Transverse stresses and modes of failure in tree branches and other beams

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The longitudinal stresses in beams subjected to bending also set up transverse stresses within them; they compress the cross section when the beam's curvature is being increased and stretch it when its curvature is being reduced. Analysis shows that transverse stresses rise to a maximum at the neutral axis and increase with both the bending moment applied and the curvature of the beam. These stresses can qualitatively explain the fracture behaviour of tree branches. Curved 'hazard beams' that are being straightened split down the middle because of the low transverse tensile strength of wood. By contrast, straight branches of light wood buckle when they are bent because of its low transverse *compressive* strength. Branches of denser wood break, but the low transverse tensile strength diverts the crack longitudinally when the fracture has only run half-way across the beam, to produce their characteristic 'greenstick fracture'. The bones of young mammals and uniaxially reinforced composite beams may also be prone to greenstick fracture because of their lower transverse tensile strength.

Keywords: bending; transverse stress; branch; bone; greenstick fracture

1. INTRODUCTION

When describing the loading regime induced in beams when they are subjected to pure bending, authors of engineering textbooks (see for instance Gere 2008) rightly concentrate on the longitudinal stresses that are set up. When a beam is bent, the concave surface is subjected to longitudinal compression and the convex surface to longitudinal tension, the stress increasing linearly away from the neutral axis which is located at the centroid of the cross section (figure 1*a*). The resistance of each element to bending is proportional to the square of its distance from the neutral axis; this is because it is both stretched more when further away from the neutral axis, and because its moment arm about the neutral axis is greater. The flexural rigidity R of a beam is therefore given by the expression

$$R = EI, \quad (1.1)$$

where E is the stiffness or Young's modulus of the material and I is the second moment of area of the beam's cross section. I is given by the expression

$$I = \int wy^2 dy, \quad (1.2)$$

where w is the chord width at a distance y from the neutral axis. For a cylindrical beam of radius r ,

$$I = \frac{\pi r^4}{4}. \quad (1.3)$$

The longitudinal stress σ_L in a part of the beam positioned a distance y from the neutral axis, when a bending moment M is applied is given by the expression

$$\sigma_L = \frac{My}{I}. \quad (1.4)$$

The maximum stress, $\sigma_{\max}$, occurs at the inner and outer edges of the beam and is given by the expression

$$\sigma_{L\max} = \frac{My_{\max}}{I} \quad (1.5)$$

where $y_{\max}$ is the greatest distance from the neutral axis. For a cylindrical beam the maximum longitudinal stress $\sigma_{L\max}$ can be readily calculated:

$$\begin{aligned} \sigma_{L\max} &= \frac{4Mr}{\pi r^4} \\ &= \frac{4M}{\pi r^3}. \end{aligned} \quad (1.6)$$

A beam is expected to fail when this maximum stress exceeds the breaking stress of the material of which the beam is composed, at which point the bending moment M is given by the expression

$$M = \frac{\pi r^3 \sigma_{\max}}{4}. \quad (1.7)$$

In typical engineering materials such as metals, ceramics and plastics, which are stiff and isotropic and have low breaking strains, this analysis is perfectly adequate. Beams made of brittle metals and ceramics will fail on the tension side and then simply break right across, while those composed of ductile metals will instead bend irreversibly when the yield stress of the metal is reached, conserving their cross section.

Some biological beams, such as the long bones of adult mammals also break across, but many fail in quite different ways. Young twigs of willow, and many herbaceous plant stems buckle inwards and bend readily without actually breaking (figure 2*a*). By contrast, the curved branches seen in some trees tend to split along their length if an attempt is made to straighten them (Mattheck & Kubler 1995). Finally, many green branches and twigs as well as the bones of young mammals (Currey 2002) only break half-way across when bent; the structure

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Figure 1. The pattern of stresses in beams loaded in pure bending. (a) Longitudinal stresses rise linearly either side of the neutral axis, the outer convex surface being loaded in tension and the concave inner surface being loaded in compression. (b) Transverse stresses rise to a maximum at the neutral axis, being compressive when the bending tends as here to increase curvature.

then splits lengthwise at its midpoint, and the unbroken half continues to bend while the split extends (figure 2b). This process is well known to doctors as ‘greenstick fracture’. Why do these beams fail in such unusual ways? Part of the answer lies in the mechanical properties of their materials, but here we show that these failure patterns are also due to the transverse stresses that are set up in beams when they are bent.

2. TRANSVERSE STRESSES IN BEAMS

In a curved beam (figure 1a), being bent in such a way as to increase curvature, the tension in the outer half of the beam will set up an inward pressure on the material inside it. Similarly the compression in the inner half will set up outward pressure on the material outside it. Transverse *compressive* stresses will therefore be set up within the beam, rising to a maximum at the neutral axis (figure 1b). By contrast if the beam is bent in such a way as to decrease curvature, transverse *tensile* stresses will be set up within the beam.

Consider the transverse stresses set up at a cross section of a beam bent into a curvature c . The inward force per unit length dF set up by a piece of material of width w and thickness dy set a distance y from the neutral axis of a beam is given by the expression

$$dF = c w \sigma_L dy, \quad (2.1)$$

where σ_L is the longitudinal stress. The total inward force per unit length F , a distance y from the centre of the beam is the integral of all the material outside this point, so

$$F = c \int w \sigma_L dy. \quad (2.2)$$

And since the stress σ_L increases linearly away from the neutral axis according to equation (1.4), substituting

this into equation (2.2) leads to

$$F = \frac{cM}{I} \int w y dy. \quad (2.3)$$

So the force per unit length is proportional to the first moment of area of the cross section. The mathematics is therefore very similar to that which calculates the pattern of shear stresses within a beam that has been subjected to shear (Gere 2008).

(a) Stresses in a circular beam

In a beam with circular cross section, the force per unit length is given by the expression

$$F = \frac{cM}{I} \int 2(r^2 - y^2)^{0.5} y dy, \quad (2.4)$$

and so integrating

$$F = \frac{cM}{I} (2/3)(r^2 - y^2)^{1.5}. \quad (2.5)$$

The inward transverse stress is the inward force per unit length F divided by the width of the chord w , so

$$\begin{aligned} \sigma_T &= \frac{cM (2/3)(r^2 - y^2)^{1.5}}{I \cdot 2(r^2 - y^2)^{0.5}} \\ &= \frac{cM(r^2 - y^2)}{3I}. \end{aligned} \quad (2.6)$$

Substituting the expression of I for a circular beam (equation (1.3))

$$\sigma_T = \frac{4cM(r^2 - y^2)}{3\pi^4}. \quad (2.7)$$

The maximum stress, $\sigma_{T\max}$, will occur at the centre of the beam where y is zero and will be given by the expression

$$\sigma_{T\max} = \frac{4cM}{3\pi r^2}. \quad (2.8)$$

The relative magnitude of this transverse stress can be compared with the maximum longitudinal stress in the beam by dividing equation (2.8) by equation (1.6) so that

$$\frac{\sigma_{T\max}}{\sigma_{L\max}} = \frac{cr}{3}$$

or

$$\frac{\sigma_{T\max}}{\sigma_{L\max}} = \frac{r}{3R}, \quad (2.9)$$

where R is the radius of curvature of the beam. The relative size of transverse stresses will therefore be greater the more sharply the beam is curved.

(b) Transverse stresses in an initially straight beam

If an initially straight beam is bent, the transverse stresses will initially be zero but since curvature will rise with the bending moment M such that

$$c = \frac{M}{I} \quad (2.10)$$

incorporating this into equation (2.6), the maximum transverse stress, which will be compressive, at least while the material behaves in a Hookean fashion, will be

Figure 2. The failure modes of branches of contrasting wood density. Branches of (a) willow *Salix alba* buckle rather than break when bent, whereas (b) ash *Fraxinus excelsior* branches, which have higher wood density undergo 'greenstick fracture', breaking on the convex side before splitting down the middle.

Figure 3. The structure of a branch (after Reiterer *et al.* 2002) (a) 80% of the wood volume is composed of tightly packed tracheids or fibres, which are oriented longitudinally, while rays in which the cells are oriented radially constitute around 20%. As a consequence mechanical tests on oriented samples (b) show that tangentially (T) wood is weaker than radially (R) and much weaker than longitudinally (L). Wood is readily split or crushed along the centre-line along which the rays and tracheids provide no reinforcement.

given by the expression

$$\sigma_{Tmax} = \frac{16M^2}{3\pi^2 r^6}. \quad (2.11)$$

The transverse stress will therefore rise with the square of the moment applied and will become increasingly large relative to the longitudinal stress, which only rises with the first power of the moment.

(c) Transverse stresses in an initially curved beam

If the beam is initially strongly curved, the maximum stress will rise approximately to the first power of the bending moment, as long as the change in curvature due to the stress is small compared with the initial curvature. In beams bent in such a way as to increase their curvature, the stresses will be compressive, whereas in beams bent to straighten them the stresses will be tensile.

3. THE FAILURE BEHAVIOUR OF BEAMS

(a) Isotropic beams

Transverse stresses will not affect the failure mechanism of beams composed of engineering materials such as ceramics, steel or plastic, which are isotropic. Such beams will fail when longitudinal strains reach the maximum tensile strain, ϵ_{max} , of the material, which is reached when

$$\epsilon_{max} = \frac{r}{R}. \quad (3.1)$$

For such materials breaking strain is typically under 1 per cent, so the transverse stresses are negligible

compared with the longitudinal ones. This is why conventional analysis can afford to ignore these stresses. This is true even for rubbery materials which have a breaking strain of greater than 1, because even if the beam were curved into a solid doughnut shape the transverse stress would still only be a third of the longitudinal stress.

(b) Anisotropic beams

If materials are anisotropic, being much stronger and stiffer longitudinally than transversely, the transverse stresses set up in bending can cause beams to fail even when the radius of curvature is greater than the radius of beam itself. Wooden branches and trunks are the most important of such beams. Wood is much stronger longitudinally than transversely because 80 per cent of the volume is composed of long narrow tracheids or fibre cells that are orientated longitudinally (Reiterer *et al.* 2002; figure 3a). In the radial direction wood is also strengthened somewhat by the rays (figure 3a) which comprise some 20 per cent of the wood volume, and in which the cells are oriented radially. However, there are no cells oriented in the tangential direction, so it is weakest in this direction; wood can be split or crushed readily along its central axis (figure 3b) as tracheid cells and rays are simply split apart or crushed laterally.

(c) Curved branches

As Mattheck & Kubler (1995) have pointed out, transverse forces can be a real problem for curved branches, which they describe as forming a 'hazard beam'. If such curved beams are bent in such a way as to straighten

Figure 4. Patterns of bending failure predicted for branches. An initially curved branch (a) should split longitudinally down the centreline if it is subjected to a force tending to straighten it. By contrast an initially straight branch made of light wood (b) should buckle transversely inwards, its section flattening out when bent. A straight cylindrical branch of reasonably dense wood (c) should break half-way across before splitting along its length at its midpoint, while a tapered branch (d) should break half-way across then split distally.

the beam (such as when a downward force is applied to a beam that has been curved upwards by reaction wood), tensile stresses will tend to split the branch along its length. Tests by Reiterer *et al.* (2002) have shown that the tangential tensile strength of green oak and ash, at around 8 MPa, are some 35 per cent lower than the radial tensile strength. Other authors give rather lower values down to 2 MPa (Panshin & de Zeeuw 1980). These values are between a fifth and one-twentieth of the longitudinal compressive yield strength of wood, which is around 40 MPa (Panshin & de Zeeuw 1980), and an even smaller fraction of the tensile strength (70–140 MPa). A strongly curved branch will therefore split before longitudinal compressive yield of the wood occurs (figure 4a). Failure will occur exactly along the midpoint where the transverse tensile stresses are greatest and along the line where the transverse tensile strength of the wood is least. Splitting will start to occur before compressive failure when the lateral stress is between a twentieth and a fifth of the longitudinal stress, so when

$$0.2 > \frac{r}{3R} > 0.05 \quad (3.1)$$

so that

$$\frac{5r}{3} < R < \frac{20r}{3}, \quad (3.2)$$

or the radius of curvature is between 0.8 and 3.5 times the diameter of the beam.

In fact, even straighter branches should split along their length, because wood does not break in compression, even after it has yielded; instead, as the tracheid cells on the compression side are compressed further they will tend to densify like other cellular solids (Gibson & Ashby 1999) and the wood's compressive resistance will increase to above the value for tensile strength (70–140 MPa). Therefore even rods with

rather a lower degree of curvature, above between 1.3 and 9 times diameter should split along their length.

It might be thought that changes in wood structure on either side of the centre of a hazard beam might help prevent splitting. Certainly Mattheck & Kubler (1995) found higher transverse strength close to the centre of their hazard beams, probably due to greater ray development. However, since the wood actually splits precisely along the centreline where there is no possibility of ray-reinforcement, this growth response cannot prevent splitting.

(d) *Straight branches*

If a straight branch is bent, compressive transverse stresses will be set up in the branch as its curvature increases. However, since the longitudinal compressive yield strain of wood is less than 1 per cent, the radius of curvature of the rod when the wood starts to yield will be more than 100 times the radius. The transverse stresses set up will therefore be less than one-three hundredth of the longitudinal stress, far too small for transverse compressive failure to occur. However, once the wood has yielded, and buckles longitudinally on the compression side, the branch can be bent much further, with greater areas of wood on the concave side buckling in compression and being densified. Consequently, the transverse compressive strain in the wood will rise rapidly, while longitudinal strain and stress will only increase slowly.

The way in which the structure eventually fails will depend on the density of the wood. According to Panshin & de Zeeuw (1980), the transverse compressive yield strength of green wood rises exponentially with density according to the equation

$$\sigma_{Tmax} = 20.7 \rho^{2.25} \text{ MPa}, \quad (3.3)$$

whereas the longitudinal compressive strength is linearly proportional to density

$$\sigma_{\text{Lmax}} = 46.4 \rho^{1.00} \text{ MPa.} \quad (3.4)$$

(e) *Failure in light wood*

Light wood consequently has a much lower transverse compressive yield strength relative to its longitudinal strength compared with denser wood; the thin-walled cells are simply easier to crush laterally. For a wood of density 0.20, the ratio of longitudinal to transverse strength will be 16.6, whereas for wood of density of 0.3 it will be 10 and for wood of density of 0.5 it will only be 5.3. Wood of density 0.2 will therefore buckle inwards as long as a radius of curvature of less than about 6 times the original radius can be reached, whereas for wood of density 0.3 and 0.5 the figures are around 3 and 1.8 times, respectively. Branches of light wood such as willow, poplar or balsa will therefore buckle inwards, flattening out rather than breaking when they are bent sharply (figure 4b).

(f) *Failure in dense woods*

By contrast, in denser wood the transverse stresses will never be large enough to cause lateral flattening. Instead, longitudinal stresses will eventually cause tensile failure, which will occur on the convex side. The branch will snap as the tracheids fail in tension.

Once failure has occurred in tension on the convex side, one might expect a broken branch to snap right across, but there are other reasons for expecting this not to happen. As we have seen, the tensile strength of wood in the tangential direction is around 8 MPa, around 6–11% of the longitudinal tensile strength. As Cook & Gordon (1964) showed, a crack running through a material sets up a tensile stress in the direction parallel to its length which is around one-fifth of that at right angles at its tip. Since the wood is anisotropic, with a transverse tensile strength that is less than one-fifth of its longitudinal strength, transverse tensile failure will occur and the crack will therefore be diverted longitudinally along the branch. The branch will split exactly down its middle, as in the hazard beam, with the line of failure travelling between the rays.

In a perfectly cylindrical branch, the crack should lengthen evenly in both directions as the bending is increased. The half cylinder of wood on the concave side will bend progressively further (figure 4c). Such a branch will be difficult to break right the way across because of the asymmetry of its cross section. The centre of area of a semicircle is located a distance $4r/3\pi$ or $0.424r$ from the flat surface, so that when the lower half of the branch is bent, the tensile stresses on the flat surface will only be equal to 0.74 times the compressive stresses on the bottom. The lower half of the branch will buckle further in compression and the branch will bend sharply as the wood densifies. It will prove extremely difficult to break this half branch all the way through. A better way to break it might be to bend it back the other way (so increasing the maximum tensile strains relative to the maximum compressive strains), but this would be prevented by the broken ends of the top half of the branch.

In a tapered branch, the crack will run more easily distally (figure 4d) than proximally, since the distal half cylinder will bend more easily than the proximal half cylinder. The bending stresses will be concentrated at the end of the crack and longitudinal splitting will continue to occur. The crack may eventually reach a knot, a bud scar or other imperfection, which might direct it laterally, and so the branch may eventually break right across.

(g) *Failure in very dense wood*

In very dense wood, the transverse tensile stiffness and strength are relatively larger compared with the longitudinal strength (Panshin & de Zeeuw 1980). Cracks might start to run almost as easily in all directions. Therefore branches made of very dense wood could break right across without splitting.

(h) *Bones and other beams*

There is less information available on the structure and anisotropy of mammalian long bones than there is about wood, but the degree of anisotropy, at least in adult bone, appears to be much lower (Currey 2002); the orientation of crystals changes rapidly through Haversian bone, so long bones are rarely more than twice as stiff longitudinally as transversely, though less is known about strength and toughness. It is not surprising, therefore, that stiff adult bones tend to break right across when bent; they will fail in tension on the convex side and the crack will run right through the bone. By contrast, it has long been observed that the more flexible bones of young children are more susceptible to 'greenstick fracture' breaking half-way across before splitting along their length. It has generally been assumed that this is because there is not enough energy for cracks to propagate right across in these more flexible, tougher bones (Currey 2002). It is possible, however, that the difference in the failure mode is because young bones are more flexible and more anisotropic than older ones. As a consequence transverse stresses might be relatively higher at failure and the bones more susceptible to longitudinal splitting. Man-made composite beams reinforced with longitudinally oriented fibres should also be prone to greenstick fracture, because of failure by the Cook–Gordon mechanism. The need to prevent such longitudinal splitting may be the reason why the shafts of feathers, which are composed almost entirely of longitudinally reinforced keratin, also have a thin outer layer of tangentially oriented keratin fibres (Earland *et al.* 1962).

4. DISCUSSION

The model developed here shows that the transverse stresses set up in isotropic materials are far too small to have any effect on the mechanical behaviour of beams loaded in pure bending. In anisotropic materials such as wood and bone, by contrast, they can greatly affect the mode of failure. The tensile stresses set up in curved branches when they are straightened are large enough to split them in their weakest tangential plane, causing the characteristic failure of Mattheck and Kubler's 'hazard beams'. The analysis also correctly predicts that in trees with light wood, transverse compressive stresses will

cause branches to buckle inwards when they are bent. This explains why willow twigs bend rather than break, and so why they can readily be woven into baskets. Such lateral buckling is similar to that seen in hollow plant stems (Spatz *et al.* 1990); in both cases the transverse strength is overcome by the compressive transverse stresses set up by the outer tissue. The model also shows that buckling should not occur in denser wood because of its greater transverse compressive strength. However, the low transverse *tensile* strength of wood will divert tensile fractures running across branches, causing them to split longitudinally down the middle. This explains why green branches are so hard to break off trees. Finally it predicts that greenstick fracture occurs in young bones because they are more flexible and show greater anisotropy than bones in adults.

The analysis presented here therefore seems to qualitatively explain our knowledge of the fracture behaviour of branches and other beams. However, the predictions are based on our rather poor knowledge of the transverse properties of wood and bone, so they cannot give a full quantitative explanation of the behaviour. Experimental studies are clearly required to quantitatively test the model.

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