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Direct determination of geometric alignment parameters for cone-beam scanners

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Abstract

This paper describes a comprehensive method for determining the geometric alignment parameters for cone-beam scanners (often called *calibrating* the scanners or performing *geometric calibration*). The method is applicable to x-ray scanners using area detectors, or to SPECT systems using pinholes or cone-beam converging collimators. Images of an alignment test object (calibration phantom) fixed in the field of view of the scanner are processed to determine the nine geometric parameters for each view. The parameter values are found directly using formulae applied to the projected positions of the test object marker points onto the detector. Each view is treated independently, and no restrictions are made on the position of the cone vertex, or on the position or orientation of the detector. The proposed test object consists of 14 small point-like objects arranged with four points on each of three orthogonal lines, and two points on a diagonal line. This test object is shown to provide unique solutions for all possible scanner geometries, even when partial measurement information is lost by points superimposing in the calibration scan. For the many situations where the cone vertex stays reasonably close to a central plane (for circular, planar, or near-planar trajectories), a simpler version of the test object is appropriate. The simpler object consists of six points, two per orthogonal line, but with some restrictions on the positioning of the test object. This paper focuses on the principles and mathematical justifications for the method. Numerical simulations of the calibration process and reconstructions using estimated parameters are also presented to validate the method and to provide evidence of the robustness of the technique.

1. Introduction

The evolution of scanner calibration methods has followed that of the technologies as well as that of reconstruction methods. Thus, the early papers dealt with two-dimensional (2D) fan-beam calibration whereas the subsequent ones dealt with three-dimensional (3D) cone-beam calibration for circular trajectories, and more recently, for generalized trajectories.

When calibrating fan-beam systems, five parameters are sought: two parameters for the position of the fan vertex, two for the position of the center of the detector, and one for the angular orientation of the detector. The detector pixel size is assumed known. Similarly, when calibrating cone-beam systems, nine parameters are sought: three for the cone vertex, three for the detector position, and three for the detector orientation. However, if a circular trajectory is assumed, then eight parameters are needed, or seven if the angular spacing is

known. For circular trajectories, the detector orientation can be identified using angular parameters θ , ϕ , η defined with respect to the line joining the cone vertex perpendicular to the axis of rotation. If this line is parallel to the detector normal vector then $\theta = \phi = 0$, with θ representing the angle between the detector plane and the rotation axis, and ϕ representing the azimuthal component of the detector normal. In-plane rotation of the detector (rotation about the detector normal) is denoted by η .

Using a single point calibration object, Gullberg *et al* (1987) proposed a fan-beam calibration method by fitting the centroid curve of the projected point, using the Levenberg–Marquardt nonlinear optimization algorithm. Based on the same approach, this model was extended by Crawford *et al* (1988), where the fan-beam misalignment (i.e., the distance between the center of rotation and the mid-line) was permitted to vary with each angular view.

Assuming both an evenly spaced circular trajectory and a perfectly oriented detector ($\theta = \phi = \eta = 0$), with the same approach as for fan-beam calibration, four cone-beam parameters were estimated by Gullberg *et al* (1990). This method was extended to the case of variable orientation of the detector normal by Li *et al* (1993), where six parameters were to be found. (Only the in-plane detector rotation η was assumed to be zero.) The case of a circular trajectory with no constraints was handled by Rizo *et al* (1994), where the three intrinsic parameters (specifying the detector position relative to the vertex) were resolved analytically, whereas the other parameters were estimated using the iterative optimization algorithm of Powell. Based on a simple calibration object, but assuming no detector tilt ($\phi = 0$), Béqué *et al* (2003) used the iterative Powell algorithm to estimate seven parameters. Defrise *et al* (2008) have recently augmented the work of Béqué *et al* (2003) by refining the estimates of the extrinsic parameters on a projection by projection basis. For the case of evenly spaced vertex positions, a totally analytic method was presented by Noo *et al* (2000), for estimating six parameters, under the assumption that the detector and the rotation axis were parallel ($\theta = 0$). An approach based on the circular trajectory using Fourier analysis has been proposed by von Smekal *et al* (2004) to estimate the three Euler angles of the detector, the source to detector distance, and the vector from the perfectly aligned detector center to the true detector center.

Some methods have been proposed for near-circular or near-planar trajectories. Fahrig and Holdsworth (2000) used the same approach as that of Gullberg *et al* (1990) and Li *et al* (1993), except that angle-dependent shifts were introduced so that the trajectory did not have to be perfectly circular. Strobel *et al* (2003) describe a method of finding projection matrices that link the world coordinates to the detector coordinates on a view by view basis, which is suitable for accurate forward- and back-projection in the reconstruction. In Cho *et al* (2005) all nine parameters were estimated analytically, for a trajectory bounded by the parallel planes containing the two circles of the calibration object. Using a planar calibration object, Metzler *et al* (2005) estimated the nine parameters in the sense of the statistically most-likely solution, using an MLEM algorithm. However no system analysis was provided, bringing into question the validity of the parameter estimates when using a planar test object.

The methods were developed for SPECT (Gullberg *et al* 1990, Li *et al* 1993, Rizo *et al* 1994, Béqué *et al* 2003, Metzler *et al* 2005, Defrise *et al* 2008), for CT (Noo *et al* 2000, von Smekal *et al* 2004, Cho *et al* 2005) and for C-arm systems (Fahrig and Holdsworth 2000, Strobel *et al* 2003). The calibration object was generally made of points: one steel ball or one radioactive source point (Gullberg *et al* 1987, Crawford *et al* 1988, Gullberg *et al* 1990, Li *et al* 1993, Fahrig and Holdsworth 2000), three source points (Béqué *et al* 2003) or more

(Rizo *et al* 1994, Metzler *et al* 2005). The points were arranged in two parallel circles in Noo *et al* (2000) and Cho *et al* (2005), and in a helical arrangement in Strobel *et al* (2003).

The method we propose is fully analytic, and general in the sense that all nine parameters are determined with no restriction on the trajectory or detector positions. An advantage of this work relative to previous advances is that the choice of calibration phantom admits a particularly simple mathematical development. A disadvantage of our approach is the requirement of a specific calibration object consisting of point-markers laid out along perpendicular lines. For some cone-beam scanners, such as micro-CT, accurate construction of this calibration object might be difficult. In contrast to this situation, some existing methods have very mild requirements on the calibration object; in Noo *et al* (2000) for example, calibration is performed by scanning two small balls placed at the user's convenience, and five geometric parameters can be estimated from the measurements without knowing the distance between the balls. On the other hand, the benefits of our method are (i) all nine parameters can be determined for each scanner position, (ii) the scanner positions are processed independently of each other, (iii) direct analytic formulae provide the parameter estimates in terms of the image locations of the calibration markers and (iv) the method can be applied to any scanning geometry. The only other work listed above that achieves these objectives is that of Cho *et al* (2005). However, that work was directed at an application in CT imaging, and would suffer from overlapping features of their calibration object for certain source trajectories (trajectories that would be unrealistic in their application), whereas our work can be applied without any constraints on the scanner geometry. This degree of flexibility can be useful in certain applications such as C-arm systems with complicated motions to achieve accurate volume imaging.

A significant contribution of our method is the simplicity both of the mathematical derivation and of the explicit inversion formulae for estimating the calibration parameters. Numerical software packages such as optimization tools, or large linear pseudo-inversion techniques are not required. We are able to analyze the system and verify that a unique solution is always obtained no matter what scanner geometry is presented. Furthermore we provide comprehensive analyses of the possible cases of missing information due to overlapping images of the calibration markers, and we give explicit constraints on the size of these markers in order to avoid catastrophic loss of information. The calibration is therefore robust and allows fully automated processing.

The mathematical principles behind our method are similar to those of certain calibration methods used in computer vision. Methods for camera calibration have been developed by resolving a linear system followed by analytical calculations for the parameters (Faugeras and Toscani 1986, Tsai 1987, Faugeras and Toscani 1987, Wang *et al* 1998). Other more recent computer vision approaches are based on sophisticated projective geometry concepts such as circular points and the absolute conic (Zhang 2000, Frémont and Chellali 2002, Meng and Hu 2003). The computer vision methods apply to a context where surfaces are imaged rather than projections (integrated densities), so the considerations for a suitable calibration object are quite different. Also, in scanner calibration, a much wider range of angular views must be handled, further constraining the choice of calibration object and requiring mathematical analysis of more general cases, such as viewing the object from opposing sides.

We describe the mathematical principles of our calibration method in section 2. The imaging equations with the nine geometric parameters buried within are converted into new imaging equations containing three vector parameters **a**, **b** and **c**. These vectors become the new nine unknowns, and they can readily be estimated using projection measurements of the calibration phantom. The calibration procedure consists of first determining the vectors **a**, **b**,

$\mathbf{c}$ from measurements, then using these vectors to find the original nine calibration parameters. In section 3 we discuss the practical application of this procedure to circular, near-circular and near-planar trajectories. For these cases a simple calibration phantom of six balls arranged on three orthogonal lines can be used without fear of interference between the balls in the projections. For fully general trajectories however, a fourth line is required to handle all geometrical possibilities. Furthermore, extra balls must be added to provide enough redundancy to handle all possible interference patterns between the balls. These general cases are discussed in section 4 along with minor extensions of the equations used to estimate the calibration parameters. In section 5, a detailed simulation study is presented to illustrate the calibration procedure and to provide an experimental validation. A pixelated detector was modeled, and the calibration phantom with finite diameter balls was simulated, thus introducing both partial volume and discretization effects. Fully automated calibration was performed and the parameter estimates found were used for reconstruction in a subsequent simulated production scan of the 3D Shepp–Logan phantom. The last section contains a brief summary of the method and discussion, including restrictions of the method and possible extensions.

2. Theory

2.1. Introduction and the cone-beam imaging model

We assume that a finite number of positions of the scanner are to be determined with respect to a single reference frame. It is also assumed that some common field of view is known, that is completely imaged (not truncated) for all positions. The detector data are assumed to correspond to an ideal flat-panel detector with a known number of rows and columns of square pixels. If the pixel size is not known, the calibration parameters will only be determined up to units of the pixel size. The precise data storage format does not need to be known, but is assumed to be in a conventional row by row or column by column format (and not, for example, back and forth rastering).

A calibration object is positioned within the common field of view of the scanner. The location and orientation of the calibration object *define* the laboratory reference frame $\{\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_z\}$ and its origin $\mathbf{r}_0 = (0, 0, 0)$. The calibration object is imaged for each scan position, and the parameters of these scan positions are found independently of each other. We therefore only describe the calibration procedure for one general position of the scanner. In practice, the calibration object is scanned in the same way as for subsequent production scans, and the calibration procedure will produce all the geometric parameters needed for whatever reconstruction algorithm is used. There is no requirement that the scanning motion be smooth or that the geometric parameters of sequential positions be close to each other.

We describe the scanner geometry using three scalars u_s , v_s and f as described in the next paragraph, and four vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{n}}$ and $\mathbf{s}$ all expressed in terms of the laboratory reference frame. See figure 1. The detector orientation is given by the triple of unit vectors $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}})$, where vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are parallel to the rows and the columns of the detector. When reading the data from a file, we do not know which physical corner of the detector we start from, and whether we are navigating horizontally or vertically. We arbitrarily choose $\hat{\mathbf{v}}$ to be the direction associated with the data component coordinate which first increases when reading the data file, and $\hat{\mathbf{u}}$ to be the direction of increase of the other data component. The vector $\hat{\mathbf{n}}$ is defined as $\hat{\mathbf{n}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$ and represents the direction perpendicular to the detector plane. Note that the triple $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}})$ forms a right-handed orthonormal basis, and can be specified using three Euler angles θ , ϕ and η . We will refer to the focal point or cone vertex of the system as the *source position* $\mathbf{s}$ using terminology appropriate for x-ray transmission scanners. In SPECT applications, $\mathbf{s}$ would refer to a pinhole location, or the focal point of a cone-beam converging collimator. Since the origin of the laboratory frame is a point within

the calibration object, it is safe to assume that $\mathbf{s} \neq \mathbf{0}$. The detector position is specified relative to the source using three scalars u_s , v_s and f .

A 3D point in the laboratory frame is denoted by $\mathbf{r}$. Its cone-beam projection, in pixel units, is $(\tilde{u}, \tilde{v})$ (sometimes more specifically $(\tilde{u}(\mathbf{r}), \tilde{v}(\mathbf{r}))$), where $(\tilde{u}, \tilde{v}) = (0, 0)$ indicates the coordinates of the first detector pixel when reading the data. The coordinates, in pixel units, of the orthogonal projection of the source onto the detector are given by (u_s, v_s) , although u_s and v_s are not necessarily integers and do not necessarily correspond to a physical position on the detector. The detector location (u_s, v_s) is called the *principal point*. The *focal length*, denoted by f , is a signed perpendicular distance from the source to the detector, and is such that $\mathbf{s} + f\hat{\mathbf{n}}$ corresponds to the principal point on the detector. (The sign of f is used to compensate for the direction of the detector normal $\hat{\mathbf{n}}$ which is controlled by the orientation $\hat{\mathbf{u}}, \hat{\mathbf{v}}$, of the detector pixels, i.e. by the unknown order of data storage.)

The mathematical consequences of certain physical features of the system are now summarized. The source cannot lie in the detector plane, so the focal length f is nonzero. In order for a point $\mathbf{r}$ to be imaged, it cannot lie in the plane passing through the source and parallel to the detector, so we have $(\mathbf{r} - \mathbf{s}) \cdot \hat{\mathbf{n}} \neq 0$ for all points $\mathbf{r}$ in the common field of view of the scanner. In particular, we see that the source must lie outside this field of view ($\mathbf{s} \neq \mathbf{r}$). Also, since the calibration object contains the origin $\mathbf{r}_0 = (0, 0, 0)$ of the laboratory frame, we immediately see that $\hat{\mathbf{n}} \cdot \mathbf{s} \neq 0$.

We are now ready to present the cone-beam imaging model. First note that location $\mathbf{p}$ of the detector pixel $(\tilde{u}, \tilde{v})$ is given by $\mathbf{p} = \mathbf{s} + f\hat{\mathbf{n}} + (\tilde{u} - u_s)\hat{\mathbf{u}} + (\tilde{v} - v_s)\hat{\mathbf{v}}$ in the laboratory frame; see figure 1. Every 3D point $\mathbf{r}$ that projects to the same detector location $\mathbf{p}$ belongs to the line passing through the source point $\mathbf{s}$ and this detector location. This line is the set $\{\mathbf{r} | \mathbf{r} = \mathbf{s} + t(\mathbf{p} - \mathbf{s}), t \in \mathbb{R}\} = \{\mathbf{r} | \mathbf{r} = \mathbf{s} + t((\tilde{u} - u_s)\hat{\mathbf{u}} + (\tilde{v} - v_s)\hat{\mathbf{v}} + f\hat{\mathbf{n}}), t \in \mathbb{R}\}$. Easy manipulations then show that the cone-beam projection of an arbitrary point $\mathbf{r}$ onto the detector is given by

$$\begin{aligned}\tilde{u}(\mathbf{r}) &= u_s + f \frac{(\mathbf{r} - \mathbf{s}) \cdot \hat{\mathbf{u}}}{(\mathbf{r} - \mathbf{s}) \cdot \hat{\mathbf{n}}} \\ \tilde{v}(\mathbf{r}) &= v_s + f \frac{(\mathbf{r} - \mathbf{s}) \cdot \hat{\mathbf{v}}}{(\mathbf{r} - \mathbf{s}) \cdot \hat{\mathbf{n}}}.\end{aligned}\tag{1}$$

These are the fundamental imaging equations. The nine unknown geometric parameters are buried within them. The idea is to use a collection of N known points $\mathbf{r}$ with their measured cone-beam projections $(\tilde{u}(\mathbf{r}), \tilde{v}(\mathbf{r}))$ to solve the $2N$ equations for the geometric parameters. In broad terms, all scanner calibration methods use this approach.

2.2. Direct imaging and pinhole imaging

No matter how sophisticated the calibration phantom, there will always be two solutions that satisfy the imaging equations. These two geometries differ in the placement and orientation of the detector, as illustrated in figure 2. Mathematically the two solutions differ by a simultaneous change of sign in the parameters $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ and f . This sign change has no effect on equations (1). Note that one of the geometries corresponds to the situation where the source position lies between the object and the detector which we call the *pinhole imaging* configuration, and the other geometry has the object between the source and the detector which we call *direct imaging*. The possibility of the detector lying between the object and the source is not excluded in the set of possible geometric parameters, and this case would also be considered direct imaging.

We assume that the scanner remains either in direct configuration or in pinhole configuration for all scan positions. We require the user to specify which physical

configuration is being calibrated, and we use the binary variable s_p to describe the configuration: $s_p = +1$ for a pinhole system, and $s_p = -1$ for a direct cone-beam imaging system.

We now examine how the ambiguity of the two imaging configurations manifests itself mathematically in terms of the geometric parameters. We choose a representative point of the object to be $\mathbf{r}_0$ (the center of the laboratory frame), and a representative point on the detector to be the principal point $\mathbf{p}_s = \mathbf{s} + f\hat{\mathbf{n}}$, expressed in the laboratory frame. Along the direction normal to the detector, we examine the order of the three points $\mathbf{r}_0$, $\mathbf{s}$ and $\mathbf{p}_s$, that is, we examine the sequence $(\hat{\mathbf{n}} \cdot \mathbf{r}_0, \hat{\mathbf{n}} \cdot \mathbf{s}, \hat{\mathbf{n}} \cdot \mathbf{p}_s) = (0, \hat{\mathbf{n}} \cdot \mathbf{s}, \hat{\mathbf{n}} \cdot \mathbf{s} + f)$. For a pinhole system (where the source lies between the object and detector) this sequence is monotonic: either strictly increasing or strictly decreasing (recalling that $\hat{\mathbf{n}} \cdot \mathbf{s} \neq 0$ and $f \neq 0$). Otherwise the sign of $\hat{\mathbf{n}} \cdot \mathbf{s}$ is opposite to that of f , and we have a direct system. Thus the sign of $\hat{\mathbf{n}} \cdot \mathbf{s}$ is the same as the sign of f if and only if we have a pinhole system.

We therefore have one further equation to be used in finding the geometric parameters,

$$s_p = \text{sgn}(f\hat{\mathbf{n}} \cdot \mathbf{s}). \quad (2)$$

2.3. Nine intermediate parameters: the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

We convert the imaging equations (1) into a set of simpler equations, involving three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ which play the role of nine intermediate unknowns. The nine geometric parameters will be obtained directly from these three vectors as shown below. In the next section we will see how to conveniently obtain these vectors from measurements of the calibration object.

The main simplification consists of shifting the origin of the detector pixel coordinates by defining

$$u = \tilde{u} - u_c, v = \tilde{v} - v_c, \quad (3)$$

where (u_c, v_c) is the cone-beam projection of $\mathbf{r}_0$ onto the detector plane. Setting $\mathbf{r} = \mathbf{r}_0$ in equation (1) gives

$$u_c = u_s + f \frac{\hat{\mathbf{u}} \cdot \mathbf{s}}{\hat{\mathbf{n}} \cdot \mathbf{s}}, v_c = v_s + f \frac{\hat{\mathbf{v}} \cdot \mathbf{s}}{\hat{\mathbf{n}} \cdot \mathbf{s}}. \quad (4)$$

Note that (u_c, v_c) , like (u_s, v_s) , is expressed in raw detector pixel units. Unlike (u_s, v_s) however, (u_c, v_c) is assumed known from a direct cone-beam measurement of $\mathbf{r}_0$. (The calibration object must be designed to identify the projection of $\mathbf{r}_0$.) After shifting the detector pixels in this way, the center of the laboratory frame $\mathbf{r}_0$ will project to the pixel $(u_0, v_0) = (0, 0)$.

The modified imaging equations that will be used instead of equations (1) are

$$u = \frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{c} \cdot \mathbf{r} + 1}, v = \frac{\mathbf{b} \cdot \mathbf{r}}{\mathbf{c} \cdot \mathbf{r} + 1}, \quad (5)$$

where the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are defined by

$$\mathbf{a} = \frac{(u_s - u_c)\hat{\mathbf{n}} + f\hat{\mathbf{u}}}{-\hat{\mathbf{n}} \cdot \mathbf{s}} \quad \mathbf{b} = \frac{(v_s - v_c)\hat{\mathbf{n}} + f\hat{\mathbf{v}}}{-\hat{\mathbf{n}} \cdot \mathbf{s}} \quad \mathbf{c} = \frac{\hat{\mathbf{n}}}{-\hat{\mathbf{n}} \cdot \mathbf{s}}. \quad (6)$$

Equations (5) are derived from equations (1), and the definitions (3), (4) and (6). For example, equations (3) and (4) can be used to eliminate $\hat{\mathbf{u}}$ and $\mathbf{s} \cdot \hat{\mathbf{u}}$ from equation (1). The result is $u = ((u_s - u_c)\hat{\mathbf{n}} + f\hat{\mathbf{u}}) \cdot \mathbf{r} / \hat{\mathbf{n}} \cdot \mathbf{r} - \hat{\mathbf{n}} \cdot \mathbf{s}$ from which we immediately obtain $u = \mathbf{a} \cdot \mathbf{r} / (\mathbf{c} \cdot \mathbf{r} + 1)$ using the definitions of $\mathbf{a}$ and $\mathbf{c}$ in equations (6). The v and $\mathbf{b}$ variables follow the same pattern.

It has already been noted that $\hat{\mathbf{n}} \cdot \mathbf{s} \neq 0$ so the denominators of equations (6) will not cause any difficulty. Looking now at equations (5), we observe that $\mathbf{c} \cdot \mathbf{r} + 1 = (\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{s})) / (\hat{\mathbf{n}} \cdot \mathbf{s})$. Recalling that $\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{s}) \neq 0$ for $\mathbf{r}$ inside the field of view of the scanner, we are assured that $\mathbf{c} \cdot \mathbf{r} + 1 \neq 0$.

Geometric interpretations of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and of the modified imaging equations (equation (5)) are presented in figure 3.

We now show that equations (6) can be inverted to obtain the nine geometric calibration parameters once the intermediate vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are known.

We define $\varepsilon_f = \text{sgn}(f)$ to be the sign of the focal length. Recalling definition (2) of the known quantity s_p , and noting from equations (6) that the triple product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = -f^2 / (\hat{\mathbf{n}} \cdot \mathbf{s})^3$ is finite, non-zero, and has the same sign as $-\hat{\mathbf{n}} \cdot \mathbf{s}$, we obtain $s_p = \text{sgn}(f)\text{sgn}(\hat{\mathbf{n}} \cdot \mathbf{s}) = \varepsilon_f \text{sgn}(-\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})$ from which we obtain

$$\varepsilon_f = -s_p \text{sgn}(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}).$$

Now, from equations (6) we obtain the geometric parameters in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$\begin{aligned} \hat{\mathbf{u}} &= \varepsilon_f \frac{\mathbf{b} \times \mathbf{c}}{\|\mathbf{b} \times \mathbf{c}\|} \hat{\mathbf{v}} = -\varepsilon_f \frac{\mathbf{a} \times \mathbf{c}}{\|\mathbf{a} \times \mathbf{c}\|} \\ \hat{\mathbf{n}} &= \hat{\mathbf{u}} \times \hat{\mathbf{v}} = -\varepsilon_f s_p \frac{\mathbf{c}}{\|\mathbf{c}\|} \\ f &= \varepsilon_f \frac{\|\mathbf{a} \times \mathbf{c}\|}{\|\mathbf{c}\|^2} = \varepsilon_f \frac{\|\mathbf{b} \times \mathbf{c}\|}{\|\mathbf{c}\|^2} \\ u_s &= \frac{\mathbf{a} \cdot \mathbf{c}}{\|\mathbf{c}\|^2} + u_c, \quad v_s = \frac{\mathbf{b} \cdot \mathbf{c}}{\|\mathbf{c}\|^2} + v_c \\ \mathbf{s} &= \frac{-s_p}{\|\mathbf{c}\|} \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\|\mathbf{a} \times \mathbf{c}\|} \hat{\mathbf{u}} + \frac{\mathbf{b} \cdot \mathbf{c}}{\|\mathbf{b} \times \mathbf{c}\|} \hat{\mathbf{v}} - \varepsilon_f \hat{\mathbf{n}} \right). \end{aligned} \quad (7)$$

The equation for $\mathbf{s}$ was derived from the starting point $\mathbf{s} = (\mathbf{s} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{s} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} + (\mathbf{s} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$ with additional help from equations (4). There are other ways of expressing $\mathbf{s}$, including some that use only the variables $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (and ε_f and s_p). The form given in equation (7) corresponds to the version that was implemented as described in section 5.

2.4. Obtaining $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ from calibration measurements

Consider a calibration object designed so that the raw detector images (u_c, v_c) , $(\tilde{u}_1, \tilde{v}_1)$, $(\tilde{u}_2, \tilde{v}_2)$, ..., $(\tilde{u}_6, \tilde{v}_6)$ of the points $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$, ..., $\mathbf{r}_6$ can be individually identified, where for some known positive constant k ,

$$\begin{aligned} \mathbf{r}_1 &= (k, 0, 0) & \mathbf{r}_2 &= (-k, 0, 0) \\ \mathbf{r}_3 &= (0, k, 0) & \mathbf{r}_4 &= (0, -k, 0) \\ \mathbf{r}_5 &= (0, 0, k) & \mathbf{r}_6 &= (0, 0, -k). \end{aligned} \quad (8)$$

After applying equation (3) to get $(u_i, v_i) = (\tilde{u}_i, \tilde{v}_i) - (u_c, v_c)$ ($i = 1, \dots, 6$), we obtain six pairs of equations (see equation (5)) from which to determine $\mathbf{a} = (a_x, a_y, a_z)$, $\mathbf{b} = (b_x, b_y, b_z)$, $\mathbf{c} = (c_x, c_y, c_z)$.

First, considering only $\mathbf{r}_1$ and $\mathbf{r}_2$, we note that

$$u_1 = \frac{ka_x}{kc_x + 1} u_2 = \frac{ka_x}{kc_x - 1} \quad (9)$$

are linear equations in the unknowns a_x and c_x . Similarly the linear equations

$$v_1 = \frac{kb_x}{kc_x + 1} v_2 = \frac{kb_x}{kc_x - 1} \quad (10)$$

in the unknowns b_x and c_x arise from considering the v -components of the measurements. Treating each pair of equations separately we find the solutions

$$a_x = \frac{2u_1 u_2}{k(u_2 - u_1)} b_x = \frac{2v_1 v_2}{k(v_2 - v_1)} c_x = \frac{u_1 + u_2}{k(u_2 - u_1)} = \frac{v_1 + v_2}{k(v_2 - v_1)}. \quad (11)$$

The y -components a_y, b_y, c_y are obtained in the same fashion from $\mathbf{r}_3$ and $\mathbf{r}_4$, and the z -components a_z, b_z, c_z from $\mathbf{r}_5$ and $\mathbf{r}_6$.

These solutions are only valid if the calibration points $\mathbf{r}_1$ to $\mathbf{r}_6$ are in sufficiently general positions that $u_1 \neq u_2, v_1 \neq v_2$ (and $u_3 \neq u_4, v_3 \neq v_4$; and $u_5 \neq u_6, v_5 \neq v_6$). Physically, the projections of points $\mathbf{r}_1$ and $\mathbf{r}_2$ should not lie in the same row or in the same column on the detector (and similarly for $\mathbf{r}_3$ and $\mathbf{r}_4$, and for $\mathbf{r}_5$ and $\mathbf{r}_6$). In this situation, none of the denominators of equations (11) vanish.

However, the redundancy in $\mathbf{c}$ allows for a more useful general solution. We first note that the equality of the two expressions for c_x in equation (11) is equivalent to $u_1 v_2 = u_2 v_1$ which simply means that $(u, v) = (0, 0)$ lies on the line connecting (u_1, v_1) to (u_2, v_2) on the detector. In the event of noisy measurements, we could take a weighted average of these two expressions, using weights chosen in some heuristic fashion. For example, using weights equal to the square of the denominators in the expressions (11) we would obtain a single solution for c_x which could then be plugged into equations (9) for two new (but equivalent, in the noise-free case) solutions for a_x . Then, averaging these two solutions (which have the same denominators) gives a new expression for a_x . Similarly, a new expression for b_x is obtained by plugging the new c_x into equations (10) and averaging. The outcome is

$$\begin{aligned} a_x &= \frac{2u_1 u_2 (u_2 - u_1) + (u_1 v_2 + u_2 v_1)(v_2 - v_1)}{k((u_2 - u_1)^2 + (v_2 - v_1)^2)} \\ b_x &= \frac{2v_1 v_2 (v_2 - v_1) + (u_1 v_2 + u_2 v_1)(u_2 - u_1)}{k((u_2 - u_1)^2 + (v_2 - v_1)^2)} \\ c_x &= \frac{(u_2 + u_1)(u_2 - u_1) + (v_2 + v_1)(v_2 - v_1)}{k((u_2 - u_1)^2 + (v_2 - v_1)^2)}, \end{aligned} \quad (12)$$

which turns out to be exactly what would be obtained using the standard matrix pseudo-inverse on the 4×3 system of equations (9) and (10). One notable advantage even in the noise-free case, is that the denominators do not vanish unless *both* $u_1 = u_2$ and $v_1 = v_2$.

We now examine more closely the possibility of vanishing denominators. If $u_1 = u_2$, we see by a direct examination of equation (9) that $a_x = 0$ and therefore $u_1 = u_2 = 0$ independent of

the value of c_x , and that c_x cannot be determined from this equation. (Also, a brief contemplation of figure 3(b) shows geometrically that $u_1 = u_2$ implies $u_1 = u_2 = 0$ therefore $a_x = 0$.) Provided $v_1 \neq v_2$, the solution for a_x, b_x, c_x can be reliably found using equations (12). But if however, we simultaneously have $u_1 = u_2$ and $v_1 = v_2$ then we know that $a_x = b_x = 0$, that $(u_1, v_1) = (u_2, v_2) = (0, 0)$, and that c_x cannot be determined from equations (9) and (10). Geometrically, the situation is that the x -axis of the calibration object projects to a single point (namely (u_c, v_c) , in raw detector coordinates). (Consider again figure 3(b).) We call this situation a *degenerate* case. The two other degenerate cases, corresponding to the y and z directions respectively, are $(u_3, v_3) = (u_4, v_4)$ and $(u_5, v_5) = (u_6, v_6)$.

Further mathematical analysis of degenerate cases is possible. In particular, it can be shown that for virtually all degenerate cases, analytic solutions for **a**, **b** and **c** can nevertheless be found. Such solutions will not be useful in practice however, because truly perfect alignment (e.g. $(u_1, v_1) = (u_2, v_2)$) is not detectable.

Other kinds of alignment (such as (u_1, v_1) with (u_3, v_3)) cause no mathematical difficulties, but the points can be considered to ‘overlap’ if they are too close, meaning that they cannot be distinguished and/or located accurately by the detector system. We assume that all points of the calibration phantom are simultaneously imaged for each scanner position. In general, an *overlap* refers to two points that are too close on the detector to be resolved, with the consequence that they cannot be located. Four (scalar) pieces of information are therefore lost. In some situations it is possible to arrange the calibration object to ensure that no overlaps occur during the scan, see section 3. For the general scanning motions however, more information will be needed to offset the possibility of overlaps, so a more elaborate calibration object will be proposed, as discussed in section 4.

3. Planar and nearly planar trajectories

Serious difficulties can arise when points of the calibration phantom are superimposed on the detector. For some scanning geometries it is possible to position the calibration phantom to avoid all possible overlaps in the measurements. In particular for trajectories that we call *nearly planar* trajectories, all overlaps can be avoided using purely geometric considerations, and a simple six-point calibration object is possible. The main characteristic of nearly planar trajectories is that the source positions lie close to an identifiable plane, and that this plane passes close to the center of the common field of view.

Although planar trajectories do not satisfy Tuy’s condition for accurate fully three-dimensional reconstruction, their mechanical simplicity is appealing. Circular trajectories are still common in cone-beam imaging, so it is useful to describe the specific handling for this case and for other planar or nearly planar trajectories. We refer to this specialized treatment as *near-planar calibration* (for ‘calibration for nearly planar source trajectories’).

3.1. The calibration object

Minimum requirements for the calibration object are that there should be six points corresponding to $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_6$ (see equation (8)) whose cone-beam images $(\tilde{u}_i, \tilde{v}_i), i = 1, \dots, 6$ can be accurately localized on the detector and can be reliably matched with the calibration points. The distance k between the pairs of adjacent points should be chosen as large as possible while assuring that the phantom can easily be positioned within the scanner field of view.

A complete discussion of how the calibration object should be constructed and how the projection images should be processed to obtain the $(\tilde{u}_i, \tilde{v}_i)$ is beyond the scope of this paper. The issues mostly relate to the specific imaging characteristics of the scanner being

calibrated. For the rest of this paper, we assume an x-ray scanning system with a flat, pixelated detector. For such systems, the calibration object usually consists of a collection of small dense balls. The balls must employ a coding mechanism or have some identifying characteristics to be able to distinguish them uniquely from the other balls, preferably in an automated fashion. In our simulations we used balls of the same radius but varying density. Many schemes are possible in practice such as varying the ball diameters, or using information in the image of the support structure for the phantom.

In addition to the problem of an automated matching, the elliptical images of the small balls need to be processed to estimate where the center of the ball, $\mathbf{r}_i$, projects onto the detector. A standard technique is to take the centroid of the projected values (after suitable background subtraction) or the centroid of a binary thresholded image (using only the elliptical outline of the projected balls). It is now known ((Desbat *et al* (2006) that except for very special cases both of these centroids do not correspond mathematically to the projection of the center of the ball. However the errors incurred are very small except in the extreme situation that the ball is close to the source and the projected image is a highly elongated ellipse. Generally, centroids can be used to locate the projection of the center of a ball to well within a few tenths of a pixel.

3.2. Positioning the calibration object

For near-planar calibration we assume that a plane P has been identified that passes centrally through the common field of view such that all source positions lie on or near the plane. Other than this broad field-of-view requirement, there is no constraint on the position or orientation of the detector.

The points $\mathbf{r}_1$, $\mathbf{r}_3$ and $\mathbf{r}_5$ of the calibration phantom form a triangle T_{135} that lies in a plane parallel to that of triangle T_{246} formed by the points $\mathbf{r}_2$, $\mathbf{r}_4$ and $\mathbf{r}_6$. The idea is that if the calibration phantom is placed so that triangles T_{135} and T_{246} lie parallel to, and one each side of plane P , then no overlaps can occur. Overlaps can only occur if the source lies in the plane of one of the triangles (or if the source unrealistically lies outside the region bounded by these two planes). The gap between the two triangles provides a margin which can be used to accommodate near-planar source trajectories, errors in identification of the plane P , and deviations in positioning the calibration object. From the definition given in equation (8) we see that triangles T_{135} and T_{246} lie a distance $k/\sqrt{3}$ from the center of the calibration phantom. Ideally, no overlaps occur if the source lies within $k/\sqrt{3}$ of the plane P . However, this analysis does not take into account the finite radius of the balls of the phantom, or the fact that the test object might not be perfectly positioned with respect to plane P . These two cases are treated in the next two paragraphs.

Consider now that the perfectly positioned test object is made up of six small balls each of radius ε , and that the sources lie on the plane P . For source locations at increasing radius from the center of the test phantom, the images on the detector of triangles T_{135} and T_{246} move closer and closer to the planes of the triangles, and the elliptical images of the balls may eventually overlap. The smallest such source location radius can be calculated by assuming a line from the source is tangent to two balls of the same triangle. Since the triangle side lengths are $\sqrt{2}k$, the angle of this tangent line with respect to plane P is $\arcsin(\sqrt{2}\varepsilon/k)$. The sides of the triangles are all $k/\sqrt{6}$ from their centers, which allows us to calculate the radius of the closest source location that might cause an overlap. To avoid overlaps, we therefore require that the maximum radius $R_{(p)}$ of all source points not exceed this calculated distance, which provides the relation

$$R_{(p)}^2 < \frac{k^2}{6} + \frac{k^2}{3} \left(\frac{k^2}{2\varepsilon^2} - 1 \right) \approx \frac{k^4}{6\varepsilon^2}. \quad (13)$$

According to condition (13), all sources should lie roughly within $k^2/(\sqrt{6}\varepsilon)$ from a perfectly positioned calibration phantom. For fairly large balls of radius say $\varepsilon = k/10$, no overlapping can occur for sources lying within $40k$ of the phantom (within about 20 times the size of the field of view). Alternatively, if the source positions are estimated *a priori* to lie within a radius $R_{(p)}$ of the calibration phantom, then the ball radius of the phantom should be no greater than $k^2/(\sqrt{6}R_{(p)})$.

We now consider that the calibration object is not perfectly positioned with respect to the plane P , and also that the sources might lie a short distance from P (due to nearly planar trajectories and/or errors in identifying P). Suppose that center of the calibration phantom is positioned to within δ_C of plane P , and the triangles are tilted by angle θ_C or less with respect to P . Assume that the source positions are known *a priori* to lie within a distance δ_S of the plane P . Imagining the sources to now lie between two planes parallel to P and separated by $2\delta_S$, and that the center of the nearest triangle has advanced by δ_C to one of these planes, we observe the restriction $\delta_S + \delta_C < k/\sqrt{3}$. The effect of the phantom tilt angle θ_C is to increase the angle of the tangent line to the balls with respect to the translated plane for the nearest problematic source location. We assume the tilt is not excessive, so

$\theta_C + \arcsin(\sqrt{2}\varepsilon/k)$ is much less than $\pi/2$. Taking all these factors into account, we obtain the condition that ensures the source location of maximum distance $R_{(np)}$ from the origin will not cause any overlap of the images of any of the six balls. (The subscript *np* refers to the case of nearly planar sources with non-perfect positioning of the phantom.)

$$\delta_S + \delta_C < \frac{k}{\sqrt{3}} \\ R_{(np)}^2 < \frac{k^2}{6} + \left(\frac{k}{\sqrt{3}} - \delta_S - \delta_C \right)^2 \cot^2 \left(\theta_C + \arcsin \left(\frac{\sqrt{2}\varepsilon}{k} \right) \right). \quad (14)$$

Note however that for equations (13) and (14), the source positions are considered to be point locations; the finite extent of the sources is not taken into account.

3.3. Near-planar calibration

To perform calibration of a planar or nearly planar geometry, a preliminary step is to examine the parameters k and ε of the calibration phantom and to verify that *a priori* estimates of the source positions never lie further than the square root of the expression given on the right side of equation (13). The phantom could then be positioned such that its center lies on the plane P of the source trajectory and such that the triangles T_{135} and T_{246} are parallel to P . Recall that we have assumed the plane P passes through the center of the common field of view but that this requirement can be relaxed provided the phantom will be completely visible in all projection views. Once the phantom has been positioned, an estimate of the positioning errors δ_C and θ_C can be made, and combined with an estimate δ_S of the largest deviation of any source from the plane P . If equation (14) is satisfied, then no overlaps can occur and the calibration can safely be carried out; otherwise there is a (probably small) risk that some scanner positions can not be calibrated using the method below. The phantom remains immobilized in the scanner, and a full sequence of projections (calibration positions) are measured and processed independently of each other.

For each position, the images are processed to identify the locations $(\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2), \dots, (\tilde{u}_6, \tilde{v}_6)$ in units of detector pixel coordinates. Recall that there is no requirement that the origin of the detector coordinates be known, nor the direction of the detector axes. The data are processed as follows to obtain the calibration parameters for each position in terms of the reference frame of the calibration phantom.

- i. The first step is to identify the position (u_c, v_c) which is the image in detector coordinates of the center of the calibration phantom. Ideally, this point lies on the intersection of the three lines that connect $(\tilde{u}_1, \tilde{v}_1)$ to $(\tilde{u}_2, \tilde{v}_2)$, $(\tilde{u}_3, \tilde{v}_3)$ to $(\tilde{u}_4, \tilde{v}_4)$ and $(\tilde{u}_5, \tilde{v}_5)$ to $(\tilde{u}_6, \tilde{v}_6)$, but in practice these lines might not intersect perfectly, and a small triangle will be formed instead. Labeling the vertices of this triangle as $(\tilde{u}_A, \tilde{v}_A)$, $(\tilde{u}_B, \tilde{v}_B)$ and $(\tilde{u}_C, \tilde{v}_C)$ with opposite side lengths A , B and C respectively, the least-squares estimate for (u_c, v_c) can be shown to be $(A^2(\tilde{u}_A, \tilde{v}_A) + B^2(\tilde{u}_B, \tilde{v}_B) + C^2(\tilde{u}_C, \tilde{v}_C))/(A^2 + B^2 + C^2)$. This point is known as the symmedian point (and sometimes as the Lemoine point) of the triangle (Honsberger 1995). (A slightly simpler choice would be the centroid of the triangle, but the centroid corresponds to a particular *weighted* least-squares distance, and is not recommended.)
- ii. By subtracting (u_c, v_c) (see equation (3)), $(u_1, v_1), \dots, (u_6, v_6)$ are obtained from $(\tilde{u}_1, \tilde{v}_1), \dots, (\tilde{u}_6, \tilde{v}_6)$.
- iii. The nearly planar geometry ensures that the points $(u_1, v_1), \dots, (u_6, v_6)$ are all distinct, so equations (12) can be used to obtain a_x, b_x and c_x from (u_1, v_1) and (u_2, v_2) . The y - and z -coordinates of **a**, **b**, **c** are found similarly using (u_3, v_3) , (u_4, v_4) , and (u_5, v_5) , (u_6, v_6) .
- iv. From the vectors **a**, **b** and **c**, the geometric calibration parameters are found directly using equations (7).

The geometric calibration parameters can now be fed directly into the reconstruction algorithm. However, the reference frame of the calibration phantom is unlikely to be suitable for image display after reconstruction, so the system would need to be rotated to a more natural orientation, either during the output of the reconstruction, or before the parameters are given to the reconstruction program. For example, a more natural coordinate system might have two axes centered in the plane (or in the least-squares best plane) of the source trajectory, with one axis pointing at the first source position. This coordinate system could be readily identified from the source trajectory in the calibration/laboratory reference frame, and the appropriate rotation and translation of the geometric parameters could be performed before passing them to the reconstruction program. See Mennessier and Clackdoyle (2008) for more on this aspect.

4. General trajectories

This section concerns cone-beam calibration of completely general geometries. No restrictions on the source positions will be made so it will not be possible to systematically avoid overlaps of the phantom. For some positions of the scanner, a loss of information will be unavoidable. To address this difficulty, we propose an augmented version of the calibration phantom to provide redundant information. In practice, the automated detection, processing and identification of the image of the augmented phantom will have to be more elaborate not only because the phantom involves additional points, but because the processing must be robust with respect to overlaps. On the other hand, an advantage is that the calibration phantom can be placed in any preferred orientation in the scanner allowing for a convenient definition of the laboratory frame (unlike the nearly planar case where the artificial orientation would likely require a general rotation before image display).

The augmented phantom includes eight additional points. Six of these points will lie along the axes of the phantom, and provide enough redundancy to overcome the information lost due to overlaps. These six points will be discussed first. The 7th and 8th additional points, which make up the 13th and 14th points of the phantom, are included specifically to handle degenerate cases.

4.1. The extended calibration phantom

When two of the phantom points cause an overlap, four scalar pieces of information are lost. For the six-point phantom, 12 scalars (the u and v values of the detected images of the points) are reduced to eight, so identification of the nine calibration parameters is not possible. The calibration phantom therefore needs additional points, which in turn can increase the possibilities for multiple overlaps. We first analyze the case of a simple extension of the phantom to 12 points. In this analysis, we assume that the radius ϵ of the balls used in the physical phantom is very small, so the balls can be treated as ideal points.

In addition to the six points defined in equation (8), the following points are included:

$$\begin{aligned} \mathbf{r}_7 &= (2k, 0, 0) & \mathbf{r}_8 &= (-2k, 0, 0) \\ \mathbf{r}_9 &= (0, 2k, 0) & \mathbf{r}_{10} &= (0, -2k, 0) \\ \mathbf{r}_{11} &= (0, 0, 2k) & \mathbf{r}_{12} &= (0, 0, -2k). \end{aligned} \quad (15)$$

The degenerate case of a source lying along any of the three axes of the calibration phantom will be handled in the next section. For all other cases, we establish here that no more than two points along each line of the phantom will be lost due to overlaps. Suppose that the source lies on a line connecting calibration points (two points of the calibration phantom). By definition these points must lie on two different coordinate axes and thus the source must lie in the plane of these two axes, so none of the points on the remaining axis can generate an overlap. Therefore we can concentrate on the two axes and assume, without loss of generality, that they are the x and y axes. The source lies on any one of the 16 lines connecting calibration points on the x -axis to calibration points on the y -axis. In examining all possible intersections of these 16 lines, and eliminating those that lie within (the convex hull of) the calibration phantom, we find precisely 20 locations where the source can simultaneously cause two overlaps. No locations with more than two overlaps exist. The 20 locations are $(\pm 2k, \pm 2k, 0)$, $(\pm 4k, \pm 3k, 0)$, $(\pm 3k, \pm 4k, 0)$, $(\pm 6k, \pm 4k, 0)$, $(\pm 4k, \pm 6k, 0)$, and they all correspond to the situation where two points on the x -axis interfere with two points on the y -axis. Furthermore, the two points on the x -axis must lie on opposite sides of the origin, and similarly for the two y -axis points. In summary, there can be at most two overlaps (four 'lost' points, eight 'lost' scalars) in the image of the 12 point calibration phantom, with at least one calibration point visible for each of the positive and negative coordinate axes.

The formulae to obtain $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ from (a visible subset of) $(u_1, v_1), \dots, (u_{12}, v_{12})$ require a small modification of equation (12). Solving for the x -components of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ only involves (u_1, v_1) , (u_2, v_2) , (u_7, v_7) , (u_8, v_8) where at least one of (u_1, v_1) or (u_7, v_7) will be available, and at least one of (u_2, v_2) or (u_8, v_8) will be available. Equations (9) and (10) become

$$\begin{aligned} u_i &= \frac{k_1 a_x}{k_1 c_x + 1} & u_j &= \frac{k_2 a_x}{k_2 c_x - 1} \\ v_i &= \frac{k_1 b_x}{k_1 c_x + 1} & v_j &= \frac{k_2 b_x}{k_2 c_x - 1}, \end{aligned} \quad (16)$$

where $i \in \{1, 7\}$, $j \in \{2, 8\}$ and $k_1, k_2 \in \{k, 2k\}$. Following the same path as lead to equations (12) gives

$$\begin{aligned}
 a_x &= \frac{\bar{k}u_j(u_j-u_i)+(k_1+k_2)(k_1^2u_jv_j+k_2^2u_jv_i)(v_j-v_i)}{k_1k_2(k_1^2+k_2^2)((u_j-u_i)^2+(v_j-v_i)^2)} \\
 b_x &= \frac{\bar{k}v_jv_i(v_j-v_i)+(k_1+k_2)(k_1^2u_jv_j+k_2^2u_jv_i)(u_j-u_i)}{k_1k_2(k_1^2+k_2^2)((u_j-u_i)^2+(v_j-v_i)^2)} \\
 c_x &= \frac{(k_1u_j+k_2u_i)(u_j-u_i)+(k_1v_j+k_2v_i)(v_j-v_i)}{k_1k_2((u_j-u_i)^2+(v_j-v_i)^2)},
 \end{aligned} \tag{17}$$

where $\bar{k}=k_1^3+k_1^2k_2+k_1k_2^2+k_2^3$. This solution also corresponds to the standard pseudo-inverse applied to the linear formulation of equations (16). Note that equation (17) has only used two of the points (u_1, v_1) , (u_2, v_2) , (u_7, v_7) , (u_8, v_8) whereas three or four may be available. Pseudo-inverse solutions for using n points ($n \geq 2$) can be readily found by forming the $2n \times 3$ matrix M of linear equations for a_x, b_x, c_x from equations (16) and applying $(M^t M)^{-1} M^t$ to the right-hand side of the system. The 3×3 matrix inverse $(M^t M)^{-1}$ exists because M is full rank for any subset of at least two points (four corresponding equations), as demonstrated by equations (17). The y - and z -components of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are found the same way, using the visible subsets of the points (u_3, v_3) , (u_4, v_4) , (u_9, v_9) , (u_{10}, v_{10}) and (u_5, v_5) , (u_6, v_6) , (u_{11}, v_{11}) , (u_{12}, v_{12}) , respectively.

4.2. Degenerate cases

In the situation where the source lies along one of the axes of the calibration phantom, the resulting overlaps imply the loss of (u, v) information for the corresponding coordinate of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. The remaining measurements only generate equations involving the other two coordinates of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Our solution to this difficulty is to add two more points to the phantom, which will effectively couple the information along the axes directions and link the equations that involve the x, y and z components of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The thirteenth point and fourteenth points are

$$\mathbf{r}_{13}=(k, k, k) \quad \mathbf{r}_{14}=(-k, -k, -k). \tag{18}$$

Before analyzing the information introduced by $\mathbf{r}_{13}$ and $\mathbf{r}_{14}$, we illustrate the absolute necessity of such points to achieve general cone-beam calibration. We show that in the physically idealized situation where the source lies exactly on one of the axes, an unresolvable calibration geometry can occur unless such supplementary points are present in the calibration phantom. Suppose the axis in question is the z -axis, so the source position is $(0, 0, s_z)$ (with $s_z \neq 0$), and the calibration points $\mathbf{r}_5, \mathbf{r}_6, \mathbf{r}_{11}, \mathbf{r}_{12}$ $((0, 0, \pm k), (0, 0, \pm 2k))$ all project to the single point (u_c, v_c) on the detector. Now if the detector is perpendicular to the z -axis ($\hat{\mathbf{n}} = (0, 0, 1)$), and (for convenience) the pixels are parallel to the x - and y -axes ($\hat{\mathbf{u}} = (1, 0, 0)$, $\hat{\mathbf{v}} = (0, 1, 0)$) then for any point $\mathbf{r} = (0, 0, r_z)$ on the z -axis, equation (1) immediately gives us $(u_s, v_s) = (\tilde{u}, \tilde{v}) = (u_c, v_c)$. All other points on the calibration phantom lie in the x - y plane; so, letting $\mathbf{r} = (r_x, r_y, 0)$ equation (1) gives us $(\tilde{u}, \tilde{v}) = (u_s, v_s) + (f/s_z)(r_x, r_y)$. Thus all calibration points only provide information on the ratio f/s_z , and the focal length f and position of the source s_z cannot be individually determined by the phantom. Geometrically, if the detector is perpendicular to one axis of the phantom, then the image of all the calibration points remains unchanged when the detector and source are moved conjugate to each other as illustrated in figure 4. So, for this situation, calibration points on the z -axis and in the $z = 0$ plane are not sufficient. Since the same argument can be applied to the x - and y -axes, we are forced to include calibration points not lying in any of the $x = 0$, the $y = 0$, or the $z = 0$ planes. We show below that by including $\mathbf{r}_{13}$ and $\mathbf{r}_{14}$, fully general calibration is ensured.

We remark here that a similar argument shows that for any *planar* calibration phantom, there are *three* degrees of freedom of conjugate motion of the source and detector, if the detector is parallel to the phantom. Specifically, for a planar phantom, the image of the phantom on the detector remains unchanged when the source is moved and the detector moves appropriately to compensate: all three parameters (s_x, s_y, s_z) can be varied if the parameters (u_s, v_s), f are adjusted accordingly. Therefore a planar calibration phantom cannot be reliably used for general calibration; the source position becomes completely undetermined if the detector happens to align parallel to such a calibration phantom.

Considering now the points $\mathbf{r}_{13}$ and $\mathbf{r}_{14}$, and applying equations (5) we obtain

$$\begin{aligned} u_{13} &= \frac{k(a_x + a_y + a_z)}{k(c_x + c_y + c_z) + 1} & u_{14} &= \frac{k(a_x + a_y + a_z)}{k(c_x + c_y + c_z) - 1} \\ v_{13} &= \frac{k(b_x + b_y + b_z)}{k(c_x + c_y + c_z) + 1} & v_{14} &= \frac{k(b_x + b_y + b_z)}{k(c_x + c_y + c_z) - 1}. \end{aligned} \quad (19)$$

Supposing again that the degenerate axis is the z -axis (the source lies on the z -axis), we are assured of solutions for a_x, b_x, c_x and a_y, b_y, c_y , since the x - and y -axes will be visible in the detector. Therefore, using the method of equations (12) we solve the system (19) for the quantities a_s, b_s, c_s where $a_s = a_x + a_y + a_z$ (similarly for b_s and c_s) and the unknowns are then found by $a_z = a_s - a_x - a_y$ and similarly for b_z and c_z . (This approach shows directly how a solution can be obtained; an alternative method would be to incorporate all the equations into a single large system matrix M of size $2(n_x + n_y + n_z + n_s) \times 9$ and to apply the pseudo-inverse to obtain all nine unknowns $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Here n_x is the number of visible points lying on the x -axis of the calibration phantom, similarly for n_y and n_z , and n_s is the number of visible points of the set $\{\mathbf{r}_{13}, \mathbf{r}_{14}\}$.)

We now verify that the two additional points of the phantom do not cause untenable overlaps. In order to use the method described here, we require that at least two points on each of three of the four calibration lines (x, y and z axes and the additional diagonal axis) be clearly visible in the projection image. First, for the source along any axis of the phantom, it is clear that the other two axes will be visible in the detector, and that the additional points $\mathbf{r}_{13}, \mathbf{r}_{14}$ also cause no overlaps. For any other position of the source, note that the additional points are not needed, so no harm is done if they are obscured, but we must examine what further overlaps might be generated along the x, y , or z axes. We begin by considering source positions that produce an overlap involving points $\mathbf{r}_{13}$ or $\mathbf{r}_{14}$. By symmetry, it is enough to just consider overlaps generated by $\mathbf{r}_{13}$ with points on the z -axis, $\mathbf{r}_5, \mathbf{r}_6, \mathbf{r}_{11}, \mathbf{r}_{12}$. In three out of these four cases, we find that a second point on the z -axis can be simultaneously obscured by $\mathbf{r}_{14}$, but that no points on the x - or y -axes are involved. For example, the source at position $(3k, 3k, 7k)$ lies on the line of sight between $\mathbf{r}_{13}$ and $\mathbf{r}_{12}$ and that of $\mathbf{r}_{14}$ and $\mathbf{r}_5$. Disregarding symmetries, the only other such case is for the source at position $(-3k, -3k, k)$ with lines of sight joining $\mathbf{r}_{13}$ and $\mathbf{r}_5$, and joining $\mathbf{r}_{14}$ and $\mathbf{r}_{12}$. In all such cases, both points $\mathbf{r}_{13}$ and $\mathbf{r}_{14}$ are obscured along with two points on the z -axis, and these two z -axis points are on opposite sides of the origin. In the fourth case, we have the only situation involving points on the x - y plane. If the source is at $(2k, 2k, 0)$ then $\mathbf{r}_{13}$ will obscure the point $\mathbf{r}_{11}$ on the z -axis, and furthermore there will be two points on each of the x - and y -axes obscured, as indicated in section 4.1. This represents the most extreme case possible, with two points lost from each of the x - and y -axes, one point from the z -axis, and one of the two diagonal points. Nevertheless, two or more points remain visible on each of the x, y and z axes, with at least one point on each side of the origin. Thus all cases involving an overlap caused by the two additional points have been analyzed, bearing in mind that source positions inside the convex hull of the calibration object are not being considered. If the source is positioned such that neither of the additional points are obscured, then the

situation reverts to that treated in section 4.1. In summary, the additional points are capable of resolving the degenerate cases and do not introduce harmful overlaps, so the calibration equations can be resolved for every possible geometric configuration.

4.3. Overlaps caused by balls of non-zero diameter

The analysis of possible overlapping given in the previous section assumed point-like balls. We now consider the balls to have finite radius ε . A finite radius can generate problematic overlaps if this radius is too large with respect to the spacing k of the balls, or if the source is very far from the phantom.

For an example of the first situation, consider the source at location $(4k, -6k, 0)$ in which case two x -axis points $\mathbf{r}_1$ and $\mathbf{r}_8$ are aligned with y -axis points $\mathbf{r}_9$ and $\mathbf{r}_{10}$, respectively. Consider now the double-cone $C_{2,4}$ that is tangent to the ε -balls at locations $\mathbf{r}_2$ on the x -axis and $\mathbf{r}_4$ on the y -axis (and whose vertex lies midway between $\mathbf{r}_2$ and $\mathbf{r}_4$). We call this cone a shadow cone and likewise for the 90 $(C(14, 2) - 1)$ other cones defined between the other pairs of balls. It is easily verified that if $\varepsilon > k/\sqrt{202}$ then shadow cone $C_{2,4}$ will contain location $(4k, -6k, 0)$ and three points will be lost on each of the x -axis and on the y -axis, namely $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_8$ and $\mathbf{r}_4, \mathbf{r}_9, \mathbf{r}_{10}$, respectively. However, this calculation applies only to the intersection of the cone $C_{2,4}$ with the *axes* of cones $C_{1,9}$ and $C_{8,10}$. The simultaneous intersection of these three shadow cones occurs whenever ε is larger than $\sim 0.045k$, at approximately $(4.4k, -6.05k, 0)$, and can be avoided by ensuring a smaller ball radius:

$$\varepsilon < 0.045k. \quad (20)$$

An exhaustive analysis of the possible overlaps of all 91 shadow cones reveals that, with one exception, condition (20) is strong enough to eliminate all such triple overlaps, and that the only other intersections of three or more shadow cones occurs for cones with parallel axes (which we examine below), or for intersections that are so close to the object that the overlap region contains one of the balls of the phantom. We disregard these nearby intersections by assuming that the source position lies at least a distance k from the convex hull of the phantom. (The reason condition (20) is strong enough to rule out other cases is because the case analyzed was the furthest non-parallel triple overlap from the origin, and because one of the cones in this overlap had the largest cone aperture involved in all such triple overlaps.)

The one exception is the unavoidable triple shadow cone overlap of the type $C_{11,13} \cap C_{1,9} \cap C_{8,4}$ involving a diagonal ball $\mathbf{r}_{13}$ which was discussed in section 4.2. In this case the *axes* of the three cones coincide so no constraint on the size of the ε -balls will help. As analyzed previously, this case (and the symmetrically equivalent cases) causes the loss of six balls, of which one is on the diagonal line (either position $\mathbf{r}_{13}$ or $\mathbf{r}_{14}$), and of the other five points no more than two are on any one of the x, y or z axes. The visibility condition (at least two balls on at least three lines) is thus satisfied for this exceptional situation.

Before proceeding with the remaining cases of shadow cones with parallel axes, we note that an overlap of two shadow cones results in at most four balls being obscured, and if any 10 of the 14 balls are visible, then the visibility condition is satisfied. Therefore only regions of three or more intersecting shadow cones can break the visibility condition³.

³It is not strictly true that the visibility condition is assured whenever 10 balls are visible. There is one exception, namely when one diagonal ball and three balls on an axis are all obscured. However, this exceptional case cannot arise from just two overlapping shadow cones, so the conclusion still holds that the visibility condition can only be broken by three or more intersecting shadow cones.

The cases of three or more parallel cones only occur in diagonal directions in the x - y , x - z or y - z planes; or along the x , y or z axes where the cone axes themselves overlap. Examining first the diagonal cases, and focusing on one of the two cases in the x - y plane we observe that cones $C_{7,9}$, $C_{1,3}$, $C_{2,4}$, $C_{8,10}$ have parallel axes and will all overlap at large enough distances from the origin. At such distant source locations, all 8 points on the x and y axes are obscured by overlapping. If any three of the cones simultaneously intersect, then 6 points are lost on the axes which already causes catastrophic overlapping. For the example at hand, the four locations closest to the origin where three of the cones intersect are $\pm(k^2/\varepsilon)(1/\sqrt{2}, -1/\sqrt{2}, 0) \pm k(1/2, 1/2, 0)$ at distance $k\sqrt{k^2/\varepsilon^2 + 1/2}$ from the origin.

Recognizing that k^2/ε^2 dominates $1/2$, we can avoid catastrophic overlapping by requiring that the sources be closer than k^2/ε to the origin, or by requiring that the ball radius be less than $k^2/s_{\max}$ where $s_{\max}$ is the length of the source vector furthest from the origin. We express this as either a condition on the ball radius ε or on the distance of the sources $s_{\max}$

$$\varepsilon s_{\max} < k^2. \quad (21)$$

Finally, we examine the overlap of the shadow cones along the x , y or z axes. Along the x -axis for example, all 6 shadow cones $C_{1,2}$, $C_{1,7}$, $C_{1,8}$, $C_{2,7}$, $C_{2,8}$ and $C_{7,8}$ share the same cone axis, so for source points near the x -axis all points $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_7$, $\mathbf{r}_8$ overlap in the projections. However, none of the other 85 cones intersect these six cones except at nearby locations (within k of the calibration phantom), so no other points are lost.

In summary, condition (20) assures that no triple-cone intersections occur at intermediate distances from the calibration phantom, and condition (21) limits the source distances from the center of the phantom to avoid catastrophic overlaps. We also note that if the conditions are only moderately violated, the regions of space in which source positions cause catastrophic overlaps are nevertheless small. For condition (20) they consist of isolated finite regions centered in the $z = 0$, $y = 0$, $x = 0$ coordinate planes at distances less than about $8k$ from the phantom. For condition (21) they consist of four slowly widening tubular regions in each coordinate plane that in the limit far away from the origin, become four cones. These cones all have aperture $\arcsin(\varepsilon/(\sqrt{2}k))$ and each cone subtends $\pi\varepsilon^2/(2k^2)$ of solid angle. At large distances the fraction of space occupied by the cones is approximately $3\varepsilon^2/2k^2$, typically under 1%.

The two conditions ensure that the visibility condition is satisfied, and mild violation of either of the conditions introduces relatively small regions in and near the coordinate planes where a source could cause catastrophic overlaps.

4.4. Calibration of general geometries

The calibration procedure for a general trajectory uses the 14-ball phantom. The balls are arranged four each along three perpendicular axes and two on a diagonal axis as expressed in equations (8), (15) and (18). If the conditions given in section 4.3 on the diameter of the balls are satisfied, then at least two balls will be visible on each of at least three of the four lines, and the method given below can be executed for every possible scanner geometry. Even if the condition on the diameter of the balls is not satisfied, it is highly likely that the 'two balls on each of at least three lines' condition will still be satisfied for all or nearly all positions of the source. In the rare case that less than two balls are visible on each of two lines then calibration for this scanner position cannot be performed using the method described here. We consider that all scanner positions satisfy this two balls on three lines condition, and apply the following procedure to perform calibration.

The 14-ball phantom is fixed in the common field of view for all positions of the scanner. Each scanner position (each projection) is treated independently. The images are processed to identify locations $(\tilde{u}_i, \tilde{v}_i)$ of the visible subset of $\{\mathbf{r}_i\}$ $i = 1, 2, \dots, 14$. The required processing involves segmentation of the elliptical images of the balls, identification and elimination of overlapping ellipses, estimation to sub-pixel accuracy of the projected centers of the balls, and identification of which ball corresponds to which detector point according to some predetermined coding mechanism. The following steps are performed to obtain the geometric calibration parameters for each scanner position.

- i. The first step is to identify three ‘working’ lines for the calibration. By preference these will be the three perpendicular lines, but if one of these lines is obscured (has only one or zero balls visible in the image) then the other two axes are used with the diagonal axis.
- ii. The next step is to identify the position (u_c, v_c) which is the image in detector coordinates of the center of the calibration phantom. First, for each working line consisting of more than two points, an effective line must be computed such as by performing an orthogonal least squares fit (least squares with perpendicular offsets) of the three or four detector points (it is helpful to note that the centroid of the three or four points will always lie on the orthogonal least squares line). Second, the triangle found from the intersection of the three effective lines is identified. Third, the symmedian point of the triangle provides the point with least squares distance to the effective lines, and is calculated by $(A^2(\tilde{u}_A, \tilde{v}_A) + B^2(\tilde{u}_B, \tilde{v}_B) + C^2(\tilde{u}_C, \tilde{v}_C)) / (A^2 + B^2 + C^2)$ where the vertices of the triangle are $(\tilde{u}_A, \tilde{v}_A)$, $(\tilde{u}_B, \tilde{v}_B)$ and $(\tilde{u}_C, \tilde{v}_C)$ with opposite side lengths A , B and C , respectively. (See ‘the symmedian point’ in Honsberger (1995).) In our simulations we did not find a strong need for an optimal estimate of (u_c, v_c) and in many instances a simpler version is possible, such as taking any point inside the triangle found by joining the two most distant points on each working line.
- iii. Translated detector coordinates $\{(u_i, v_i)\}$ for the visible subset are found from the raw detector coordinates $\{(\tilde{u}_i, \tilde{v}_i)\}$ by subtracting (u_c, v_c) as indicated by equation (3).
- iv. From the visible subset $\{(u_i, v_i)\}$, the vectors **a**, **b** and **c** are found as follows. If the three working lines are perpendicular then **a**, **b**, **c** are found from equations (17) according to the explanation given in section 4.1. If one of the working lines is the diagonal line, then **a**, **b**, **c** are found according to the discussion following equation (19) in section 4.2. In all cases a unique solution for **a**, **b**, **c** is found directly from explicit formulae.
- v. From the vectors **a**, **b** and **c**, the geometric calibration parameters are found directly using equations (7).

In practice the calibration parameters would then be passed to a suitable reconstruction program that can accommodate general source trajectories and general detector positions. In the simulations described in section 5 we used the general CB-FBP algorithm of Defrise and Clack (1994) which assumes that the object to be reconstructed lies within the field of view (no truncated projections) but is otherwise general.

We emphasize that the positioning and orientation of the calibration phantom is important because the geometric description of the scanner will be made with respect to the calibration phantom’s reference frame, and that without a further change of reference frame, the reconstruction object will be obtained in the calibration reference frame. Alternative reference frames could be obtained before reconstruction by identifying a new frame in

terms of the source trajectory, and performing the appropriate change of coordinates to the geometric parameters.

5. A simulation example

A simulated calibration was performed using a source trajectory selected to illustrate the full generality of the procedure. In a temporary reference frame called the trajectory frame, the source traced a saddle trajectory (sine on a cylinder) at a fixed distance from the z axis, followed by a circular motion about the x axis in the y - z plane, with a discontinuous jump between the two trajectory components. There were 360 projections in total. For the first 180 projections the source trajectory was $\mathbf{s} = (170 \cos \omega, 170 \sin \omega, 30 \sin(3\omega))$ and for the second 180 projections it was $\mathbf{s} = (0, 170 \cos \omega, 170 \sin \omega)$ expressed in the trajectory frame. In these expressions, ω was given by $\omega = \pi/180 + 2\pi n/180$ where n refers to the projection index number $n = 0, 1, \dots, 359$. All units are in millimeters, with the source to rotation axis of 170 mm and an amplitude of 30 mm for the sinusoidal curve of three periods. The first trajectory component was perturbed by adding a small translation to each source position, selected randomly from the 1 mm unit cube (uniform probability distribution). The detector consisted of known size 512×512 with 1 mm square pixels. The focal length was fixed at 300 mm with respect to the unperturbed source. The detector was oriented parallel to the rotation axes and parallel to the tangent of the source motion, and translated so that the cone-beam projection of the center of the trajectory reference frame always intersected the center of the detector. The detector orientations were then perturbed for the first trajectory component by adding uniformly random angular noise of $[-2.5^\circ, 2.5^\circ]$, and the detector positions were perturbed for the second trajectory component by a random displacements selected uniformly from the 10 mm unit cube. The purpose of the calibration was to estimate these 360 source and detector geometries from measurements of the calibration object.

Projections of the standard 3D Shepp-Logan phantom (full elliptical axes of 138 mm, 180 mm, 184 mm) and the calibration test object were simulated using standard analytic line-length calculations for single lines from the point source locations to the centers of each detector pixel. The calibration object consisted of 14 balls of radius 3 mm with axis spacing $k = 40$ mm (see equations (8), (15) and (18)). The balls were coded with different densities for identification in the projections. With respect to the trajectory reference frame, the center of the calibration object was placed at (0, 6.98, 13.95) and was aligned with the trajectory frame axes. A perfect alignment is unrealistic in practice but this orientation and displacement were chosen to invoke as much overlapping as possible while still being able to conveniently compare equivalent slices with a gold-standard reconstruction. The reconstruction algorithm was taken from Defrise and Clack (1994) with the implementation described in Noo (1998). The reconstruction array consisted of $200 \times 200 \times 200$ cubic voxels of side 1 mm.

Matlab code was written to perform the preprocessing and the calibration procedure from the scan of the calibration object. Each projection was processed independently. The 14 balls projected onto 14 elliptical regions on the detector that occupied between 40 and 400 pixels each. The preprocessing involved sophisticated thresholding, segmentation, rejection of overlapping ellipses and identification of the visible balls from the density coding to yield a visible subset $\{(\hat{u}_i, \hat{v}_i)\}$. This subset of raw detector positions was immediately processed according to the five-step procedure given in section 4.4 to yield the nine geometric parameters in the format $f, (u_s, v_s), \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}}$ and $\mathbf{s}$. This format was converted to a more redundant version used by the reconstruction program. All processing of the 360 projections was performed completely automatically to provide a single file containing the geometric calibration information in a format suitable for the pre-existing reconstruction program.

A second file of geometric information expressed in terms of the trajectory reference frame had been used for simulating the calibration and Shepp–Logan projection data, and was also used for a baseline reconstruction of the Shepp–Logan phantom. Two reconstructions of the Shepp–Logan phantom were performed, one using the known parameters in trajectory reference frame format and one using the parameters estimated from the calibration procedure.

Figure 5 shows the results obtained from the calibration procedure, and figure 6 shows two perpendicular slices of the two reconstructions of the Shepp–Logan phantom with the appropriate translation in order to visualize the same physical slices.

In the simulations, the values of the ball radius $\varepsilon = 3$ mm, ball spacing $k = 40$ mm and maximum source distance $s_{\max} \approx 175$ mm did not satisfy all the conditions of section 4.3 to ensure a minimum of visible balls for calibration for every projection. Nevertheless, no catastrophic overlapping occurred and all projections were calibrated. No more than four balls of the 14 balls were obscured by overlaps in any one of the projections. For the first (sinusoid) component of the trajectory, 13 of the 180 projections involved degenerate cases of either the x -axis or the y -axis being completely obscured and in 54 other cases, there was one ball or two balls missing from the axes, although always at least three balls visible on the z -axis. In seven cases, one of the diagonal balls was obscured. The overlapping was more pronounced for the second (circular) component of the trajectory. All 14 balls were visible for only 23 of the projections (compared to 103 for the sinusoidal component) and there were 22 degenerate cases where the y -axis or z -axis was obscured. In six more cases three of the four balls were lost on either the y -axis or z -axis. In all 180 projections, both balls along the diagonal were visible, and all four balls along the x -axis were visible.

The results of the calibration show that the detector positions and source locations were consistently found to within about 0.5 mm (about half a detector pixel) except in the axial direction where even better accuracy was achieved for the source location. The orientation of the detector was found to within about 1/20 of a degree for the normal direction and to better than 1/50 of a degree for the in-plane rotation of the detector (η). However, a small bias of about 0.1 mm was observed for the source radius and the focal length, i.e., the calibration procedure seemed to find a focal length of 169.9 mm instead of 170.0 mm. A careful study revealed that the origin of this bias was completely due to the preprocessing of the detected elliptical images of the balls. Centroid calculations are used to estimate the point inside the ellipse corresponding to the cone-beam projection of the center of the ball. This calculation is known (Desbat *et al* 2006) to cause an extremely slight systematic bias on the detector in the direction away from the principal point (u_s, v_s). For the highly exaggerated cone-beam geometry simulated here (with a cone-aperture approaching 40°) this slight error in centroid calculation was causing the system to believe the source was slightly closer to the detector than it actually was. This source of bias is easily removed by either using balls of smaller radius (since the centroid bias increases with the square of the radius) or by applying a preliminary calibration step to approximately identify the source and detector positions, followed by small known centroid corrections to the $(\tilde{u}_i, \tilde{v}_i)$ values. We do not report further on this point here.

The reconstructed images are shown in figure 6. Since the calibration reference frame was only offset (and not rotated) with respect to the scanner reference frame, the reconstructed images can be compared by taking into account this offset. The results show that the reconstruction using the estimated geometric parameters is remarkably similar to the baseline image, to the point of reproducing the low-level visual artifacts in the baseline image that are presumed to be due to the high jitter in all components of the simulated geometry. However, an examination of the profiles reveals that the reconstruction from

estimated parameters appears to have a slight downward shift in intensity by about 0.005. A more detailed examination of the images revealed that the values in the reconstructed image are scaled downwards by about 0.4% to 0.5% with respect to the baseline image. The reason for this slight intensity scaling has not yet been determined, although it might be linked to the small geometric bias discussed in the previous paragraph.

We have also performed simulations for planar and nearly planar trajectories. In broad terms, the results showed similar accuracy to those of figure 6. We noted that with a trajectory radius of 300 mm (rather than the 170 mm used here), no bias of the focal length was detected, and no intensity shift was observed in the reconstructed images. These results can be found in Mennessier and Clackdoyle (2008). Furthermore, the simulations shown there used a more realistic, arbitrary positioning of the calibration phantom, and suitable processing of the source trajectory was applied to realign the calibration parameters with the scanner geometry.

6. Discussion and conclusions

We have presented a comprehensive analytic method of estimating geometric calibration parameters for general cone-beam scanners that employ planar detectors. It is assumed that a field of view common to all scanner positions is known for the purposes of conveniently positioning a calibration phantom. The size of the detector pixels is assumed to be known. For each scanner position, the method estimates nine geometric parameters describing the locations of the source (the cone-vertex) and the detector, and the orientation of the detector. The source position and the detector orientation angles are defined with respect to a common reference frame defined by the position and orientation of the calibration test phantom; the detector location however is expressed relative to the position of the source. Our proposed calibration phantom consists of marker points along each of three orthogonal lines, and points along a fourth diagonal line to be able to handle completely general scanner geometries. Small balls are used in the phantom, centered on the marker points. Since each projection contains the images of all the balls of the phantom, there is a significant risk of some balls obscuring others in the projection images; the 14-ball test phantom proposed in this work has enough redundancy to overcome any such loss of measurement points. For scanners that employ planar or nearly planar trajectories, a simpler 6-ball phantom can be used safely, subject to quantitative (but mild) constraints on the size and spacing of the balls. The calibration theory and the analysis of overlapping images of the balls was presented in sections 2, 3 and 4.

A simulation study was described in section 5. With 1 mm square pixels, and balls of diameter 6 mm, calibration of the sinusoid-and-circle geometry was performed with an accuracy of better than 0.5 mm for positioning information, and better than 0.1° in angular information. The accuracy of the estimate of the in-plane rotation angle of the detector was better than 0.02° . This parameter is critical for accurate reconstructions, and the fact that it was so well recovered supports the view that parameters that matter to the reconstruction step are the ones most easily found, whereas parameters with less sensitive effects on the reconstruction are harder to obtain accurately by imaging measurements.

The cornerstone of the mathematical theory applied in this work was described in equations (5) which relate points in space $\{\mathbf{r}_i\}$ to detector positions $(\tilde{u}_i, \tilde{v}_i)$ via intermediate imaging parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The measurement of each point in space provided two linear equations in the nine unknowns (vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$). The choice of a calibration object using points on three orthogonal lines considerably simplified the equations, and allowed for a rigorous analysis of the linear system. The main issue is to ensure a unique solution to the system using a fixed set of points $\{\mathbf{r}_i\}$ for all scanner positions. It was noted that even under idealized

(‘mathematical’) imaging conditions with no interference from overlaps (for example, if the balls were somehow imaged sequentially rather than simultaneously), there can be ambiguous imaging geometries such as shown in figure 4. We remark here that the orthogonality of the lines of the phantom were not the underlying cause: for any three linearly independent lines in space, a similar degenerate situation arises. (A simple affine transformation of figure 4 provides the demonstration.) The only option was to include a fourth line, which then ensures unique solutions for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ for any scanner geometry. Subsequent inversion from the intermediate parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to the calibration parameters is given in equations (7).

In sections 3 and 4, a rigorous analysis was provided to establish that the overlaps of the balls in the calibration phantom could be completely circumvented by a combination of choosing the ball radius according to specific criteria and by using redundant information in the 14-ball phantom. The simulation study was designed to provoke as much overlapping as possible, and even violated some of the mathematical conditions on the ball radius. Nevertheless, all 360 scanner positions were fully resolved by the calibration procedure in a fully automated fashion.

After the circle, the second most common trajectory in cone-beam tomography is the helix. However, helical scanning does not maintain a field of view common to all projections so the calibration method described here would not be directly applicable. A possible modification would be to place a series of calibration phantoms in the scanner, and pass the coordinate system sequentially between phantoms during the course of the calibration scan.

Scanning trajectories such as the helix or the circle are often precisely engineered so that only the parameters of the trajectory (e.g., radius, axis location, pitch) need to be estimated, rather than a full set of parameters for each position of the trajectory. The method described in this paper is not well-adapted to such situations unless there are unknown perturbations of the trajectory to be detected. With fixed, parameterized source trajectories, calibration with using unknown test objects becomes possible, which has advantages over the method presented here. Conversely, the use of unknown or partially-known test objects would be unsuitable for trajectories that jitter, such as the artificial movements simulated in section 5.

One of the referees suggested that a simple eight-point calibration phantom consisting of $\mathbf{r}_1, \dots, \mathbf{r}_6, \mathbf{r}_{13}, \mathbf{r}_{14}$ may be sufficient for general calibration. We concur with the referee that at most two balls can cause an overlap (provided that balls are small enough) and therefore at least six of the points will always be available for calibration. Unfortunately, the visibility condition (at least two balls visible for each of at least three lines) is not assured so further theoretical analysis would be required to determine if calibration can be performed for all geometric configurations. Specifically, it would have to be established that the 12 equations from any six of the visible balls can always be inverted stably to determine the nine intermediate parameters of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. (We recall from figure 4 that in some circumstances, eight visible balls (16 equations) of the 12-point phantom is not sufficient to find the nine unknowns.) We have performed this analysis and verified that indeed the eight-point phantom can be safely applied for fully general calibration. However the explicit inversion equations are more complicated than those of equations (12) and (17) and we do not present them here. At this point, we have not performed simulation studies to verify the efficacy of the eight-point calibration method in practice. Calibration using the eight-point phantom would have several practical advantages despite the more complicated formulae needed to solve for the geometric parameters. The reduced number of balls would immediately imply some simplification in the construction of the phantom, and also in the coding system needed for identification of the images of the balls. Furthermore, the processing required to identify overlapping balls in the projections would be considerably

simplified since at most two balls can overlap. However, if the balls are not small enough (equivalently, if the source is too far away) then any additional overlapping (i.e., a second pair of overlapping balls) would be catastrophic; there would simply not be enough information available from the remaining balls from which to recover the geometric parameters.

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Figure 1.

The cone-beam geometric calibration parameters f , (u_s, v_s) , $\mathbf{s}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{n}}$ and the cone-beam imaging model. From the cone-beam vertex point $\mathbf{s}$ (the ‘source’), the point $\mathbf{r}$ is projected onto the detector at position $\mathbf{p}$ given by $\mathbf{p} = \mathbf{s} + f\hat{\mathbf{n}} + (\tilde{u} - u_s)\hat{\mathbf{u}} + (\tilde{v} - v_s)\hat{\mathbf{v}}$. Note that the detector normal $\hat{\mathbf{n}}$ points away from the source so the focal length f is positive. A different orientation of the detector pixel directions $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ could change the direction of $\hat{\mathbf{n}}$ and force the sign of f to be negative.

Figure 2.

Direct cone-beam imaging and pinhole imaging. Given a source position, a known object, and an image on the detector, there are always two possible positions for the detector, as illustrated. The detector on the left corresponds to a direct imaging configuration and the detector on the right corresponds to a pinhole imaging configuration. Note that the sign of f and the signs of $\hat{u}$ and $\hat{v}$ all change when alternating between the two detector possibilities. The other geometric parameters (u_s, v_s) , s , and $\hat{n}$ all remain unchanged. The pinhole configuration is characterized by $f\hat{n} \cdot s > 0$ and the direct configuration by $f\hat{n} \cdot s < 0$.

Figure 3.

Geometric interpretation of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. (a) Imaging equations. Only the u direction is considered here. Recall that by definition $\mathbf{c} = \hat{\mathbf{n}}/(-\hat{\mathbf{n}} \cdot \mathbf{s})$, $\mathbf{a} = ((u_s - u_c)\hat{\mathbf{n}} + f\hat{\mathbf{u}})/(-\hat{\mathbf{n}} \cdot \mathbf{s})$ and that $\mathbf{r}_0 = \mathbf{0}$ is the center of the lab frame. Note that $f > 0$ in the diagram since we have a direct imaging system. From its definition, vector $\mathbf{a}$ is perpendicular to the line joining s to $\mathbf{r}_0$ (and is also perpendicular to $\hat{\mathbf{v}}$), and lies in the direction shown, where $\hat{\mathbf{a}} = \mathbf{a}/\|\mathbf{a}\|$. Note that $l_1 = \hat{\mathbf{a}} \cdot \mathbf{r}$ is negative, as is $l_2 = (f/d)l_1$, because $d = \hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{s}) > 0$. Finally, $l_3 = l_2 \sqrt{f^2 + (u_s - u_c)^2}/f$ and $u = l_3$ is also negative (recalling that $u = \hat{\mathbf{u}} \cdot \mathbf{r} - u_c$). Assembling these pieces we have

$$u = \frac{\sqrt{f^2 + (u_s - u_c)^2}(\hat{\mathbf{a}} \cdot \mathbf{r})}{\hat{\mathbf{n}} \cdot \mathbf{r} - \hat{\mathbf{n}} \cdot \mathbf{s}} = \frac{(\sqrt{f^2 + (u_s - u_c)^2})/(-\hat{\mathbf{n}} \cdot \mathbf{s})(\hat{\mathbf{a}} \cdot \mathbf{r})}{(\hat{\mathbf{n}} \cdot \mathbf{r})/(-\hat{\mathbf{n}} \cdot \mathbf{s}) + 1} = \frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{c} \cdot \mathbf{r} + 1} \text{ (because } -\hat{\mathbf{n}} \cdot \mathbf{s} > 0 \text{).}$$

Thus we obtain the imaging equations directly from the definitions of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and the imaging geometry (the diagram). (b) Directions and lengths. A rectangle with sides parallel to $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ and with diagonally opposing points (u_s, v_s) and (u_c, v_c) is formed on the detector. The pyramid with this rectangular base and vertex at $\mathbf{s}$ is shown. Two of the sides rise perpendicularly from the detector, with unit normal vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. The other two sides (shaded) are slightly inclined, and their unit normal vectors are $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ for the more hidden vertical side and for the nearly horizontal side, respectively. The rectangular base has a unit normal vector $\hat{\mathbf{c}}$. For the lengths of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, we note that $1/\|\mathbf{c}\| = |\hat{\mathbf{n}} \cdot \mathbf{s}|$ is the length from $\mathbf{s}$ to $\mathbf{r}_0$ projected along the vertical edge of the pyramid. We also note that

$\|\mathbf{b}\|/\|\mathbf{c}\| = \sqrt{(v_s - v_c)^2 + f^2}$ which is the full length of the pyramid edge with base point (u_s, v_c) . Similarly $\|\mathbf{a}\|/\|\mathbf{c}\|$ is the length of the pyramid edge with base point (u_c, v_s) .

Figure 4.

Geometric demonstration of the need for points r_{13} and r_{14} of the calibration phantom. Without such points, the configuration shown above can occur, with a calibration points only along the normal to the detector, such as r_A , or parallel to the detector face, such as r_B and r_C . For such configurations, we find multiple positions of the source and detector that match the data (u_A, v_A) , (u_B, v_B) , (u_C, v_C) . Two such positions are indicated, with possible sources at s_1 or s_2 , and the detector at focal lengths of f_1 or f_2 , respectively.

Figure 5.

Graphs showing true and estimated calibration parameters for the general (sinusoid+circle) geometry. Top two lines: detector position parameters f , (u_s, v_s) . Top line shows the true values, and second line shows the difference between the true values and those estimated by the calibration procedure. Note that the detector was not perturbed for the first 180 positions, but the parameters are measured with respect to the perturbed source position, and (u_s, v_s) also react to the angular perturbations. Next two lines: detector orientation parameters θ , ϕ , η . To more clearly visualize the changes in ϕ , the angular component ϕ_{rotation} (tracking the source motion) has been removed. Last two lines: source position s . The rotational component ω_{rotation} has been removed so that the tangential perturbations of the source can

be visualized. In all estimated parameters the errors found are small, although a 0.1 mm bias was found in the source radius and the focal length. See text.

Figure 6.

Reconstructed images of the three-dimensional Shepp Logan phantom for the general (sinusoid+circle) geometry. All display scales are [1.00, 1.04]. Top row: baseline reconstruction performed with perfect knowledge of the geometry. The low level image artifacts are attributed to the difficulty of the reconstruction algorithm to cope with the vibrating (perturbed) geometry. Bottom row: the same physical slices of the reconstruction performed using geometric parameters estimated by the calibration procedure. The shift in the position of the phantom (by 7 slices in y and 14 slices in z) is due to the position of the calibration phantom, which defines the reference frame for calibration parameters. The

bottom images also show a slight intensity decrease of about 0.005 units with respect to the baseline images. See text.