Acoustic scattering by arbitrary distributions of disjoint, homogeneous cylinders or spheres

Andrew J. Hesford^{a)} and Jeffrey P. Astheimer

Department of Electrical and Computer Engineering, University of Rochester, Rochester, New York 14627

Robert C. Waag

Department of Electrical and Computer Engineering and Department of Imaging Sciences, University of Rochester, Rochester, New York 14627

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A T -matrix formulation is presented to compute acoustic scattering from arbitrary, disjoint distributions of cylinders or spheres, each with arbitrary, uniform acoustic properties. The generalized approach exploits the similarities in these scattering problems to present a single system of equations that is easily specialized to cylindrical or spherical scatterers. By employing field expansions based on orthogonal harmonic functions, continuity of pressure and normal particle velocity are directly enforced at each scatterer using diagonal, analytic expressions to eliminate the need for integral equations. The effect of a cylinder or sphere that encloses all other scatterers is simulated with an outer iterative procedure that decouples the inner-object solution from the effect of the enclosing object to improve computational efficiency when interactions among the interior objects are significant. Numerical results establish the validity and efficiency of the outer iteration procedure for nested objects. Two- and three-dimensional methods that employ this outer iteration are used to measure and characterize the accuracy of two-dimensional approximations to three-dimensional scattering of elevation-focused beams.

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I. INTRODUCTION

Forward scattering calculations in three dimensions can be very expensive for acoustically large scatterers. This cost is especially problematic in inverse scattering calculations that require repeated solution of the forward problem. To compensate, measurement schemes have been employed that probe tissue with elevation-focused incident fields concentrated in a plane and that detect reflected fields using receiver sensitivity patterns that are concentrated in the same plane.¹ Measurements acquired using transmit and receive focusing are interpreted as two-dimensional scattering by variations in planar cross sections of the media. This dimensional reduction allows three-dimensional forward and inverse scattering calculations to be carried out as sequences of more efficient, two-dimensional calculations for a succession of parallel slices.

A number of studies have previously investigated the effects of two-dimensional approximations to three-dimensional, elevation-focused scattering in the context of linear^{2,3} and nonlinear^{4,5} inverse scattering involving electromagnetic or acoustic fields. The more thorough and recent studies^{3–5} each considered a series of experiments involving scattering from and reconstructions of a single, radially symmetric, homogeneous scatterer. Quantitative evaluations of two-dimensional methods were made by computing the error of two-dimensional slice reconstructions relative to known

scatterer profiles in the slice planes. A natural extension of these studies is an investigation of the quality of two-dimensional approximations to three-dimensional, elevation-focused scattering for more complicated, realistic scattering objects.

Scattering models composed of distributions of homogeneous circular cylinders or spheres are appealing because descriptions of such models are much simpler than descriptions of general scattering media. A solution for scattering of acoustic waves by a homogeneous cylinder or sphere appears to have been first published⁶ in 1945. Among the early studies of multiple cylinders is one by Twersky⁷ in 1952. Electromagnetic scattering in the range of optical wavelength has been analyzed for a sphere with spherical inclusions.^{8,9}

Nonconcentric nesting of cylinders or spheres permits formation of compound scattering objects that exhibit asymmetric scattering due to asymmetry in structural arrangement. Furthermore, tissue-mimicking phantoms that are used in experimental studies of ultrasonic propagation can be constructed by embedding spherical inclusions in an otherwise homogeneous background sphere.¹⁰ Scattering from distributions of cylinders or spheres represents a compromise that sacrifices the ability to describe arbitrarily complex media for a significant reduction in the number of degrees of freedom. Furthermore, algorithms with reduced computational complexity can be used in place of more general integral equation methods.^{11,12}

An efficient solver that is applicable to arbitrary distributions of homogeneous scattering spheres, optionally embedded within a larger sphere, was developed in Ref. 13.

^{a)} Author to whom correspondence should be addressed. Electronic address: hesford@ece.rochester.edu

This method is an adaptation of the T -matrix approach.^{14–20} In this approach, reflection coefficients are computed at the boundary of each sphere as quotients relating the coefficients for incident and scattered spherical harmonics of the same degree. To make use of these reflection coefficients, the spherical harmonic expansions for the scattered fields must be translated from center to center. Recursive spherical harmonic translators have been used by others²¹ to carry out these re-expansions. However, a faster approach was developed in Ref. 13 by employing diagonal translation operators.²²

This paper presents a unification and extension of the work undertaken in Refs. 5 and 13. The method presented in Ref. 13 is here generalized in a form that is easily adapted to scattering problems in both two and three dimensions. A new nested iterative approach is additionally presented to decouple the solution of scattering from interior cylinders or spheres from the additional constraints imposed by an enclosing cylinder or sphere. The resulting technique is conceptually simpler than a coupled solution and improves the efficiency of the solution when interactions among the interior objects are significant. Finally, the efficient technique is used in two and three dimensions to conduct numerical experiments that assess the quality of two-dimensional approximations to three-dimensional scattering of elevation-focused beams by geometries that are more realistic and, hence, complicated than can be described with the analytic solutions of Ref. 5.

II. THEORY

The problem of acoustic scattering from an arbitrary medium characterized by a spatially varying complex sound speed $c(\mathbf{r})$ is considered. The imaginary part of the sound speed describes loss in the medium. The polar coordinates of the position vector $\mathbf{r}$ are given by (r, ϕ) in two dimensions and by (r, θ, ϕ) in three dimensions. The angular part of the vector $\mathbf{r}$ is defined as $\hat{\mathbf{s}} = \mathbf{r}/r$. The two-dimensional problem can be considered a special case of the three-dimensional scattering problem when the medium does not vary along the z axis. If all pressure waves are time harmonic with a frequency ω and an $e^{-i\omega t}$ time dependence, the total acoustic pressure is described by a complex-valued spatial field $\varphi(\mathbf{r})$ that satisfies the Helmholtz equation

$$[\nabla^2 + k^2(\mathbf{r})]\varphi(\mathbf{r}) = 0, \quad (1)$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$ for two-dimensional geometries, $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ for three-dimensional geometries, and $k(\mathbf{r}) = \omega/c(\mathbf{r})$ is the wavenumber.

The total pressure can be decomposed as

$$\varphi(\mathbf{r}) = \xi(\mathbf{r}) + \varphi_s(\mathbf{r}), \quad (2)$$

where ξ is an incident field that satisfies the Helmholtz equation for the background wavenumber $k=k_0$, and φ_s is the field scattered by the medium. If c varies only over a domain of finite support, then the scattered field satisfies the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r^{(d-1)/2} [\partial_r - ik] \varphi_s(\mathbf{r}) = 0, \quad (3)$$

FIG. 1. Left: Scattering geometry for disjoint, homogeneous objects embedded in a background. Right: Close-up view of a single object and all fields involved.

where $d=2$ or $d=3$ describes the dimension of the scattering geometry.

The boundary conditions between two distinct, homogeneous regions require continuity of pressure and normal particle velocity across the interface Γ separating the regions. Hence, if φ_A and φ_B describe the total pressure fields inside the first and second regions, respectively, then

$$\varphi_A(\mathbf{r}) = \varphi_B(\mathbf{r}), \quad \mathbf{r} \in \Gamma, \quad (4a)$$

$$\partial_{\hat{\mathbf{n}}} \varphi_A(\mathbf{r}) = \frac{\rho_A}{\rho_B} \partial_{\hat{\mathbf{n}}} \varphi_B(\mathbf{r}), \quad \mathbf{r} \in \Gamma. \quad (4b)$$

The direction $\hat{\mathbf{n}}$ of differentiation in Eq. (4b) is that of the outward normal to Γ at the point $\mathbf{r}$.

A. Scattering from multiple cylinders or spheres

Denote by $\{S_1, \dots, S_N\}$ a collection of disjoint spheres or axial cross sections of infinite circular cylinders embedded in an infinite, homogeneous background material with wavenumber k_0 and density ρ_0 . Each object S_i is described by a unique wavenumber k_i , density ρ_i , radius a_i , and center position $\mathbf{c}_i$. For generality, the objects S_i are called “radial objects.” A representative scattering geometry is shown in the left panel of Fig. 1 for three radial objects.

The T -matrix approach is used to calculate scattering by writing the total scattered field φ_s as

$$\varphi_s(\mathbf{r}) = \sum_{i=1}^N \varphi_{s,i}(\mathbf{r}), \quad (5)$$

where $\varphi_{s,i}$ is a contribution attributed to S_i . A local scattering operator R_i is constructed that defines the contribution $\varphi_{s,i}$ in terms of all other fields:

$$\varphi_{s,i} = R_i(\xi, \{\varphi_{s,j}\}_{j \neq i}). \quad (6)$$

An example of the local scattering operator is shown in the right panel of Fig. 1, where the fields $\varphi_{s,j}$ and ξ induce an internal field $\varphi_{\text{int},i}$ and a scattering contribution $\varphi_{s,i}$.

Solutions of the Helmholtz equation (1) based on separation of variables use orthogonal wave functions as a basis to expand the field scattered by a radial object. These expansions provide a simple form for the local scattering operator. The similarities between the two-dimensional and three-dimensional problems allow the waves to be expanded in general forms that are easily specialized:

$$\xi(\mathbf{r}) = \sum_{l,m} A_{lm}^i \mathcal{J}_l(k_0 r_i) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_i), \quad r_i > a_i, \quad (7a)$$

$$\varphi_{s,i}(\mathbf{r}) = \sum_{l,m} B_{lm}^i \mathcal{H}_l(k_0 r_i) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_i), \quad r_i > a_i, \quad (7b)$$

$$\varphi_{\text{int},i}(\mathbf{r}) = \sum_{l,m} C_{lm}^i \mathcal{J}_l(k_i r_i) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_i), \quad r_i < a_i, \quad (7c)$$

where r_i and $\hat{\mathbf{s}}_i$ are, respectively, the magnitude and direction of the local position vector $\mathbf{r}_i = \mathbf{r} - \mathbf{c}_i$. The radial wave functions $\mathcal{J}_l$ are regular cylindrical (in two dimensions) or spherical (in three dimensions) Bessel functions of order l . Likewise, the radial wave functions $\mathcal{H}_l$ represent in two and three dimensions, respectively, the singular cylindrical or spherical Hankel functions of the first kind of order l . In two dimensions, the angular wave function is $\mathcal{Y}_{lm}(\hat{\mathbf{s}}) = e^{il\phi}$; in three dimensions, the angular wave function is the spherical harmonic $\mathcal{Y}_{lm}(\hat{\mathbf{s}}) = \bar{P}_l^m(\cos \theta) e^{im\phi}$, in which $\bar{P}_l^m$ is the normalized associated Legendre function described in Ref. 23. The order m is not considered in two dimensions, so the compact summation in Eq. (7) expands to $\sum_{l,m} \rightarrow \sum_{l=-\infty}^{\infty}$ in two dimensions and $\sum_{l,m} \rightarrow \sum_{l=0}^{\infty} \sum_{m=-l}^l$ in three dimensions.

The scattered fields $\varphi_{s,j}$, $j \neq i$, are regular outside of S_j . Consequently, each field $\varphi_{s,j}$ may be expanded in the vicinity of S_i using the same regular form as Eq. (7a):

$$\varphi_{s,j}(\mathbf{r}) = \sum_{l,m} D_{lm}^{ij} \mathcal{J}_l(k_0 r_i) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_i), \quad a_i < r_i < d_{ij}, \quad (8)$$

where $d_{ij} = |\mathbf{c}_i - \mathbf{c}_j| - a_j$. The upper bound on r_i ensures that the expansion does not encounter the interior of S_j , over which $\varphi_{s,j}$ is not defined. The linear relationship between regular coefficients D_{lm}^{ij} and singular coefficients B_{lm}^j is known as an outer-to-inner translation. The translation operator is readily derived from addition theorems for Bessel functions in two dimensions and spherical harmonics in three dimensions.^{11,24} In three dimensions, where more degrees of freedom require improved efficiency, recursive formulas have been developed for the addition theorem that may be employed to develop efficient translation operators.^{14,15,25} The approach presented here uses direct application of the addition theorem in two dimensions and an efficient, diagonal approach thoroughly described in Ref. 13 for three-dimensional scattering from spheres.

The boundary conditions in Eq. (4) and the field expansions (7) and (8) can be reduced to a series of well-known scattering quotients^{6,26} that define the local scattering operator R_i in terms of harmonic expansion coefficients. Since the angular functions $\mathcal{Y}_{lm}$ are orthogonal, the individual terms of the expansions separate to yield

$$B_{lm}^i = Q_i^l \left(A_{lm}^i + \sum_{j \neq i} D_{lm}^{ij} \right). \quad (9)$$

The scattered field coefficients B_{lm}^i are related to the incident field coefficients A_{lm}^i and the scattered field coefficients D_{lm}^{ij} by the quotients

$$Q_i^l = \frac{\gamma_i \mathcal{J}_l(k_0 a_i) \mathcal{J}_l'(k_i a_i) - \mathcal{J}_l'(k_0 a_i) \mathcal{J}_l(k_i a_i)}{\mathcal{H}_l'(k_0 a_i) \mathcal{J}_l(k_i a_i) - \gamma_i \mathcal{H}_l(k_0 a_i) \mathcal{J}_l'(k_i a_i)}, \quad (10)$$

where $\gamma_i = \rho_0 k_i / \rho_i k_0$ and $(\cdot)'$ represents differentiation with respect to the whole argument. The local scattering operator R_i is therefore represented by the composition of Eq. (9) with

the translation operators mapping coefficients B_{lm}^j to D_{lm}^{ij} .

The form of the local scattering operator R_i represented by Eq. (9) is not useful because the coefficients D_{lm}^{ij} depend on the scattered field coefficients B_{lm}^j . The simple rearrangement

$$B_{lm}^i - Q_i^l \sum_{j \neq i} D_{lm}^{ij} = Q_i^l A_{lm}^i \quad (11)$$

yields a system of linear equations relating the unknown scattered field coefficients from all spheres to quantities that are a function of the known incident field expansion. Iterative inversion requires repeated evaluation of the left-hand side of Eq. (11), which can be performed in an efficient, matrix-free fashion using efficient outer-to-inner translations and multiplication by the scattering quotients Q_i^l .

B. Truncation of harmonic expansions

Numerical solutions of scattering problems require truncation of the infinite summations in Eq. (7). The maximum harmonic degree L_i required to represent the field scattered by S_i to a desired precision is called the approximate harmonic bandwidth and is roughly estimated using the excess bandwidth formula^{11,27}

$$L_i \approx k_0 a_i + 1.8 d_0^{2/3} (k_0 a_i)^{1/3}, \quad (12)$$

in which $d_0 = -\log \epsilon$ is a function of the desired accuracy ϵ of the expansion. The approximation (12) assumes that scattered fields are represented by arbitrary source distributions contained within S_i and the fields are observed in the far zone. The excess bandwidth formula may, therefore, overestimate the harmonic bandwidth of fields scattered by radial objects because these fields tend to be smoother (have lower harmonic bandwidth) than fields from general source distributions.

A more significant problem can arise when objects are too close to satisfy the far-field assumption of the excess bandwidth formula (12). In such cases, high-degree terms in harmonic expansions (7b) that do not radiate in the far zone may be significant to nearby scatterers. The excess bandwidth formula fails to capture the significance of these high-degree terms. A brute-force determination of the approximate harmonic bandwidth of scattered field expansions (7b) can be performed based on the relationship (10) between the scattering quotients Q_i^l and the incident field coefficients. The coefficients Q_i^l vanish as l increases, and the incident field expansion (7a) has a known approximate bandwidth. The brute-force determination of harmonic expansion truncation points is thoroughly discussed in Refs. 13 and 23.

C. Enclosing objects

Addition of a radial object to enclose the collection of objects $S_{\text{int}} = \{S_i: 1 \leq i \leq N\}$ complicates the scattering calculation. The enclosing object, denoted S_0 , is completely described by center $\mathbf{c}_0$, radius a_0 , wavenumber k_0 , and density ρ_0 . The choice of parameters k_0 and ρ_0 ensures that expansions of the form (7b) and the scattering quotients (10) re-

main unchanged for the inner objects S_i immersed in S_0 . The medium outside the enclosing object will now be described by wavenumber k_∞ and density ρ_∞ .

The incident field may be expanded relative to the center $\mathbf{c}_0$ using the regular harmonic expansion

$$\xi(\mathbf{r}) = \sum_{l,m} A_{lm}^0 \mathcal{J}_l(k_\infty r_0) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_0), \quad r_0 > a_0, \quad (13a)$$

where $\mathbf{r}_0 = \mathbf{r} - \mathbf{c}_0$ in analogy with the previously described position vectors $\mathbf{r}_i$. As with the inner objects, S_0 admits a singular scattered wave $\varphi_{s,0}$ and a regular internal wave $\varphi_{\text{int},0}$ with harmonic expansions

$$\varphi_{s,0}(\mathbf{r}) = \sum_{l,m} B_{lm}^0 \mathcal{H}_l(k_\infty r_0) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_0), \quad r_0 > a_0, \quad (13b)$$

$$\varphi_{\text{int},0}(\mathbf{r}) = \sum_{l,m} C_{lm}^0 \mathcal{J}_l(k_0 r_0) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_0), \quad r_0 < a_0. \quad (13c)$$

The scattered field $\varphi_{s,0}$ is also the scattered field for the entire collection of objects.

Equation (9) still describes the local scattering operators for all objects in S_{int} , except the coefficients A_{lm}^i must be replaced with coefficients C_{lm}^{i0} that expand the internal field as

$$\varphi_{\text{int},0}(\mathbf{r}) = \sum_{l,m} C_{lm}^{i0} \mathcal{J}_l(k_0 r_i) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_i), \quad \mathbf{r} \in S_0 \setminus S_i, \quad (14a)$$

where $S_0 \setminus S_i$ represents the intersection of the interior of S_0 and the exterior of S_i . The shifted coefficients C_{lm}^{i0} are related to the coefficients C_{lm}^0 in Eq. (13c) by a linear operation, known as an inner-to-inner translation, thoroughly described in Ref. 11 for problems in two and three dimensions. The shifted expansion is valid over the intersection of the interior of S_0 and the exterior of S_i .

The addition of two unknown fields, $\varphi_{s,0}$ and $\varphi_{\text{int},0}$, to the scattering problem requires the addition of two sets of equations to ensure a well-determined system. These equations are produced by enforcing the boundary conditions (4) at the surface of S_0 . The total field in the exterior of S_0 is the superposition of the incident field ξ and the scattered field $\varphi_{s,0}$. The total field in the interior of S_0 is the superposition of $\varphi_{\text{int},0}$ and the fields $\varphi_{s,i}$ scattered by objects in S_{int} . Before they can be summed, the fields $\varphi_{s,i}$ must be expressed as a singular expansion centered on $\mathbf{c}_0$:

$$\varphi_{s,i}(\mathbf{r}) = \sum_{l,m} B_{lm}^{0i} \mathcal{H}_l(k_0 r_0) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_0), \quad r_0 > a_0. \quad (14b)$$

The linear operator relating the coefficients B_{lm}^{0i} to the coefficients B_{lm}^i is known as an outer-to-outer translation and is derived in Ref. 11 in two and three dimensions.

Using the expansions (13) and (14) in the boundary conditions (4) yields the relationships

$$C_{lm}^0 = T_l A_{lm}^0 + R_l \sum_{i=1}^N B_{lm}^{0i}, \quad (15a)$$

$$B_{lm}^0 = T_l^e \sum_{i=1}^N B_{lm}^{0i} + R_l^e C_{lm}^0. \quad (15b)$$

The relation (15a) describes the internal coefficients C_{lm}^0 as a transmission of the incident field superimposed with a reflection of the scattered fields produced by interior objects. The relation (15b) describes the coefficients B_{lm}^0 of the scattered field as a superposition of a transmission of the fields scattered by the interior objects and a “reflection” of the field internal to S_0 . The transmission and reflection coefficients are

$$T_l = \frac{\mathcal{J}_l'(k_\infty a_0) \mathcal{H}_l(k_\infty a_0) + \mathcal{J}_l(k_\infty a_0) \mathcal{H}_l'(k_\infty a_0)}{\gamma \mathcal{J}_l'(k_0 a_0) \mathcal{H}_l(k_\infty a_0) - \mathcal{J}_l(k_0 a_0) \mathcal{H}_l'(k_\infty a_0)}, \quad (16a)$$

$$R_l = \frac{\gamma \mathcal{H}_l(k_\infty a_0) \mathcal{H}_l'(k_0 a_0) - \mathcal{H}_l(k_0 a_0) \mathcal{H}_l'(k_\infty a_0)}{\gamma \mathcal{J}_l'(k_0 a_0) \mathcal{H}_l(k_\infty a_0) - \mathcal{J}_l(k_0 a_0) \mathcal{H}_l'(k_\infty a_0)}, \quad (16b)$$

$$T_l^e = \frac{\gamma \mathcal{J}_l(k_\infty a_0) \mathcal{H}_l'(k_0 a_0) - \mathcal{J}_l'(k_\infty a_0) \mathcal{H}_l(k_0 a_0)}{\mathcal{J}_l(k_\infty a_0) \mathcal{H}_l'(k_\infty a_0) - \mathcal{J}_l'(k_\infty a_0) \mathcal{H}_l(k_\infty a_0)}, \quad (16c)$$

$$R_l^e = \frac{\gamma \mathcal{J}_l'(k_0 a_0) \mathcal{J}_l(k_\infty a_0) - \mathcal{J}_l(k_0 a_0) \mathcal{J}_l'(k_\infty a_0)}{\mathcal{J}_l(k_\infty a_0) \mathcal{H}_l'(k_\infty a_0) - \mathcal{J}_l'(k_\infty a_0) \mathcal{H}_l(k_\infty a_0)}, \quad (16d)$$

where $\gamma = \rho_\infty k_0 / \rho_0 k_\infty$.

Equation (15) can be used with the previously developed Eq. (9), with coefficients A_{lm}^i replaced with C_{lm}^{i0} , to yield a global matrix system for solving the scattering problem. However, the typically large size of the enclosing sphere demands evaluation of Eq. (15a) to high harmonic degrees. If interactions within S_{int} are computed with lower bandwidth, the fields represented by coefficients B_{lm}^i must be interpolated to a higher bandwidth prior to translation to the bounding sphere. In an iterative solution, these high-bandwidth calculations must be performed repeatedly as the solver approaches the final solution.

D. The successive-bounce approach

As an alternative to development of a single, large system of equations to describe scattering from a collection of radial objects embedded within another radial object, a technique called the successive-bounce approach is presented. The regular, internal field $\varphi_{\text{int},0}$ can be represented as a superposition of the incident field transmitted through the boundary and a series of successive reflections of the field scattered by objects in S_{int} :

$$\varphi_{\text{int},0}(\mathbf{r}) = T_s \xi(\mathbf{r}) + \sum_{n=1}^{\infty} [R_s D^{-1}]^n T_s \xi(\mathbf{r}). \quad (17)$$

The transmission operator T_s is defined such that, for the incident field (14a) and the scattering quotients (16a),

$$T_s \xi(\mathbf{r}) = \sum_{l,m} T_l A_{lm}^0 \mathcal{J}_l(k_0 r_0) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_0), \quad r_0 < a_0. \quad (18)$$

The reflection operator R_s is similarly defined using the wave expansions (13b) and the scattering quotients (16b) such that

$$R_s \sum_{i=1}^N \varphi_{s,i}(\mathbf{r}) = \sum_{l,m} R_l \left[\sum_{i=1}^N B_{lm}^{0i} \right] \mathcal{H}_l(k_0 r_0) \mathcal{Y}_{lm}(\hat{\mathbf{s}}_0), \quad r_0 > a_0. \quad (19)$$

The scattering operator D maps the superposition of scattered fields $\varphi_{s,i}$ to the regular field $\varphi_{\text{int},0}$:

$$\varphi_{\text{int},0}(\mathbf{r}) = D \sum_{i=1}^N \varphi_{s,i}(\mathbf{r}). \quad (20)$$

The operator D is the same operator defined by the system of Eq. (11) without regard to S_0 . Since D does not involve reflection of the scattered field from the boundary of S_0 , no need exists to interpolate and translate the scattered fields $\varphi_{s,i}$ to the center $\mathbf{c}_0$. Hence, on-demand computation of D is more efficient than the computation of the scattering operator for the entire geometry.

The series expansion (17) can be used to develop an outer iteration which uses at the n th step an approximation

$$\varphi_{\text{int},0}(\mathbf{r}) \approx \varphi_{\text{int},0}^{(n)}(\mathbf{r}) = \sum_{p=0}^n [R_s D^{-1}]^p T_s \xi(\mathbf{r}). \quad (21)$$

This estimate can be updated using the expression

$$\varphi_{\text{int},0}^{(n+1)}(\mathbf{r}) = T_s \xi(\mathbf{r}) + R_s D^{-1} \varphi_{\text{int},0}^{(n)}(\mathbf{r}). \quad (22)$$

A strategy for tracking the convergence of this method is presented in Sec. IV.

In the case where the enclosing object has the same material parameters as the infinite background (i.e., there is no enclosing object), the transmission operator $T_s = 1$, while the reflection operator $R_s = 0$. Therefore, $\varphi_{\text{int},0} = \varphi_{\text{int},0}^{(0)} = \xi$, and all updates to the interior field vanish. Therefore, the successive-bounce approach reduces to a single iteration when no enclosing sphere exists.

Scattering problems that are more sensitive to interactions between interior scatterers benefit from high-accuracy evaluation of important interactions using reduced harmonic bandwidth and a low number n of outer iterations involving

FIG. 2. Scattering by an arbitrarily positioned sphere of an elevation-focused incident beam with -6 dB levels noted (dash-dotted lines). The central plane of the beam (dotted line) passes through the coordinate origin ($\times$), and its intersection with the scattering spheres defines the two-dimensional scattering geometry.

the larger S_0 . The efficiency of the successive-bounce approach improves as the radius of S_0 increases relative to the radii of the interior scatterers, since the ratio of the harmonic bandwidths of the interior fields and the fields contributed by S_0 will diminish in this case. In addition, problems in which the dominant effect may be attributed solely to the enclosing object, or when lossy materials are involved, will tend to see successive updates diminish rapidly in magnitude and are ideally suited to the successive-bounce approach.

E. Elevation-focused beams and two-dimensional scattering

Elevation-focused beams with an incident azimuthal angle ϕ can be represented as a superposition of plane waves with coefficients dependent on the polar angle θ :

$$\xi(\mathbf{r}) = \int_0^\pi d\theta T(\theta) e^{ik \cdot \mathbf{r}}, \quad (23)$$

where $\mathbf{k} = k(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ and k is the wave-number of the water background. Complex focusing coefficients $T(\theta)$ represent the Fourier transform of the lens factor given in Eq. (18) of Ref. 5. As illustrated in Fig. 2, elevation focusing concentrates most of the incident energy in a narrow strip around the central plane of the beam. The same lens factor may also be used to focus the received scattered field, resulting in a region of interest that is confined to an

FIG. 3. (Color online) Left: Arrangement of twelve tissue-mimicking spheres of various acoustic properties embedded in a spherical reference medium. Right: Cross section at $z=0$ of the sphere arrangement. Tissue types are labeled “F” for fat, “M” for muscle, and “C” for connective tissue.

TABLE I. Material properties of spheres designed to mimic human tissue.

Tissue	Sound speed (m/s)	Density (kg/m ³)	Absorption (dB/cm MHz)
Water	1509.0	997.0	0.00
Reference	1570.0	970.0	0.30
Fat	1478.0	950.0	0.52
Muscle	1547.0	1050.0	0.91
Skin	1613.0	1120.0	1.61

even narrower strip around the central plane of the beam. Focused received fields $\varphi_{s,f}$ are related to the unfocused scattered field φ_s by

$$\varphi_{s,f}(\mathbf{r}) = \int_0^\pi d\theta T^*(\theta) \varphi_s(\mathbf{r}), \quad (24)$$

where T^* is the complex conjugate of the transmission coefficient distribution T . The azimuthal integrations on transmit and receive reduce the three-dimensional scattering problem to a two-dimensional problem with a scattering object that is approximated by the cross section of the three-dimensional object in the focal plane.

III. NUMERICAL RESULTS

A preliminary numerical result is presented to validate the successive-bounce approach for the calculation of scattering from a sphere containing other spheres. Additional results utilize adaptations of this technique to problems in both two and three dimensions. Canonical studies investigate the error of a two-dimensional result as an approximation to three-dimensional scattering in the presence of an elevation-focused, incident plane wave. These studies are a continuation of the result in Ref. 5 that explore more complicated configurations than possible using only analytic solutions to the sphere scattering problem. Finally, a more complex result is presented that compares two-dimensional scattering to three-dimensional scattering of an elevation-focused beam. Comparisons of scattered fields are made using the mean squared error (MSE) between the patterns defined as

$$\text{MSE} = \sqrt{\frac{\int_{\Omega} d\hat{s} |\varphi_s(\hat{s}) - \varphi_{s,r}(\hat{s})|^2}{\int_{\Omega} d\hat{s} |\varphi_{s,r}(\hat{s})|^2}}, \quad (25)$$

where Ω is the unit circle in two dimensions or the unit sphere in three dimensions, φ_s is a scattered field, and $\varphi_{s,r}$ is the corresponding reference field.

A. The successive-bounce approach

Validation of the successive-bounce approach used a collection of homogeneous spheres contained within a sphere. The configuration of these spheres is shown in Fig. 3. The properties of these objects were selected to mimic the properties of various human tissues. The reference tissues and their properties are listed in Table I, while the characteristics of the internal spheres are listed in Table II. The enclosing sphere, centered at the origin of the coordinate system, had a radius of 24 mm and was composed of the listed reference material.

TABLE II. Characteristics of the spheres in the tissue-mimicking phantom.

Radius (mm)	Center (mm)			Tissue
	x	y	z	
4.0	0.0	0.0	0.0	Fat
5.0	14.0	2.0	4.0	Skin
5.0	5.0	-10.0	-4.0	Fat
3.0	17.0	-7.0	0.0	Fat
7.8	-10.0	10.0	7.2	Muscle
7.8	5.0	12.0	-7.2	Muscle
5.0	14.0	12.0	3.0	Muscle
5.0	-5.0	-18.0	-3.0	Skin
2.5	7.5	-18.0	-2.0	Skin
1.5	-4.0	20.0	0.0	Skin
2.5	-18.0	4.0	2.0	Skin
9.1	-12.5	-5.0	-5.2	Muscle

Scattering of a 2.5 MHz plane wave incident from the $\theta=90^\circ$, $\phi=0^\circ$ direction was calculated. The resulting polar and azimuthal scattering patterns are shown in Fig. 4 for both the successive-bounce approach and a single-system solution. The MSE between the solutions is 0.017%, with no distinguishable difference between the results produced by the two methods.

The matrix equations formulated in both the successive-bounce and single-system approaches were solved using the generalized minimal residual (GMRES) iterative solver.²⁸ The associated residual norms for each method are shown in Fig. 5 as a function of iteration count. Convergence of the iterative method was declared when the residual norm fell below 5×10^{-6} . For the single-system approach, this limit was reached after eight iterations, and no further action was required. In the successive-bounce approach, the inner iterations compute the fields scattered by the internal spheres for a fixed approximation to the internal field of the parent sphere. Once the iterations have converged, the internal field must be updated and the process repeated. For this problem, the internal field required three updates before an overall convergent solution was achieved, for a total of four outer iterations. This required a total of 16 GMRES iterations. De-

FIG. 4. (Color online) The polar and azimuthal scattering patterns computed for 12 spheres embedded in a larger reference sphere using both a single system of equations and the successive-bounce approach.

FIG. 5. (Color online) Convergence rate for the successive-bounce approach to scattering from spheres embedded in a parent sphere. Also shown is the convergence rate of the reference solution, computed using a single system of equations to relate all field components.

spite the increased iteration count, the reduced complexity of the inner iterations and reduced evaluation of the boundary conditions of the parent sphere produced a successive-bounce solution in less than half the time of the single-system approach. On a four-processor computer system, shared-memory parallel implementations of both the successive-bounce and single-system approaches produced final solutions in 62 and 135 s, respectively.

B. Canonical focusing studies

Canonical experiments were conducted with simple arrangements of spheres to evaluate the quality of two-dimensional approximations to scattering of elevation-focused incident plane waves by three-dimensional spheres, as depicted in Fig. 2. All studies were performed first with spheres embedded in an infinite water background, and then with the arrangements enclosed in a larger sphere, centered at the coordinate origin, composed of the reference material listed in Table I and with a radius of 24 mm. Experiments involving an enclosing sphere offer insight into the error contributed to two-dimensional approximations to elevation-focused scattering by interior scatterers in laboratory phantoms such as that shown in Fig. 3.

The incident (23) and receive (24) focusing coefficients were selected to simulate the elevation-focused plane wave produced by the 75-mm-radius ring transducer array described in Ref. 5. The elements of this array were 15 mm

FIG. 6. (Color online) Amplitude of plane wave as a function of elevation and range for transmit-only (top) and transmit-receive (bottom) focusing. Solid and dashed lines represent the -3 and -6 dB beam heights, respectively. The dynamic range of the plots is 20 dB.

high and were apodized with a Gaussian window. All presented canonical studies assumed a 2.5 MHz plane wave incident from $\phi=0$. The coordinate system was chosen so that the peak amplitude of the focused beam coincided with the origin. The amplitude of the elevation-focused incident beam is shown in the top panel of Fig. 6 over the 48 mm range of interest for these studies. The -6 and -3 dB beam heights were very uniform over the range of interest, with an -6 dB beam height of approximately 5 mm. The product of the elevation-focused incident beam and the focused receiver sensitivity is shown in the bottom panel of Fig. 6. The receiver sensitivity reduced the height of the effective illumination region, but also reduced the uniformity of the field near the edges of the range of interest. For comparison, experiments were also conducted with no receiver focusing by sampling the three-dimensional scattered field confined to the $\theta=90^\circ$ plane.

The first set of experiments considered scattering from a solitary sphere composed of fat. The sphere was centered on the coordinate origin, and the radius of the sphere was varied from 0.6 mm (one wavelength) to 24 mm. As shown in the top-left panel of Fig. 7, when there was no enclosing sphere, the MSE of the two-dimensional approximation decreased as the radius increased for both transmit-only and transmit-receive focusing. This improvement reflected the reduced variation of cross-section diameters of larger spheres throughout the focal zone. The bottom-left panel of Fig. 7

FIG. 7. (Color online) Error of two-dimensional scattering computations for a single sphere with a variable radius (left) or elevation offset (right). The results when a 24 mm reference sphere surrounds the variable sphere are also shown.

FIG. 8. Classes of scattering experiments involving two spheres. The x axis represents the direction of propagation of the incident beam. Class (a): One sphere is vertically offset from the second sphere, centered on the origin. Class (b): Two spheres are each separated by a lateral offset from the origin, parallel to the direction of wave propagation ("inline"). Class (c): Two spheres are each separated by a lateral offset from the origin, perpendicular to the direction of wave propagation ("perpendicular").

shows results from the same computations when there was an enclosing sphere. In the presence of the enclosing sphere, the error initially decreased as the sphere grew beyond the boundaries of the beam. However, for very large inner spheres, the two-dimensional approximation began to diverge from the three-dimensional results. This effect is attributed to increased out-of-beam interactions between the inner and outer spheres that were not represented by the two-dimensional model.

In the second set of experiments, the radius of the fat sphere was fixed at 7 mm and the location of the sphere was placed at $(0,0,z)$, where z varied between 0 and 6.9 mm. The results of the offset experiments are presented in the right panel of Fig. 7. As expected, the error of the two-dimensional result increased with increasing offset, because the sphere tended to scatter more energy outside of the measurement zone, and also because the diameters of the sphere cross sections had more variation within the focal zone. The decline in error in the transmit-receive experiment as the offset approached 7 mm can be explained by the narrower beam height, which resulted in weaker scattering that was better approximated by the diminishing radii of cylinders in two dimensions.

Three distinct classes of scattering experiments were conducted with two interior scattering spheres. As illustrated in Fig. 8, these experiments consider vertically offset spheres (separation in z), inline laterally offset spheres (parallel to x and the direction of propagation), and perpendicular (to x and the direction of propagation) laterally offset spheres. In all experiments, the spheres had fixed radii of 7 mm and were composed of fat. As with the previous experiments, these variations were also tested when confined within an enclosing sphere composed of reference material.

For the experiment involving vertically offset spheres, one sphere was centered on the origin and the other was centered on the z axis with a vertical offset ranging from 8.7 to 16.9 mm. The second sphere was never within the -6 dB height of the beam. Regardless of the separation between the spheres (provided the two spheres did not touch), the differences between single-sphere scattering and two-sphere scattering resulted in MSE values below 1% for transmit-receive focusing and below 2% for transmit-only focusing. Because the incident beam did not interact with the second sphere, the quality of the two-dimensional approximation did not differ from that for the corresponding single-sphere experiment.

FIG. 9. (Color online) Error of two-dimensional scattering computations for two spheres with variable lateral offsets in inline and perpendicular arrangements. The results when the two spheres are enclosed in a reference sphere with a 24 mm radius are also shown.

The errors of two-dimensional approximations to the three-dimensional experiments involving both inline and perpendicular lateral offsets are shown in Fig. 9 for transmit-receive focusing. The reported lateral offset is the distance from the center of each sphere to the origin. The two-dimensional inline configuration suffered from significantly more error, because scattering from the far sphere was strongly influenced by propagation of the incident and scattered fields through the near sphere. When the two spheres were surrounded by an enclosing sphere, the error began to increase as the lateral offset approached 16.9 mm, placing the interior spheres close to the boundary of the reference sphere. This phenomenon is attributed to out-of-beam interactions between the objects that were not modeled by two-dimensional approximations. In the perpendicular arrangement, this tendency was much more pronounced, since interactions among the interior objects were much less significant to the overall pattern, increasing the significance of interactions with the enclosing sphere.

In all experiments but the two-sphere, inline configuration, the results with an enclosing sphere exhibited lower error than that of the reference sphere with no enclosing object. The reason for this is shown in Fig. 10, which depicts the field inside the reference sphere. The enclosing sphere acted as a lens, focusing the incident energy into a region with a much narrower height than either the transmit-only or the transmit-receive focused beams shown in Fig. 6. When the interior object was located in the vicinity of the strong

FIG. 10. (Color online) The field magnitude inside a reference sphere with a radius of 24 mm as a function of elevation and range in the direction of travel of the incident field. The solid and dashed lines represent the -3 and -6 dB beam levels, respectively. The dynamic range of the plot is 21 dB, with each contour step representing a magnitude difference of 3 dB.

FIG. 11. (Color online) Scattering pattern for a tissue-mimicking phantom as computed with transmit-receive focusing, no focusing, and the equivalent two-dimensional approximation.

field, scattering by the interior object became more significant, and the extremely small beam height improved the quality of the two-dimensional approximation.

C. A tissue-mimicking phantom

The presented approach was used to compare the three-dimensional scattering of the phantom in the left panel of Fig. 3 to the two-dimensional scattering of the $z=0$ slice shown in the right panel of Fig. 3. A nonfocused result was generated in three dimensions using a single plane wave incident from $\theta=90^\circ$, $\phi=0$ and point-source reception in the $\theta=90^\circ$ plane. The transmit-receive focused result was generated using the elevation-focused incident field and receiver sensitivity used for the previously described canonical experiments. The computed patterns are shown in Fig. 11. The MSE between the two-dimensional result and the nonfocused, three-dimensional pattern was 63.4%. With transmit-receive focusing in three dimensions, the MSE of the two-dimensional calculations was reduced to 25.5%. While the error was higher than error observed in canonical studies with an enclosing sphere, transmit-receive focusing yielded a dramatic reduction in two-dimensional approximation error, with patterns exhibiting coarse-scale agreement within the region centered around the forward-scattering direction $\phi=180^\circ$. This agreement was not observed with the nonfocused result.

The canonical studies for one and two spheres suggest that considerable error was caused either by elevation offsets that shifted scattering spheres outside of the incident beam or by small spheres that were entirely contained within the beam. To eliminate the impact of elevation offsets, the phantom was modified to ensure that all spheres were centered at $z=0$. The x and y coordinates of each sphere in the phantom were identical to those listed in Table II, but the radii of all spheres were reduced so that the $z=0$ cross section of the modified phantom is identical to the right panel of Fig. 3. A comparison of the modified three-dimensional results and the two-dimensional approximation, with a significantly reduced MSE of 17.9%, is shown in the top panel of Fig. 12. The effects of small scattering spheres were eliminated by remov-

FIG. 12. (Color online) Top: Scattering pattern for the tissue-mimicking phantom modified such that all spheres are centered in the incident beam and the $z=0$ cross section is identical to that of the original phantom. Bottom: Scattering pattern for the modified phantom after the removal of spheres with radii less than 4 mm.

ing spheres with radii less than 4 mm from the modified phantom. The MSE of the two-dimensional approximation to scattering from this restricted phantom, shown in the bottom panel of Fig. 12, was further reduced to 8.0%.

IV. DISCUSSION

Three aspects of the described research warrant further discussion. The first aspect is the monitoring of convergence of the solution when the successive-bounce approach is applied. Iteration based on a single, large system of equations can employ the traditional residual norm to monitor convergence of the solution. However, in the successive-bounce approach, no single scattering operator exists for which the residual norm can be computed. The second aspect is the suitability of the demonstrated method for more generalized scattering geometries. A hierarchical implementation of this method facilitates the solution of complicated arrangements of homogeneous that are not restricted to a simple embedding of such structures in a larger object. The third aspect is the error observed in the elevation-focused scattering experiments. The results suggest two-dimensional approximations are unlikely to yield high-accuracy solutions to elevation-focused scattering problems even when large scatterers are centered in the focal zone.

A. Convergence tracking in the successive-bounce approach

Iterative methods often attempt to approximately invert a matrix system of the form $Ax=b$ by minimizing the cost functional

$$f(\tilde{x}) = \|A\tilde{x} - b\|, \quad (26)$$

where $\tilde{x}$ is an approximation to x and $\|\cdot\|$ is an appropriate norm. Convergence of the method is monitored by computing the residual norms associated with each iterative solution x_i . These residual norms are defined as

$$\|r_i\| = \|Ax_i - b\|. \quad (27)$$

However, such convergence monitoring is not possible when using the successive-bounce approach, because the internal scattering operator and the scattering from S_0 are decoupled.

Convergence of the overall solution can be monitored by considering the norms of the updates to the approximation of the internal field. From the definition of the approximate internal field (21), the approximation may be written as

$$\varphi_{\text{int},0}^{(n+1)}(\mathbf{r}) = \varphi_{\text{int},0}^{(n)}(\mathbf{r}) + [R_s D^{-1}]^{n+1} T_s \xi(\mathbf{r}). \quad (28)$$

The term $[R_s D^{-1}]^{n+1} T_s \xi(\mathbf{r})$ constitutes an update u_{n+1} to the field $\varphi_{\text{int},0}^{(n)}$. The form of Eq. (28) suggests that the successive-bounce approach can be identified as a modification of the Born series. In the Born series, scattering computations are separated into terms that have the same number of reflections. Thus, the n th partial sum of the Born series includes all scattering that results from sequences of at most n reflections. Similarly, the scattering accumulated from n successive-bounce computations includes all terms that result from sequences with n or fewer reflections from the boundary of S_0 , as well as any number of reflections from child scatterers. Since these terms include all of the reflection sequences in the n th partial sum of the Born series, the successive-bounce iterations can be expected to converge more rapidly than the Born series. The presence of lossy materials will further diminish the contribution of high-order terms, accelerating the convergence of the successive-bounce approach. In either case, the updates $u_n \rightarrow 0$ as $n \rightarrow \infty$. For the purposes of this research, convergence of the successive-bounce approach was deemed achieved once the amplitude of an update u_n fell below a prescribed level.

B. Generalizations of the scattering geometry

Although the formulation presented above was limited to homogeneous radial objects embedded in a single, larger homogeneous radial object, the method is readily adapted to compute fields scattered by more complicated structures. The presented technique is not applicable to truly arbitrary scattering media, but a hierarchical adaptation of this method is capable of computing scattering from multiply nested spherical or cylindrical objects. The T -matrix method requires for each object only the evaluation of an arbitrary local scattering operator to an incoming field. If a radial object is homogeneous, analytic quotients of the form (9) are used to construct the local scattering operator. However, if a radial object contains inclusions of its own (forming a local scattering group), the local scattering operator is represented as the scattering of a given incident field by the group, neglecting all scatterers outside of the group. This scattering may be calculated from a single system of equations, as described in Sec. II C, or by using a localized application of the successive-bounce approach discussed in Sec. II D. This hierarchical procedure is analogous to the multilevel fast multipole methods^{11,23} that recursively subdivide groups of scattering elements.

A special case of multiply nested scatterers is that of layered objects. If a scatterer consists of a finite number of radially symmetric layers, then the relationship between the

scattered field and the incident field expansion can be derived analytically and analogously to the relationship for homogeneous radial objects. This analytic relationship allows the presented approach to handle inclusions consisting of multiple layers with no loss in efficiency relative to the case of homogeneous inclusions.

C. Two-dimensional approximations to elevation-focused scattering

In both the canonical studies and the scattering experiments involving tissue-mimicking phantoms, significant contributions to the error of two-dimensional approximations to elevation-focused scattering can be attributed either to scatterers with diameters smaller than the incident beam height, or to scatterers whose centers deviate from the elevation center of the incident beam. Both of these factors contribute to significant out-of-plane interactions that are not captured by the two-dimensional problem. However, even when these factors are absent from the scattering geometry, considerable two-dimensional approximation error (approaching 10%) was still observed even for simplistic geometries. Much of this error is present in the backscattering region, but appreciable error remains in the forward direction. In applications where the backscattering measurements are not significant, two-dimensional approximations can be expected to perform somewhat better. However, when backscattering measurements are important, the need for fully three-dimensional scattering models is apparent.

V. CONCLUSION

A method for solving for scattering in the presence of nested, homogeneous, radially symmetric objects has been presented in a generalized form that is easily adapted to two-dimensional and three-dimensional geometries. In addition, a technique was developed to solve the scattering problem efficiently for objects contained within larger objects. This method, based on successive updates to an approximation of the field inside the large object, has been demonstrated to be more efficient for a complex scattering geometries than a more traditional approach that relates all fields in a single, large system of equations. The methods were used to conduct numerical experiments that illustrate the error associated with two-dimensional approximations to three-dimensional scattering of elevation-focused beams for both canonical and complicated, realistic geometries. The experiments suggest that considerable error can be attributed to small scatterers or scatterers offset from the beam center. However, controlling for these factors is not sufficient to achieve arbitrary accuracy in the approximate, two-dimensional solution.

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