360° range is used as a reference image [Fig. 2(b)]. The number of gradient computations is counted as the total number of iterations, and the number of iterations for ϕ evolution iterations is recorded separately. The normalized RMSE and histograms of the absolute error for reconstructions from the proposed algorithm and three other algorithms (IR, IRPS, and FBP) are computed and compared in a manner similar to that used for the numerical phantom.

Parameters for the proposed algorithm are listed in Table I. Specific values for the *a priori* constraint parameters and the $\phi(x)$ evolution parameters used for reconstructions are listed in Table II. Both the numerical phantom and the Catphan phantom are comprised of more than two materials (e.g., air, water, and high contrast material) and, therefore, do

TABLE II. Experiment parameter values.

	Parameter	Specific value
Numerical phantom	β_j	$\beta_1 = 9 \times 10^{-2}, \beta_2 = 4.5 \times 10^{-2}$
	ν	2.5×10^4
	$[l_1, h_1], [l_2, h_2]$	$[400, 1300], [-1000, 400]$
	dt_ϕ	1×10^{-3}
	N_ϕ	120
	ϵ	1×10^{-1}
Catphan phantom	β_j	$\beta_1 = 1.0 \times 10^{-4}, \beta_2 = 5.0 \times 10^{-2}$
	ν	2.0×10^4
	$[l_1, h_1], [l_2, h_2]$	$[125, 625], [-1000, 125]$
	dt_ϕ	1×10^{-5}
	N_ϕ	25
	ϵ	1×10^{-2}

FIG. 3. Reconstructions of the numerical phantom using (a) the proposed algorithm, (b) IR, (c) IRPS, and (d) FBP. Window and level: $[-1000, 1300]$ HU for (a)–(c) and $[-1645, 1322]$ HU for (d).

not strictly match the two-phase PSM assumption used in the reconstruction algorithm. In our experiments, we set the algorithm parameters such that u_1 captures the details of the high contrast material accurately, and u_2 lumps together the rest of the materials. In general, the proposed algorithm has the flexibility to segment and reconstruct different primary materials of interest, depending on the imaging task, by changing the algorithm parameters.

III. RESULTS

Tomographic reconstructions of the numerical phantom were computed from 15 noisy projections. Figure 3(a) shows reconstructions using the proposed algorithm with the specific parameter values shown in Table II. The proposed algorithm for the numerical phantom converged after 132 u_j update iterations and $480(=4N_\phi)\phi$ evolution iterations. Because the bound constraints were incorporated into the proposed algorithm, the reconstructed CT numbers are within the specified range of $[-1000, 1300]$ HU. As a comparison, a FBP reconstruction from the 15 noisy projections is shown in Fig. 3(d). Its reconstructed CT numbers are in the range of $[-1645, 1322]$ HU and in the range of $[-1112, 1322]$ HU when limited to the object region of support.

Typical IR methods do not use the additional region encoding information from the level set function ϕ and, therefore, assumes one global smoothness prior parametrized by β . Figure 3(b) shows a GPCG reconstruction (IR) using a smoothness constraint parameter $\beta=0.05$, which converged in 218 iterations. Figure 3(c) shows a GPCG reconstruction, followed by a level set method segmentation (IRPS). Level

FIG. 4. Difference image between four different reconstructions and the numerical phantom. The four reconstructions are (a) the proposed algorithm, (b) IR, (c) IRPS, and (d) FBP. All images are displayed using the same window and level $[-500, 500]$ HU.

set method segmentation with piecewise smooth function prior denoises and segments the input image, and Fig. 3(c) is the resulting image when Fig. 3(b) was used as the input image for the level set method segmentation. Additionally, all the iterative reconstructions in Fig. 3 exhibit artifactual mass in the left lung (right side of the images), caused by the limited number of projections.

Figure 4 shows difference images between the four reconstructions and the numerical phantom. The proposed reconstruction exhibits sharp transitions at the region boundaries and has suppressed the limited view reconstruction artifacts. Although IRPS reduces some of the artifacts in IR, it does not remove as much as the proposed algorithm.

Figure 5 compares the segmented regions from the proposed reconstruction, numerical phantom, and the IRPS. Figures 5(a), 5(c), and 5(e) show zoomed u_1 , and Figs. 5(b), 5(d), and 5(f) show u_2 . High contrast material intensity gradients in u_1 , which were intentionally embedded in the numerical phantom, are well recovered using the proposed reconstruction algorithm [Fig. 5(a)]. Simultaneous segmentation and reconstruction encodes regions with in-region smoothness constraints and, thus, enables the recovery of such in-region variations. On the other hand, a separate iterative reconstruction followed by segmentation underestimates the pixel intensities and cannot fully recover the in-region variations of u_1 [Fig. 5(e)]. Limited view reconstruction artifacts are better suppressed using the pro-

FIG. 5. Zoomed views of the segmented u_1 from the (a) proposed reconstruction, (c) original numerical phantom, and (e) IRPS, shown in a display window and level [400, 1300] HU. Segmented u_2 from the (b) proposed reconstruction, (d) original numerical phantom, and (f) IRPS, shown in a display window and level [-1000, 400] HU.

posed algorithm [Fig. 5(b)] because of the separate region-specific smoothing constraints enabled by the simultaneous segmentation during reconstruction.

The normalized RMSE of reconstructions from the proposed algorithm and three other algorithms are shown in Table III. The proposed algorithm achieved approximately 6% and 9% improvements over IR and IRPS, respectively. Figure 6 shows the histograms of the absolute error between

TABLE III. Normalized RMSE ($\times 10^{-1}$).

	Proposed	IR	IRPS	FBP
Numerical phantom	1.33	1.42	1.46	1.83
Catphan phantom	4.16	5.41	4.79	5.39

FIG. 6. Absolute error histogram for pixels within the numerical phantom region of support.

different reconstructions and the known reference object. For example, a perfect reconstruction would result in all of the errors in the bin containing zero, and as the reconstruction deviates from an ideal reconstruction, more pixels would exhibit error values greater than zero. Reconstruction from the proposed algorithm has a large number of pixels with absolute error of less than 50 HU, and a small number of pixels with absolute error between 50 and 300 HU. IR and IRPS reconstructions exhibit a similar distribution of error values. On the other hand, FBP reconstruction exhibits a wide distribution of errors with a relatively large number of pixels in the range of [100, 300] HU.

Tomographic reconstructions were performed from 42 projections of the Catphan phantom. Figure 7(a) shows a reconstruction using the proposed algorithm with the specific parameter values shown in Table II. The algorithm converged after 30 initial GPCG iterations (to reach u_0), 73 u_j update iterations, and 100 ($=4N_\phi$) ϕ evolution iterations. (Having used precomputed forward projection weights, the computation took approximately 3 min on an Intel dual core 3.0 GHz Linux machine with 4 Gbytes of RAM.) Because u_2 consists of all the material except for the high contrast material (u_1), smoothing was applied ($\beta_2=5.0 \times 10^{-2}$), and therefore, the resulting reconstruction shows blurring for edges within u_2 . On the other hand, the high contrast material exhibits sharp edges because the high contrast material was set to be the object of interest and assigned to u_1 . The proposed algorithm not only creates sharper boundaries for the high contrast material but also effectively reduces the limited view reconstruction artifacts, confirming the numerical phantom reconstruction results.

Figure 7(b) shows an IR (88 iterations, $\beta=5.0 \times 10^{-2}$) reconstruction. Because of the global smoothing, the high contrast material exhibits blurred edges and CT numbers underestimation. Figure 7(c) shows an IRPS reconstruction, where a nonregularized IR (GPCG reconstruction algorithm) was first performed (214 iterations) with no smoothing ($\beta=0$) and then a post level set method two-phase segmentation³⁵ was performed on the IR. A nonregularized IR results in a reconstruction with sharper edges, but with amplified noise

FIG. 7. Reconstructions from 42 projections of the Catphan phantom using (a) the proposed algorithm, (b) IR, (c) IRPS, and (d) FBP. All images are displayed using the same window and level $[-400, 400]$ HU.

and limited view reconstruction artifacts, when compared to a regularized IR [Fig. 7(b)]. Postsegmentation smoothed out the limited view artifacts, but erroneously included an outer annulus [Figs. 7(c) and 9(e)]. Additionally, IRPS reconstruction resulted in a widening of the segmented high contrast material u_1 [Fig. 9(e)]. Figure 7(d) shows a FBP reconstruction for comparison. Interestingly, all of the reconstructions from the sparse number of projections could not recover the low contrast material visible in the reference reconstruction [Fig. 2(b)]. We attribute this characteristic to the dominant limited view reconstruction artifacts, which are better suppressed by iterative reconstruction algorithms.

Figure 8 shows absolute difference images ($|\hat{u} - u^*|$) between the four reconstructions and a reference reconstruction (from 625 projections). Figures 8(a) and 8(b) exhibit similar limited view reconstruction artifact patterns, but Fig. 8(a) shows that the proposed algorithm improves the high contrast material CT numbers and boundaries. Figures 8(c) and 8(d) exhibit similar limited view artifact patterns, but Fig. 8(c) shows that the post level set method segmentation has smoothed out the artifacts. Although the difference images for the proposed algorithm and IRPS [Figs. 8(a) and 8(c), respectively] are similar except for the limited view artifact patterns, the proposed algorithm correctly segments the high contrast material and requires fewer number of iterations to achieve similar reconstruction image quality. Annuli in the difference images in Figs. 8(a) and 8(b) come from smoothing of the water and air boundary; on the other hand, the annulus in the difference image in Fig. 8(c) comes from incorrect segmentation of the high contrast material.

FIG. 8. Absolute difference images between four reconstructions from 42 projections and reference reconstruction from 625 projections: (a) Proposed algorithm, (b) IR, (c) IRPS, and (d) FBP. All images are displayed using the same window and level $[000, 300]$ HU.

Figure 9 compares the segmented regions from the proposed reconstruction, reference reconstruction, and IRPS reconstruction. Figures 9(a)–9(c) show u_1 , and Figs. 9(d)–9(f) show u_2 . The proposed reconstruction algorithm segments the high contrast material that matches well with the reference reconstruction segmentation. Although the circular high contrast material is not a perfect circle, as in the reference reconstruction segmentation, this distortion arises from the limited view problem and can be seen in all other limited view reconstructions. On the other hand, IRPS reconstruction exhibits wider segmented region for the high contrast material and also erroneously segmented the outer annulus as the high contrast material [Figs. 9(e) and 9(f)]. Although the erroneous outer annulus segmentation can be avoided by performing a regularized IR ($\beta > 0$) before segmentation, it leads to a trade-off in underestimation of the high contrast material CT numbers and blurring of edges.

The normalized RMSE for the four different reconstructions are shown in Table III. The proposed algorithm achieved approximately 23% and 13% improvements over IR and IRPS, respectively. Figure 10 shows the absolute error histogram for the proposed, IR, IRPS, and FBP reconstructions. FBP has a higher error distribution in the range of $[0, 170]$ HU, but does not exhibit errors larger than 300 HU for the pixels within the Catphan phantom region of support. The proposed and the IR reconstructions exhibit a higher distribution of pixels with errors in the $[100, 320]$ HU range, when compared to the IRPS reconstruction. IRPS reconstruction has a smaller number of pixels with errors in the $[50,$

FIG. 9. Zoomed views of the segmented u_1 of the (a) proposed reconstruction, (c) reference reconstruction, and (e) IRPS. Segmented u_2 of the (b) proposed reconstruction, (d) reference reconstruction, and (f) IRPS. All images are displayed using the same window and level $[-1000, 700]$ HU.

330] HU range when compared to the proposed and IR reconstructions. This corresponds to the reduced level of artifact in the water and air regions of the Catphan phantom [Fig. 7(c)] when compared to Fig. 7(a). It is the result of applying an iterative reconstruction algorithm with zero smoothing prior to segmentation in the IRPS algorithm, thereby reducing the *a priori* model and data mismatch (discussed in Sec. IV). In contrast, IRPS reconstruction exhibits an increase in the number of pixels with errors in the [180, 230] HU range, which corresponds to the erroneous segmentation of the annulus visible in the IRPS reconstruction [Fig. 7(c)].

IV. DISCUSSION

Level set method reconstruction with a piecewise constant function assumption for PET and CT has been presented in Refs. 53 and 52, respectively. The piecewise constant function assumption can be thought of as a special case of the

FIG. 10. Absolute error histogram for pixels within the Catphan phantom region of support.

proposed algorithm corresponding to the case for large smoothness regularization parameter values. In other words, the proposed algorithm relaxes the strict constraint of the piecewise constant assumption to allow for smooth changes and, as a result, it improves the handling of data inconsistencies. Similar algorithms that assume a piecewise smooth function as the underlying data have been previously published for tomographic reconstruction.^{45,48,56} Alvino and Yezzi⁴⁸ showed reconstructions on CT data, but all the data used were noise-free, and the resulting segmentation and comparison to other iterative reconstructions is not provided. Kolehmainen *et al.*⁵⁶ showed interesting results from limited angle dental data, but used the level set formulation to divide valid from nonvalid regions rather than encoding the regions with individual region-specific characteristics.

Yu and Fessler⁴⁵ showed results on emission tomography using a masking function that selects the boundaries between regions, and the algorithm does not encode each region with its own region-specific characteristics. Using this approach, a good initial estimate is required for the algorithm to converge and manual initial estimates were used. We have observed that numerous attempts with different thresholds for the initial estimate are often required to converge to an accurate reconstruction and segmentation. In contrast, our proposed algorithm can make use of the region-specific bound constraints to better deal with poor initial estimates and, as a result, converges more quickly. In other words, our proposed method with nonoverlapping per-region intensity value bounds is more robust to poor initial estimates.

IR and IRPS with different parameters were computed as examples of reconstructions that could potentially match the performance of the proposed algorithm. Because the smoothness regularization parameter β in IR is a global parameter over the reconstruction domain, it controls the trade-off between accurate intensity value estimation and noise (artifact) amplification. For a large β , the noise was well suppressed at the cost of overly smoothed reconstruction; for a small β , sharper intensity values were estimated at the cost of amplified artifacts.

Currently, the level set function ϕ is time evolved with a

fixed time step size dt_ϕ repeated N_ϕ times. Because each ϕ evolution iteration computes a gradient, which requires projection and backprojection operations, computational cost can be minimized by taking a larger step size. Such time step size can be relatively large for the conventional level set method segmentation of an image (when using a nonexplicit finite difference method). In contrast, a large step size for the proposed algorithm resulted in oscillatory behavior of the updates as the estimate approached convergence, which requires smaller changes per iteration. Therefore, the time step size had to be small enough to avoid oscillatory behavior near convergence, and the number of time steps had to be large enough to reach a stable configuration of the boundaries, especially at the earlier stages of the algorithm. An improved step size determination is currently under investigation to reduce the number of ϕ evolution iterations, which, in turn, reduces the computational load.

The objective function (3), in conjunction with the two-phase function assumption using an implicit level set function (2), leads to a general framework for the level set method reconstruction algorithm. Depending on the *a priori* information of the underlying data to reconstruct, we can extend to different formulations of R_u (e.g., L_1 norm), which, in turn, would result in different reconstructions. We have chosen the gradient projection conjugate gradient method as our objective function minimization search algorithm because our objective function was formulated as a quadratic function. However, any search algorithm can be used in the framework, and the choice would depend on the objective function to minimize. Additionally, regularizing the level set function other than by its boundary curve length can be formulated into R_ϕ . For example, deformations of the region boundaries over time can be regularized to match the *a priori* information for a spatiotemporal reconstruction.

The current two-phase PSM has limitations when the reconstruction object is comprised of more than two piecewise smooth regions. This is highlighted in the different reconstruction artifacts visible in the Catphan phantom reconstructions using the proposed algorithm with different parameters. Figure 11(a) shows the proposed reconstruction using a two-phase PSM, as discussed in Sec. III, where the primary region of interest has been set to the high contrast material; therefore, u_1 incorporates the high contrast material and u_2 incorporates the water and air of the phantom. Figure 11(b) shows a reconstruction using a two-phase PSM where the primary region of interest has been set to air; therefore, u_1 incorporates the high contrast material and the water region of the phantom, whereas u_2 incorporates only the air region. Figures 11(a) and 11(b) exhibit different reconstruction artifact patterns because a single smoothness constraint is applied to a wide range of pixel values for u_2 [Fig. 11(a)] and for u_1 [Fig. 11(b)]. Therefore, reconstruction artifact patterns in Figs. 11(a) and 11(b) come from the mismatch between the assumption of a single smooth function and the object of interest (data). In contrast, Fig. 7(c) was reconstructed using an iterative reconstruction with no *a priori* information, i.e., zero smoothing, followed by a two-phase model segmenta-

FIG. 11. (a) Proposed reconstruction with parameters in Table II. (b) Proposed reconstruction with different u_1 and u_2 bound values such that the resulting reconstruction segments the air as u_2 . (c) Combined reconstruction using (a) and (b). Images (a)–(c) are shown in the same window and level $[-400, 400]$ HU. (d) Absolute difference image between (c) and reference reconstruction from 625 projections, shown in window and level $[0, 300]$ HU.

tion. It exhibits less artifact in the water region when compared to Fig. 7(a).

Two-phase PSM can provide sufficient information when only a single material is of interest (such as high contrast material) and other structures are of lesser interest. More phases are required when multiple materials are of interest. For example, material separation within regions where multiple materials were lumped into a single smooth function will remove the reconstruction artifacts observed in such regions for the two-phase PSM. As an extension to the proposed method, a four-phase PSM can be constructed by adding a second level set function and maintaining the piecewise smooth function assumption. In such a case, further encoding of the reconstruction domain is possible, resulting in better separation of different regions or materials. For example, sharper boundaries between the lung and the body regions can be obtained for the numerical phantom reconstructions and, similarly, sharper boundaries between the air and the water regions can be obtained for the Catphan phantom reconstructions. We are investigating an extension of our proposed algorithm to a four-phase PSM assumption.

As a proof of concept, the current two-phase algorithm was applied to the 42 projections twice; once to reconstruct and segment the high contrast material [Fig. 11(a)] and once to reconstruct and segment the air regions [Fig. 11(b)]. A combined reconstruction image [Fig. 11(c)] was constructed from the two two-phase PSMs by using the high contrast

material in Fig. 11(a), the air region in Fig. 11(b), and the mean of Figs. 11(a) and 11(b) for the remaining region. It has sharp edges for the high contrast material as well as the air regions, which closely resembles the reference reconstruction [Fig. 2(b)]. An absolute difference image [Fig. 11(d)] between the combined reconstruction image and the reference reconstruction verifies that both the high contrast material and air region are accurately estimated. This is confirmed by the normalized RMSE for the combined image, 2.92×10^{-1} , which is a 30% decrease from a single two-phase PSM reconstruction. Additionally, the absolute error histogram (Fig. 10) shows a decrease in the number of pixels with errors greater than 160 HU. Therefore, we conjecture that a four-phase model reconstruction will improve the segmentation of different materials (especially when more than two materials are of interest) and reconstruct in fewer number of iterations (compared to a two-step IRPS method) and, as a result, improve the suppression of reconstruction artifacts. With respect to the absolute error distribution (Fig. 10), a four-phase PSM reconstruction will greatly reduce the number of pixels with errors in the [40, 160] HU range, while increasing the number of pixels with errors in the [0, 40] HU range.

V. CONCLUSION

We have presented a tomographic image reconstruction algorithm for a limited number of projections based on the two-phase level set method segmentation. The algorithm alternates between the implicit function evolution that determines the boundaries between the two regions (phases) and the intensity value updates for each region. Assuming smooth functions within each region and bounded function intensity values, a quadratic objective function with linear bound constraints is established. The intensity value updates are achieved using the gradient projection conjugate gradient algorithm. With the addition of a level set curve length regularization in the objective function, the level set function evolutions are achieved.

The proposed algorithm simultaneously segments and reconstructs, resulting in tomographic reconstructions with segmented regions that exhibit suppressed artifacts from noise as well as from the limited number of projections in the primary region of interest, where the assumed smooth function assumption matched the data well. Improvements in image quality, such as the RMSE and segmentation, were observed when the algorithm was applied to a numerical phantom and real fan-beam data which did not strictly satisfy the two-phase piecewise smooth function assumption. Improvements of 6% and 13% in the normalized RMSE were observed for the numerical phantom and real fan-beam data, respectively, when compared to traditional iterative reconstructions. On the other hand, the proposed reconstruction algorithm is limited by the two-phase piecewise smooth function assumption; as the object of interest deviates from the assumed two-phase piecewise smooth function model, the reconstructed image exhibits artifacts caused by the assumed model and data mismatch. Finally, the proposed algo-

rithm assumed a piecewise smooth function as the underlying data, but the framework has the flexibility to be adapted to different *a priori* constraints while maintaining the benefits achieved by simultaneous segmentation and reconstruction.

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