Noncommutative Jordan Algebras with Commutators Satisfying an Alternativity Condition*

Richard E. Block

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA (RIVERSIDE)

Communicated by Adrian Albert, October 13, 1969

Abstract. The theorems of this paper show that the main results in the structure and representation theory of Jordan algebras and of alternative algebras are valid for a larger class of algebras defined by simple identities which obviously hold in the Jordan and alternative case. A new unification of the Jordan and associative theories is also achieved.

The structure and representation theories of alternative algebras on the one hand and of Jordan algebras on the other have closely parallel results. It is therefore desirable to find a suitable class of algebras which contains these two classes and for which the same results hold. The noncommutative Jordan algebras form too large a class (they include nil simple algebras, and fail to satisfy the Wedderburn principal theorem and the complete reducibility theorem), but form a natural and well-studied starting point. In the results announced in this note, we consider a class of noncommutative Jordan algebras which contains the alternative algebras and the Jordan algebras and which is defined by simple conditions on the commutators. For this class we can obtain the usual structure and representation theorems and show that the simple algebras are all alternative or Jordan. This generalizes the recent work of Schafer¹ on generalized standard algebras, which are noncommutative Jordan algebras satisfying three complicated further identities. In addition to using identities which are more natural and weaker than Schafer's, we also obtain the representation theory. Moreover, each of our theorems gives a new result on the unification of the associative and Jordan theories, generalizing the work on standard algebras of Albert and of Schafer.² Our result on the simple algebras also partially generalizes the results of several authors³ on classes of simple rings.

Before stating the theorems explicitly we give some notation and definitions. The algebras considered (while of course not necessarily associative) are assumed to be finite-dimensional over a field of characteristic $\neq 2$. As usual, (x, y, z) denotes the associator $(xy)z - x(yz)$, and $[x, y]$ the commutator $xy - yx$. Recall that an algebra A is (commutative) *Jordan* if $[x, y] = 0 = (x, y, x^2)$ ($x, y \in A$), and *alternative* if $(x, y, y) = 0$ (the right alternative law) and $(y, y, x) = 0$ (the left alternative law). A generalization of commutativity is the *flexible law*:

$$(x, y, x) = 0 \quad (x, y \in A), \tag{1}$$

which also holds in all alternative algebras. *Noncommutative Jordan algebras*

are defined as (possibly commutative) flexible algebras satisfying the Jordan identity

$$(x, y, x^2) = 0 \quad (x, y \in A). \quad (2)$$

A triple x, y, z in A is called *right alternative* if it satisfies the linearized form of the right alternative law:

$$(x, y, z) + (x, z, y) = 0.$$

We now introduce our key definition. A subset M of a flexible algebra is called *completely alternative* (respectively *semi-completely alternative*, respectively *strongly alternative*) if a triple x, y, z in A is right alternative whenever $x \in M$ or $y \in M$ (resp. $x \in M$, resp. $y \in M$).

Our main result is that the basic structure and representation theorems hold for noncommutative Jordan algebras A with $[A, A]$ completely alternative, i.e., for algebras A satisfying identities (1), (2) and

$$(x, [y, z], w) + (x, w, [y, z]) = 0 \quad (x, y, z, w \in A). \quad (3)$$

(Obviously these identities are satisfied by all alternative algebras and all Jordan algebras.) Much of the theory holds under the weaker condition that $[A, A]$ be semicompletely alternative, i.e.,⁴

$$[(x, y), z, z] = 0 \quad (x, y, z \in A), \quad (4)$$

or even just strongly alternative, i.e.,

$$([v, w], [x, y], z) + ([v, w], z, [x, y]) = 0 \quad (v, w, x, y, z \in A). \quad (5)$$

It follows from flexibility that identities (3), (4), and (5) are equivalent to their left alternative versions, $[A, A]$ is completely alternative if and only if every triple x, y, z with a member in $[A, A]$ is alternative, and if $[A, A]$ is strongly alternative then a triple is alternative whenever two members are in $[A, A]$.

Henceforth A will always denote a noncommutative Jordan algebra.

THEOREM 1. *If $[A, A]$ is strongly alternative, or if $([A, A], [A, A], A) = 0$, and if B is a nil ideal of A , then there is a chain of $B = B_0 \supset B_1 \supset \dots \supset B_n = 0$ of ideals of A such that $B_j^2 \subseteq B_{j+1}$, $j = 0, \dots, n - 1$ (i.e., B is "generalized Penico solvable").*

COROLLARY. *Under the hypothesis on $[A, A]$, if M is a minimal ideal of A then either M is simple or $M^2 = 0$.*

THEOREM 2. *If $[A, A]$ is semicompletely alternative and if A is nil then A is nilpotent.*

As is well known, if A is semisimple (i.e., has no nonzero nil ideal) then A is a direct sum of simple algebras.

THEOREM 3. *If A is simple, and if $[A, A]$ is strongly alternative or if $([A, A], [A, A], A) = 0$, then A is alternative or Jordan.*

THEOREM 4. *Suppose that A/N is separable (where N is the nil radical of A), and that $[A, A]$ is completely alternative or $([A, A], A, A) = 0$. Then A has a subalgebra $S \cong A/N$ (the Wedderburn Principal Theorem for A).*

Any variety $V(I)$ of algebras (= class of algebras satisfying a set I of identi-

ties) determines the class of I -bimodules or I -birepresentations. For Jordan algebras and for alternative algebras the birepresentations of the separable algebras are completely reducible and the irreducible birepresentations are known.⁵ For birepresentations of noncommutative Jordan algebras McCrimmon⁶ has proved that if A is separable with all simple components of degree at least three then every birepresentation is completely reducible, and has shown that this is false without the hypothesis on degree.

We now write I_{ca} and I_{in} respectively for the sets of identities given by (1), (2), (3) (resp. (1), (2) and $([A, A], A, A) = 0$); i.e. $V(I_{ca})$ (resp. $V(I_{in})$) is the class of noncommutative Jordan algebras with commutators completely alternative (resp. contained in the left nucleus). If A in $V(I_{ca})$ is separable, we shall say that an I_{ca} -birepresentation of A is *essentially alternative* (resp. *essentially Jordan*) if its restriction to the sum of the alternative (resp. Jordan) simple direct summands of A is alternative (resp. Jordan) and if it annihilates the remaining simple direct summands. Thus for simple algebras the word "essentially" may be dropped. We make a similar definition of essentially associative (and essentially Jordan) in the I_{in} case.

THEOREM 5. *Let $I = I_{ca}$ (resp. I_{in}). For any separable algebra in $V(I)$, every I -birepresentation is completely reducible, and every irreducible I -birepresentation is either essentially alternative (resp. essentially associative) or essentially Jordan.*

We have used five conditions on $[A, A]$ in the hypotheses of the above theorems: Namely (a) $[A, A]$ is completely alternative; (b) $([A, A], A, A) = 0$; (c) $[A, A]$ is semicompletely alternative; (d) $([A, A], [A, A], A) = 0$; and (e) $[A, A]$ is strongly alternative. Here (b) and (d) are associative analogs of the alternative concepts (c) and (e), respectively. Obviously $(a) \Rightarrow (c) \Rightarrow (e)$, $(b) \Rightarrow (d)$, and $(b) \Rightarrow (c)$. However there are examples of noncommutative Jordan algebras showing that $(c) \not\Rightarrow (a)$, $(e) \not\Rightarrow (c)$, $(d) \not\Rightarrow (e)$, $(b) \not\Rightarrow (a)$, and $(a) \not\Rightarrow (d)$ [the same holds for (d') $([A, A], A, [A, A]) = 0$ in place of (d)]. Each of the above theorems is best possible in the sense that Theorems 4 and 5 are false if condition ((a) or (b)) is weakened to (c); Theorem 2 is false if condition (c) is changed to ((d) and (e)); and Theorems 1 and 3 are false without some hypothesis such as (d) or (e).

It can be shown that axioms (i) and (iii) for generalized standard algebras¹ imply (a) above, and so the theorems above subsume those of Schafer. Moreover the identities used here are in fact weaker than those for generalized standard algebras, since there exist noncommutative Jordan algebras with commutators in the nucleus which satisfy neither axiom (ii) nor (iii) for generalized standard algebras. These same examples also do not satisfy the axioms of the recently announced work of Kleinfeld, Kleinfeld and Kosier³ (their Lemma 2 shows that their algebras have commutators completely alternative).

The proofs of these results will be presented elsewhere. For the most part they are quite conceptual, as might be expected since the identities used are so simple. The proofs of Theorems 1 and 2 use the corresponding known results for the commutative Jordan algebra A^+ . The proof of Theorem 3 uses the fact that simple noncommutative Jordan algebras over an algebraically closed field are either Jordan, quasiassociative, quadratic or (at characteristic p) nodal. The

proofs of Theorems 4 and 5 make use of a Peirce decomposition. It is shown that if $[A, A]$ is semicompletely alternative and if e is an idempotent of A then A has a Peirce decomposition with almost all of the properties that Schafer gets in his case,¹ and in particular

$$A = A_{11} + A_{10} + A_{\alpha\alpha} + A_{01} + A_{00},$$

where $\alpha = 1/2$ and $A_{ij} = \{x \in A \mid ex = ix, xe = jx\}$ ($i, j = 0, 1, 1/2$). The proofs of Theorems 4 and 5 also use the corresponding known results for Jordan algebras and for alternative algebras.

* This research was supported by the National Science Foundation under grant GP8713.

¹ Schafer, R. D., these PROCEEDINGS, 60, 73 (1968); *J. Algebra*, 12, 386 (1969). The author would like to thank Professor Schafer for kindly supplying a preprint of the latter paper—his work turned the author's attention to this area.

² Albert, A. A., *Trans. Amer. Math. Soc.*, 64, 552 (1948); Schafer, R. D., *Pac. J. Math.*, 29, 203 (1969).

³ Kleinfeld, E., *Canad. J. Math.*, 8, 335 (1956); San Soucie, R. L., *Amer. J. Math.*, 79, 80 (1957); Kleinfeld, E., M. H. Kleinfeld, and J. F. Kosier, *Bull. Amer. Math. Soc.*, 75, 415 (1969), details to appear in *J. Algebra*.

⁴ It is trivial that identities (1) and (3) imply (4); this was pointed out to me by A. Thedy.

⁵ Jacobson, N., *Osaka J. Math.*, 6, (1954); Jacobson, N., *Structure and Representations of Jordan Algebras* (Providence, R.I.: Amer. Math. Soc. Colloquium Publications, 1968), vol. 39.

⁶ McCrimmon, K., *Trans. Amer. Math. Soc.*, 121, 187 (1966).