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Blood Flow in Small Curved Tubes

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Abstract

Blood flow in small curved tubes is modeled by the two-fluid model where a relatively cell-free fluid layer envelops a fluid core of higher viscosity. The parameters in the model are successfully curve fitted to experimental data for straight tubes. The curved tube equations are then solved by perturbation theory. It was found that curvature in general lowers the tube resistance, but increases the shear stress near the inside wall.

1 Introduction

Blood can be considered as homogeneous in large blood vessels, but in small tubes the rheology shows considerable non-homogeneous behavior [1,2]. Due to the particulate nature of red blood cells, there is an almost cell-free layer of plasma near the vessel wall. The major effect of this layer is to decrease the apparent viscosity or resistance (the Fahraeus-Lindqvist effect), especially in medium to moderately small blood vessels. For vessels of 50-1000 microns, the phenomenon can be adequately described by the two-fluid model. First proposed by Vand [3] for general suspensions, the model assumes a homogeneous fluid core enclosed by a fluid annulus of lower viscosity. The two-fluid model has been applied to blood flow in straight tubes, notably by Hayes [4] and Sharan and Popel [5]. The theoretical derivation of the resistance formula of this model and our simple curve fit with blood flow data in small straight tubes are given in the Appendix.

In the microvasculature there are numerous instances where the blood vessels are curved or tortuous [6], especially in the diseased state. Also in biomechanical instrumentation blood may be transported through small curved tubes. It is the aim of this study to investigate the effect of curvature on the wall shear and the resistance of the flow through such a tube.

We shall use a coordinate system first proposed by Dean [7,8] to treat the flow in a circular tube of small constant curvature. The inertial effects are neglected since the Reynolds number in the microcirculation is typically very small (of order 10^{-3}). The Dean number, proportional to $(\text{curvature})(\text{Reynolds number})^2$, is even smaller. The secondary recirculation, which is prevalent for large curved tubes, would be entirely absent.

Formulation

Let the tube be of radius a , with a constant centerline curvature K . An orthogonal system can then be constructed from the elemental distance squared (Fig. 1a):

$$|d\vec{x}|^2 = (dr)^2 + r^2(d\theta)^2 + L^2(ds)^2 \quad (1)$$

where the center line scale factor is

$$L \equiv 1 - Kr \cos \theta \quad (2)$$

(r, θ) are local polar coordinates and s is the coordinate along the center line. Since the inertial terms are absent, due to symmetry one can show only the velocity component $w(r, \theta)$ in the s direction exists. Using the scale factors from Eq. (1) (e.g. Batchelor [9]) the Stokes equation reduces to:

$$\frac{\mu}{r} \left\{ \left[\frac{r}{L} (Lw)_r \right]_r + \left[\frac{1}{rL} (Lw)_\theta \right]_\theta \right\} = \frac{G}{L} \quad (3)$$

Here G is the constant center-line pressure gradient (assumed negative along s), and μ is the viscosity. Fig. 1b shows the two regions. Region *I* is the core containing red blood cells with velocity w_I and viscosity μ_I and Region *II* is the cell-free layer of thickness t from the vessel wall and with velocity w_{II} and viscosity μ_{II} . Eq. (3), with the same pressure gradient, is solved separately for the two regions. The boundary conditions are that on the axis (at $r = 0$) w_I is bounded, on the wall (at $r = a$) there is no slip, i.e. $w_{II}(a, \theta) = 0$ and velocities and shear stresses match on the interface at $r = a - t$:

$$w_I = w_{II} \quad (4)$$

$$\mu_I L \left(\frac{w_I}{L} \right)_r = \mu_{II} L \left(\frac{w_{II}}{L} \right)_r \quad (5)$$

Solution

The solution for the two-fluid flow in a straight circular tube is given in the Appendix. We shall perturb from this solution. Let the ratio of the tube radius to the radius of curvature of the center line be small, i.e.:

$$aK = \varepsilon \quad (6)$$

where $\varepsilon \ll 1$. (Actual values of ε is between 0 and 0.3 obtained from Ref. [6]). The other variable are normalized to order unity as follows:

$$\eta \equiv r/a \quad (7)$$

$$w_I = U(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots) \quad (8)$$

$$w_{II} = \alpha U(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots) \quad (9)$$

Here $U = |G|a^2/4\mu_I$ is a normalized pressure gradient, $\alpha \equiv \mu_I/\mu_{II}$, and w_0, u_0 is the normalized straight tube solution (Eqs. (A2-A4)):

$$w_0 = C - \eta^2, \quad u_0 = 1 - \eta^2 \quad (10)$$

where C is defined in Eq. (A4). Upon substitution of Eqs. (8,9) into Eqs. (3) the first-order equation is found to be

$$4\eta \cos\theta + \frac{1}{\eta} [\eta(w_1 - \eta w_0 \cos\theta)_{\eta} + \eta^2 \cos\theta w_{0\eta}]_{\eta} + \frac{1}{\eta^2} (w_1 - \eta \cos\theta w_0)_{\theta\theta} = 0 \quad (11)$$

The first-order equation for w_{II} is similar to Eq. (11), only with w replaced by u . Together with the perturbed boundary conditions Eqs. (4,5) the solution is:

$$w_1 = \cos\theta \left[C_1 \eta - \frac{3}{4} \eta^3 \right] \quad (12)$$

$$u_1 = \cos\theta \left[\frac{3}{4} (\eta - \eta^3) + C_2 \left(\eta - \frac{1}{\eta} \right) \right] \quad (13)$$

where $\delta \equiv t/a$ and

$$C_1 = \frac{\{6 + (\alpha - 1)[18\delta - (16\alpha + 21)\delta^2 + (4\alpha + 3)(4\delta^3 - \delta^4)]\}}{4[2 + 2(\alpha - 1)\delta - (\alpha - 1)\delta^2]} \quad (14)$$

$$C_2 = \frac{7(\alpha - 1)\delta(2 - \delta)(1 - \delta)^2}{4[2 + 2(\alpha - 1)\delta - (\alpha - 1)\delta^2]} \quad (15)$$

Notice the outer region velocity u_1 is zero on the tube wall but may include terms which are singular at the center. Since w_1, u_1 are periodic in θ , they do not contribute to the net flow rate, which is of higher order. However, for the second-order flow rate correction only the non-periodic part is needed.

Let an over bar denote the average with respect to the angle θ . The averaging is then applied to the second-order terms of Eq. (3). The result is:

$$2\eta^3 - \left[\eta \left(\frac{1}{2} \eta w_0 + \overline{\cos \theta w_1} - \overline{w_2 \eta} \right) \right]_{\eta} = 0 \quad (16)$$

with a similar equation for u_2 . After some work, the solution is:

$$\overline{w_2} = C_3 + \frac{C+C_1}{4} \eta^2 - \frac{11}{32} \eta^4 \quad (17)$$

$$\overline{u_2} = -\frac{7+4C_2}{16} (1-\eta^2) + \frac{11}{32} (1-\eta^4) + C_4 \ln \eta \quad (18)$$

where

$$C_3 = \frac{-1}{32[2+2(\alpha-1)\delta-(\alpha-1)\delta^2]} \times \left\{ 2+(\alpha-1)[10\delta+(8\alpha-21)\delta^2-8(2\alpha-3)\delta^3 \right. \\ \left. +2(7\alpha+8)\delta^4-6(\alpha-1)\delta^5+(\alpha-1)\delta^6 \right. \\ \left. -28\alpha\delta(2-\delta)(1-\delta)^2 \ln(1-\delta) \right\} \quad (19)$$

$$C_4 = \frac{21(\alpha-1)(1-\delta)^2\delta(2-\delta)}{8[2+2(\alpha-1)\delta-(\alpha-1)\delta^2]} \quad (20)$$

The net flow rate is then integrated

$$F = 2\pi a^2 U [q_0 + \varepsilon^2 q_2 + O(\varepsilon^4)] \quad (21)$$

where

$$q_0 = \int_0^{1-\delta} w_0 \eta d\eta + \alpha \int_{1-\delta}^1 u_0 \eta d\eta = \frac{\alpha}{4} \left[1 - (1-\delta)^4 \left(1 - \frac{1}{\alpha} \right) \right] \quad (22)$$

$$q_2 = \int_0^{1-\delta} \overline{w_2} \eta d\eta + \alpha \int_{1-\delta}^1 \overline{u_2} \eta d\eta \\ = \frac{1}{192[2+2(\alpha-1)\delta-(\alpha-1)\delta^2]} \\ \times \left\{ 2+(\alpha-1)[14\delta-(576\alpha+43)\delta^2+4(432\alpha+19)\delta^3 \right. \\ \left. -(1856\alpha+85)\delta^4+(832\alpha+62)\delta^5-(120\alpha+29)\delta^6 \right. \\ \left. -8(\alpha-1)\delta^7+(\alpha-1)\delta^8 \right\} \quad (23)$$

The above computations are facilitated by a computer program with symbolic capabilities

Results and Discussions

Figure 2 shows some typical velocity distributions. The zeroth-order is a blunted parabola form which has been observed in experiments using straight tubes [2]. The first-order correction is due to the curving of the vessel. Contrary to high Reynolds number flows, the velocity near the inner surface is increased while that near the outer surface is decreased. Consequently the shear stress is higher on the inside wall near point A in Fig. 1a. The shear stress on the vessel wall is:

$$\begin{aligned}\tau &= \mu_{II} w_{rr} = \frac{|G|a}{4} (u_{0\eta} + \varepsilon u_{1\eta} + \dots) \Big|_{\eta=1} \\ &= \frac{|G|a}{4} \left[-2 + \varepsilon \cos\theta \left(-\frac{3}{2} + \frac{7(\alpha-1)\delta(2-\delta)(1-\delta)^2}{2[2+2(\alpha-1)\delta-(\alpha-1)\delta^2]} \right) \right]\end{aligned}\quad (24)$$

Whether higher local shear stress is the cause of vessel tortuosity is still uncertain at present. Fig. 3a shows a typical zeroth-order net flow rate q_0 versus the thickness-radius ratio. For smaller vessels (larger δ) the flow rate can be as much as 50% more due to the lowered viscosity of the cell-free layer. Fig. 3b shows a typical first-order flow rate correction q_2 due to curvature effects. Notice q_2 becomes negative when δ is larger than 0.026. This means the resistance of a larger curved vessel is lower than that of a straight vessel, but the resistance of a smaller curved vessel is higher than a straight vessel of the same size. These properties would persist even when the curvature is not small.

We can also solve Eq. (3) numerically. This is not recommended since the parameter ε is indeed small and the perturbation method is adequate. Furthermore perturbation solutions, being exact, do show more clearly the dependence of the various parameters.

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Appendix-Two-Fluid Flow in a Straight Tube

For a straight tube the curvature K is zero and thus the scale factor $L = 1$. Eq. (3) becomes the Poisson equation:

$$\frac{\mu}{r} \left[(rw_r)_r + \frac{1}{r} w_{\theta\theta} \right] = G \quad (A1)$$

Let $\delta \equiv t/a$ and $\alpha \equiv \mu_I/\mu_{II}$. The axisymmetric solution that satisfies the conditions of boundedness, no slip, and Eqs. (4,5) is:

$$w_I = \frac{|G|a^2}{4\mu_I} [C - (r/a)^2] \quad (A2)$$

$$w_{II} = \frac{|G|a^2}{4\mu_{II}} [1 - (r/a)^2] \quad (A3)$$

where

$$C = (1 - \delta)^2 - \alpha[(1 - \delta)^2 - 1] \quad (A4)$$

The total flow can be integrated

$$Q = \int_0^{a-t} w_I 2\pi r dr + \int_{a-t}^a w_{II} 2\pi r dr \\ = \frac{\pi |G| a^4}{8\mu_{II}} \left[1 - (1 - \delta)^4 \left(1 - \frac{1}{\alpha} \right) \right] \quad (A5)$$

In order to compare with experiments, we define an apparent viscosity

$$\mu_{app} = \frac{\pi |G| D^4}{128Q} \quad (A6)$$

where $D = 2a$ is the inside tube diameter, and a relative apparent viscosity

$$\mu_{rel} \equiv \frac{\mu_{app}}{\mu_{II}} \quad (A7)$$

Using Eq. (A5) we find

$$\mu_{rel} = \frac{1}{1 - (1 - \delta)^4 (1 - 1/\alpha)} \quad (A8)$$

This is essentially the result of Vand [3]. The apparent viscosity is that of the core when the thickness of the cell-free layer is zero, and is the plasma viscosity when the thickness equal the radius of the tube. Using in vitro experiments (in glass tubes) of 19 sources compiled by Pries et al. [1] and Secomb [10] we curve-fitted Eq. (A8) for each hematocrit as shown in Fig. 4. The fitted values of α and $t = D \delta/2$ are shown in Table 1.

The error of the curve fit is within 3% for diameter D from 40 to 1000 microns. We keep in mind that any shear dependence from rouleaux formation, cell deformation, rotation, wall effects, nonparticle free plasma layer, etc are included in the apparent curve fit of the two parameters. Also included is the relation between the tube hematocrit and the feed hematocrit, upon which the data was based. Thus it is not necessary to consider such complicated effects for the two-fluid model. For in vivo experiments with blood vessels instead of glass tube, the experiments show higher apparent viscosity, probably due to the uneven glycocalyx wall. The curves however can be fitted similarly.

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Fig. 1.

(a) The curved tube coordinates (r, θ, s) . The direction s is along the center line. (b) Axial cross section showing two fluid regions and the blunted velocity profile.

Fig. 2. Typical velocity profiles ($\delta=0.1, \alpha=3.2, \theta=0$). The first-order correction w_1 due to curvature is anti-symmetric.

Fig. 3. The flow rate as a function of δ . ($\alpha=3.317$). q_0 is for the straight tube, q_2 is the correction due to curvature.

Fig. 4.
Relative apparent viscosity as a function of diameter — glass tube experiments [10], — — —
Eq. (A8)

Table 1
Values of α and t fitted to experimental data in Fig. 4

<i>Hct</i>	15%	30%	45%	60%
α	1.52	2.22	3.34	5.04
t	3.77	2.54	2.03	1.25