Backprojection-filtration reconstruction without invoking a spatially varying weighting factor

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Purpose: To develop a backprojection-filtration (BPF) algorithm with improved noise properties over the existing BPF algorithm through utilizing (approximate) redundant information in circular cone-beam or fan-beam scans.

Methods: The backprojection steps in the existing filtered-backprojection (FBP) and BPF algorithms for fan-beam and cone-beam projections invoke spatially varying weighting factors, which may not only increase the computational load in image reconstruction but also, more importantly, result in reconstruction artifacts. Redundant information in fan-beam projections has been exploited for eliminating the weighting factor in the existing FBP algorithm. However, the new FBP algorithm cannot be applied to image reconstruction in a region of interest from transversely truncated data. In this work, the authors identify approximate data redundancy in circular cone-beam projections and propose a new BPF algorithm in which the approximate data redundancy is exploited for eliminating the spatially varying weighting factor in the existing BPF algorithm.

Results: The authors have implemented and evaluated the proposed BPF algorithm in numerical studies of reconstructing 3D images from both the nontruncated and truncated projection data in a circular cone-beam scan. The results of numerical studies demonstrate that the proposed BPF algorithm retains the resolution property of the existing BPF algorithm, and that it can also reconstruct accurately ROI images from truncated data. More importantly, the results also indicate that the proposed BPF algorithm not only is computationally more efficient but also yields generally lower image variances than the existing BPF algorithm.

Conclusions: A BPF algorithm was proposed that not only retains the desirable properties of the existing BPF algorithm but also possesses improved computational and noise properties over the latter. © 2010 American Association of Physicists in Medicine. [DOI: [10.1118/1.3285041](https://doi.org/10.1118/1.3285041)]

Key words: circular trajectory, backprojection-filtration algorithm, region of interest, weighting factor

I. INTRODUCTION

In recent years, significant advances have been made in the development of analytic algorithms, including filtered-backprojection (FBP) and backprojection-filtration (BPF) algorithms for accurate image reconstruction from divergent-beam projections acquired with a broad class of scanning trajectories.^{1–8} Despite the fact that a circular trajectory yield data insufficient for exact 3D-image reconstruction, it remains the most widely used scanning trajectory in computed tomography (CT) applications because it is easy to implement mechanically and because it can yield images of practical utility. The recently developed analytic algorithms can be applied to reconstructing images from data acquired with a circular trajectory, and some of the recently developed FBP algorithms indeed become the well-known Feldkamp–Davis–Kress algorithm for image reconstruction from circular cone-beam data.^{9–12} Because these FBP algorithms invoke one-dimensional filtering of the data function, they

cannot reconstruct accurate images when data truncations occur on the filtering lines.^{13–15} On the other hand, the BPF algorithm can reconstruct images in regions of interest (ROIs) from data containing certain truncations because the BPF algorithm processes data through using only local data information before backprojecting the processed data onto the chords.^{2,16,17} The BPF algorithm has been applied to reconstructing images from circular cone-beam data.¹⁸ Moreover, new BPF-based algorithms have also been proposed to improve the image quality for circular scans, such as the reduction in cone-beam artifacts.¹⁹

When a cone-beam or fan-beam geometry is considered, the backprojection steps in the FBP and BPF algorithms invoke spatially varying weighting factors. When the algorithms are implemented in discrete forms, the weighting factors not only increase the computational load in image reconstruction but also, more importantly, can result in elevated noise level.^{20–24} Because the backprojection step of

FIG. 1. Illustration of a general cone-beam scanning configuration consisting of a source at $\vec{r}_0(\lambda)$ and a flat-panel detector, where λ is a parameter specifying the source trajectory. The fixed-coordinate system $\{x, y, z\}$ centers on the imaged object, whereas the rotation-coordinate system $\{u, v, w\}$ centers on the source. The detector plane is at a distance S from the source point and orients along $\hat{e}_w(\lambda)$. $\vec{\beta}$ indicates the direction of the ray connecting the source point $\vec{r}_0(\lambda)$ and the point (u, v) on the detector.

the FBP and BPF algorithms for a parallel-beam geometry does not have such a weighting factor, effort has been devoted to rebinning fan-beam data into parallel-beam data or cone-beam data into fan-parallel-beam data¹⁰ such that the FBP and BPF algorithms without spatially varying weighting factors can be applied, thus avoiding the undesirable numerical issues related to the spatially varying weighting factors. Some of the rebinning schemes achieve the desired data rebinning through the interpolation of the measured data, which may, however, result in a reduction in spatial resolution in the reconstructed images especially when the measured data are insufficiently densely sampled. Recently, it has been demonstrated that the redundant information in circular fan-beam data can be exploited for eliminating the spatially varying weighting factor in the FBP algorithm without rebinning the fan-beam data, thus avoiding the potential reduction in image resolution.²⁴

In this work, based on the insight described in Ref. 24, we explore the use of (approximate) data redundancy for eliminating the spatially varying weighting factor in the existing BPF algorithm for circular cone-beam image reconstruction. This paper is organized as follows. In Sec. II, we provide a brief overview of the existing BPF algorithm for circular cone-beam reconstruction, followed by the derivation of a BPF algorithm in which (approximate) data redundancy in circular cone-beam data is exploited for eliminating the spatially weighting factor in the existing BPF algorithm. In Sec. III, we perform computer-simulation studies to validate the proposed BPF algorithm. We compare the performance of the existing and proposed BPF algorithms in terms of image spatial resolution and noise properties. Additionally, we demonstrate ROI image reconstruction by use of the proposed BPF algorithm from data containing truncations. Finally, we discuss in Sec. IV the practical implications and possible generalization of the work.

II. THE BACKPROJECTION-FILTRATION ALGORITHMS

We display in Fig. 1 a general cone-beam scanning configuration in which the coordinate system $\{x, y, z\}$ is fixed on the imaged object and the x-ray source trajectory $\vec{r}_0(\lambda)$ is parameterized by λ . A rotation-coordinate system $\{u, v, w\}$ is

FIG. 2. A chord is defined as a straight line segment, along direction $\hat{e}_c$, connecting two points $\vec{r}_0(\lambda_a)$ and $\vec{r}_0(\lambda_b)$ on the source trajectory $\vec{r}_0(\lambda)$. The shadow area represents the object support, and x_{s1} and x_{s2} indicate two end points of the intersection between the chord and the object support.

centered on the source with three unit vectors that are expressed in the fixed-coordinate system $\{x, y, z\}$ as

$$\begin{aligned}\hat{e}_u &= (-\sin \lambda, \cos \lambda, 0)^T \\ \hat{e}_v &= (0, 0, 1)^T \\ \hat{e}_w &= (\cos \lambda, \sin \lambda, 0)^T.\end{aligned}\quad (1)$$

A flat-panel detector is placed at a distance S from the source point and orients along $\hat{e}_w(\lambda)$. At source position $\vec{r}_0(\lambda)$, the cone-beam projection of the object function $f(\vec{r})$ onto a point (u, v) on the detector can be given by

$$P(u, v, \lambda) = \int_0^\infty dt f(\vec{r}_0(\lambda) + t\vec{\beta}), \quad (2)$$

where

$$\vec{\beta} = \frac{1}{\sqrt{u^2 + v^2 + S^2}} [u\hat{e}_u(\lambda) + v\hat{e}_v(\lambda) - S\hat{e}_w(\lambda)] \quad (3)$$

is a unit vector indicating the direction of a ray that connects the source point $\vec{r}_0(\lambda)$ and the point (u, v) on the detector.

II.A. The existing BPF algorithm

The existing BPF algorithm^{2,18} can reconstruct the image on a chord, which, as shown in Fig. 2, is a straight line segment connecting two points $\vec{r}_0(\lambda_a)$ and $\vec{r}_0(\lambda_b)$ on the source trajectory. We use x_c to denote a point on the chord. Therefore, three parameters $(x_c, \lambda_a, \lambda_b)$, which we refer to as chord coordinates, can be used for specifying a point on the chord. The chord coordinates are related to the fixed coordinates through

$$\vec{r} = \frac{[\vec{r}_0(\lambda_a) + \vec{r}_0(\lambda_b)]}{2} + x_c \hat{e}_c, \quad x_c \in (-l/2, l/2), \quad (4)$$

where

$$\hat{e}_c = \frac{\vec{r}_0(\lambda_b) - \vec{r}_0(\lambda_a)}{|\vec{r}_0(\lambda_b) - \vec{r}_0(\lambda_a)|} \quad \text{and} \quad l = |\vec{r}_0(\lambda_b) - \vec{r}_0(\lambda_a)| \quad (5)$$

denote the direction and length of the chord. Furthermore, we use x_{s1} and x_{s2} to denote the two end points of the intersection between a chord and the object support. We assume

FIG. 3. The solid circle and solid line segments within the plane at $z=0$ represent the actual trajectory and actual chords, whereas the dashed circles and dashed line segments within the planes at $z \neq 0$ denote virtual trajectories and virtual chords, respectively. Vectors $\vec{r}_0(\lambda_a)$ and $\vec{r}_0(\lambda_b)$ indicate two end points of an actual chord. Without loss of generality, we assume that $-\pi \leq \lambda_a < \lambda_b \leq \pi$.

that the source trajectory does not pass through the object support, therefore, $[x_{s1}, x_{s2}] \subset [-l/2, l/2]$.

A circular trajectory within a 2D plane at $z=0$ can be expressed in the fixed-coordinate system as

$$\vec{r}_0(\lambda) = (R \cos \lambda, R \sin \lambda, 0)^T, \quad (6)$$

where λ indicates the rotation angle of the x-ray source, and R denotes the source to the rotation axis distance. In this case, actual chords can be only within the plane containing the source trajectory (i.e., $z=0$). For a nontrajectory plane (i.e., $z \neq 0$), there is no actual source trajectory and, consequently, no actual chords exist.

Virtual circular trajectories and virtual chords^{12,18} have been introduced so that the BPF algorithm can be applied to reconstructing approximately images within these nontrajectory planes. As illustrated in Fig. 3, a virtual circular trajectory is identical to the actual trajectory except that it is within a plane at $z \neq 0$, and a virtual chord is a straight line segment that connects two points on a virtual circular trajectory at $z \neq 0$. Therefore, an additional parameter z_0 is used for specifying a virtual trajectory, and the chord coordinates for a point on a virtual chord are related to the fixed coordinates through

$$\vec{r} = \frac{[\vec{r}_0(\lambda_a) + \vec{r}_0(\lambda_b)]}{2} + x_c \hat{e}_c + z_0 \hat{e}_v, \quad x_c \in (-l/2, l/2). \quad (7)$$

For simplicity of discussion below, we refer to an actual chord and a virtual chord simply as a chord.

The existing BPF algorithm that reconstructs an image $f_c(x_c, \lambda_a, \lambda_b, z_0)$ on a chord^{2,18} is given by

$$f_c(x_c, \lambda_a, \lambda_b, z_0) = \frac{1}{2\pi^2} \frac{1}{C(x_c)} \times \left[\int_{x_A}^{x_B} \frac{C(x'_c) dx'_c}{x_c - x'_c} g(x'_c, \lambda_a, \lambda_b, z_0) + 2\pi P_0 \right], \quad (8)$$

where

$$C(x_c) = \sqrt{(x_B - x_c)(x_c - x_A)}, \quad (9)$$

x_A and x_B are two points on the chord satisfying $[-l/2, l/2] \supset [x_A, x_B] \supset [x_{s1}, x_{s2}]$, the constant term P_0 represents the projection data along the chord, which is estimated

from the average of projection data for the x rays passing through the middle point of the chord at view λ_a and λ_b . The backprojection image $g(x'_c, \lambda_a, \lambda_b, z_0)$ can be written as

$$g(x'_c, \lambda_a, \lambda_b, z_0) = \int_{\lambda_a}^{\lambda_b} \frac{d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}}, \quad (10)$$

where $\vec{r}'$ is a function of parameters $(x'_c, \lambda_a, \lambda_b, z_0)$, which can be obtained by replacing x_c with x'_c in Eq. (7). Detector coordinates u and v are given by

$$u(\vec{r}', \lambda) = S \frac{\vec{r}' \cdot \hat{e}_u(\lambda)}{R - \vec{r}' \cdot \hat{e}_w(\lambda)},$$

$$v(\vec{r}', \lambda) = S \frac{\vec{r}' \cdot \hat{e}_v(\lambda)}{R - \vec{r}' \cdot \hat{e}_w(\lambda)}, \quad (11)$$

and the data derivative is expressed as

$$\frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} = \frac{\partial P(u, v, \lambda)}{\partial \lambda} + \frac{S^2 + u^2}{S} \frac{\partial P(u, v, \lambda)}{\partial u} + \frac{uv}{S} \frac{\partial P(u, v, \lambda)}{\partial v}. \quad (12)$$

The existing BPF algorithm described in Eqs. (8)–(12) yields mathematically exact, and approximate, images on actual chords and virtual chords, respectively. It can be observed that the backprojection step in Eq. (10) of the existing BPF algorithm contains a spatially varying weighting factor $1/|\vec{r}' - \vec{r}_0(\lambda)|$. When the BPF algorithm is implemented in discrete form, it is this weighting factor that can result in not only a high computational load but also increased noise in reconstructed images. As shown below, however, this weighting factor can be eliminated in the backprojection step through an appropriate use of (approximate) data redundancy.

II.B. The proposed BPF algorithm

In a circular cone-beam scan, data $P(u, v, \lambda)$ collected on the row at $v=0$ correspond to fan-beam projections within the actual trajectory plane, and thus satisfy the data consistency condition¹⁷

$$\frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} = - \frac{dP(u', v, \lambda')}{d\lambda'} \bigg|_{\hat{\beta}}, \quad (13)$$

where

$$\lambda' = \lambda + \pi - 2 \arctan \frac{u}{S}$$

$$u' = -u. \quad (14)$$

The consistency condition in Eq. (13) suggests that the fan-beam data collected over 2π on the detector row at $v=0$ contain redundant information for image reconstruction on actual chords. Specifically, for an actual chord determined by λ_a and λ_b , the fan-beam data over angular ranges $[-\pi, \lambda_a]$ and $[\lambda_b, \pi]$ are redundant to that over angular range $[\lambda_a, \lambda_b]$.

In general, data $P(u, v, \lambda)$ collected on a detector row at $v \neq 0$ in a circular cone-beam do not satisfy exactly the consistency condition in Eq. (13) because no projection ray passing through a point within a nontrajectory plane is measured twice. However, we assume that they satisfy approximately such consistency condition, i.e.,

$$\left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} \approx - \left. \frac{dP(u', v, \lambda')}{d\lambda'} \right|_{\hat{\beta}}, \quad (15)$$

where $v \neq 0$, and (u, λ) and (u', λ') are related through Eq. (14). The adequacy level of the approximation entailed in Eq. (15) depends on the value of v and the complexity of the structures within the object. Based on Eq. (15), it can also be concluded that data acquired over angular ranges $[-\pi, \lambda_a]$ and $[\lambda_b, \pi]$ are approximately redundant to that over angular range $[\lambda_a, \lambda_b]$. By use of such approximately redundant information, a weighted backprojection can be obtained as

$$g_w(x'_c, \lambda_a, \lambda_b, z_0) = \int_{-\pi}^{\pi} d\lambda \frac{w(\lambda, \vec{r}')}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}}, \quad (16)$$

where $w(\lambda, \vec{r}')$ is a function for adequately normalizing the

$$\text{“redundant” data derivative } \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}}, \text{ in}$$

a full scan, which satisfies

$$w(\lambda, \vec{r}') - w(\lambda', \vec{r}') = 1.0, \quad (17)$$

$$\lambda' = \lambda + \pi - 2 \arctan \frac{\vec{r}' \cdot \hat{e}_u(\lambda)}{R - \vec{r}' \cdot \hat{e}_w(\lambda)}. \text{ It can be shown that}$$

$\lambda' = \lambda_b$ when $\lambda = \lambda_a$.

In the Appendix, we show that

$$g(x'_c, \lambda_a, \lambda_b, z_0) = g_w(x'_c, \lambda_a, \lambda_b, z_0) \quad \text{for } z_0 = 0,$$

$$g(x'_c, \lambda_a, \lambda_b, z_0) \approx g_w(x'_c, \lambda_a, \lambda_b, z_0) \quad \text{for } z_0 \neq 0. \quad (18)$$

Therefore, using the weighted backprojection image of Eq. (16) to replace the backprojection image of Eq. (10) in the existing BPF algorithm, we obtain the proposed BPF algorithm. Clearly, the use of different weighting functions, or, equivalently, the use of different ways to exploit for data redundancy, would result in different noise properties of the proposed algorithm.

An equal weighting function has been used in previous studies,¹⁷ which is given by

$$w(\lambda, \vec{r}') = \begin{cases} \frac{1}{2}, & \lambda_a < \lambda < \lambda_b \\ -\frac{1}{2}, & \text{otherwise} \end{cases}. \quad (19)$$

Substituting the equal weighting function $w(\lambda, \vec{r}')$ into Eq. (16), we obtain the existing BPF algorithm with an equal weight for image reconstruction on the chord specified by λ_a and λ_b from full-scan data,

$$g_w(x'_c, \lambda_a, \lambda_b, z_0) = \frac{1}{2} \left[\int_{\lambda_a}^{\lambda_b} \frac{d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} - \int_{-\pi}^{\lambda_a} \frac{d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} - \int_{\lambda_b}^{\pi} \frac{d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} \right]. \quad (20)$$

The noise property of this BPF algorithm in its discrete form has been evaluated in Ref. 25. Clearly, with the weighting function in Eq. (19), the spatially varying weighting factor $1/|\vec{r}' - \vec{r}_0(\lambda)|$ remains in the backprojection step.

Recently, an FBP algorithm has been proposed for reconstructing images directly from fan-beam projections²⁴ in which the spatially varying weighting factor is removed through an appropriate utilization of data redundancy. In this work, in an attempt to eliminate the spatially varying weighting factor $1/|\vec{r}' - \vec{r}_0(\lambda)|$ in the backprojection in Eq. (20), we propose below a weighting function

$$w(\lambda, \vec{r}') = \begin{cases} \frac{|\vec{r}' - \vec{r}_0(\lambda)|}{2R \cos \gamma}, & \lambda_a < \lambda < \lambda_b \\ -\frac{|\vec{r}' - \vec{r}_0(\lambda)|}{2R \cos \gamma}, & \text{otherwise} \end{cases}, \quad (21)$$

where

$$\gamma = \arctan \frac{\vec{r}' \cdot \hat{e}_u(\lambda)}{R - \vec{r}' \cdot \hat{e}_w(\lambda)} = \arctan \frac{u}{S}. \quad (22)$$

Following the same strategy in Ref. 24, one can easily show that the weighting function in Eq. (21) satisfies Eq. (17). Substitution of Eq. (21) into Eq. (16) yields a weighted backprojection image

$$g_w(x'_c, \lambda_a, \lambda_b, z_0) = \int_{\lambda_a}^{\lambda_b} d\lambda \frac{\sqrt{u^2 + S^2}}{2RS} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} - \int_{-\pi}^{\lambda_a} d\lambda \frac{\sqrt{u^2 + S^2}}{2RS} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} - \int_{\lambda_b}^{\pi} d\lambda \frac{\sqrt{u^2 + S^2}}{2RS} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}}. \quad (23)$$

Replacing Eq. (10) with Eq. (23) in the existing BPF algorithm, we obtain the proposed BPF algorithm in which the backprojection involves, as shown in Eq. (23), no spatially varying weighting factor.

Detailed discussion concerning the implementation of differentiation, backprojection, and filtration in the BPF algorithm has been given in Ref. 25.

III. NUMERICAL STUDIES

We have performed computer-simulation studies to validate and evaluate the proposed BPF algorithm for image reconstruction from nontruncated and truncated circular

FIG. 4. Shepp-Logan images reconstructed from full-scan, nontruncated data by use of the existing (upper row) and proposed (lower row) BPF algorithms within the trajectory plane at $z=0$ mm (column 1), a transverse slice at $z=27$ mm (column 2), a coronal slice at $x=0$ mm (column 3), and a sagittal slice at $y=-27$ mm (column 4). The display gray scale is $[1.0, 1.04]$.

cone-beam data. For comparison, we have also used the existing algorithm [i.e., Eq. (20)] to reconstruct the images from the same cone-beam data.

III.A. Image reconstruction from nontruncated data

In the study, a circular trajectory with a source to rotation axis distance $R=320$ mm was used, and a 2D flat-panel detector was placed at a distance $S=580$ mm from the source. The detector array consists of 256×256 elements each of which has a size of 2.2×2.2 mm². Therefore, the maximum fan-angle subtended by the detector is 52° . A modified 3D Shepp-Logan phantom,²⁶ which has an ellipsoidal support with 49, 98, and 90 mm along the x , y , and z axes, respectively, was used to generate circular cone-beam data. We generated the nontruncated projection data at 300 views uniformly distributed over 2π , from which we have reconstructed 3D images by use of the existing and proposed BPF algorithms.

We display in Fig. 4 images reconstructed by use of the existing (upper row) and proposed (lower row) BPF algorithms within the trajectory plane at $z=0$ mm (column 1), transverse slice at $z=27$ mm (column 2), coronal slice at $x=0$ mm (column 3), and sagittal slice at $y=-27$ mm (column 4), respectively. Visually identical results in column 1 of Fig. 4 indicate that both existing and proposed BPF algorithms can reconstruct exact images from full-scan fan-beam data. Quantitative agreement between the reconstructed images and the true image can be seen in Fig. 5(a), which

FIG. 5. Profiles of true image (solid line) and the images obtained by use of the existing (dotted line) and proposed (dashed line) BPF algorithms along lines specified by (a) $y=20.2$ mm and $z=0$ mm and by (b) $x=0$ mm and $y=-27$ mm.

FIG. 6. Head images reconstructed from full-scan, nontruncated data by use of the existing (upper row) and proposed (lower row) BPF algorithms within the trajectory plane at $z=0$ mm (column 1), a transverse slice at $z=18.5$ mm (column 2), a coronal slice at $x=0$ mm (column 3), and a sagittal slice at $y=0$ mm (column 4). The display gray scale is $[1.0, 1.1]$.

compares the profiles of the reconstructed and true images along a line specified by $y=20.2$ mm and $z=0$ mm. However, when reconstructions in nontrajectory planes are considered, as shown in columns 2–4 in Fig. 4, the two algorithms yield somewhat different images. This is expected because the two algorithms exploit differently the approximate data redundancy. The profiles of the reconstructed images along a line specified by $x=0$ mm and $y=-27$ mm, as shown in Fig. 5(b), demonstrate that the existing and proposed BPF algorithms yield the images with different intensity-drop artifacts.

In order to demonstrate the performance of the proposed algorithm for different phantoms, we generated projection data from the FORBILD head phantom with the same scanning geometry except that 1200 projection views were used. Again, we display in Fig. 6 the images reconstructed by use of the existing (upper row) and proposed (lower row) BPF algorithms within the transverse slices at $z=0$ mm (column 1) and $z=18.5$ mm (column 2), coronal slice at $x=0$ mm (column 3), and sagittal slice at $y=0$ mm (column 4), respectively. It can also be observed that the reconstructed images within the trajectory plane obtained with both existing and proposed BPF algorithms are visually identical, while the images obtained with these two algorithms within nontrajectory planes are different. In an attempt to display quantitatively the reconstruction accuracy, we have also shown in Fig. 7 the profiles of the reconstructed and true images along lines specified by $y=27$ mm and $z=0$ mm, and $x=0$ mm and $y=0$ mm. We have also estimated the

FIG. 7. Profiles of true image (solid line) and the images obtained by use of the existing (dotted line) and proposed (dashed line) BPF algorithms along lines specified by (a) $y=27$ mm and $z=0$ mm and by (b) $x=0$ mm and $y=0$ mm.

FIG. 8. Illustration of the scanning configuration in the ROI image reconstruction study. The regions enclosed by the white solid box indicates the ROI to be reconstructed within the transverse slice at $z=0$ mm. Clearly, the projection data are transversely truncated at certain views.

computational time for the current implementation of the proposed and existing algorithms. For the reconstruction of the modified Shepp–Logan phantom, the CPU time for each backprojection step in the proposed algorithm is 0.42 s, while it is 0.46 s for the existing algorithm.

III.B. ROI image reconstruction from truncated data

We have also performed a study on image reconstruction within a 3D ROI enclosed by the white solid box, as shown in Fig. 8, by use of the existing [i.e., Eq. (20)] and proposed [i.e., Eq. (23)] BPF algorithms from truncated circular cone-beam data. The Shepp–Logan phantom and scanning configuration parameters identical to those described in Sec. III A were used to generate cone-beam projection data at 300 views uniformly distributed over 2π . However, we shifted the phantom so that the selected ROI is centered on the rotation center, and we have narrowed transverse coverage of the x-ray beam so that it illuminates only the ROI, thus resulting in transversely truncated data at certain views. In Fig. 9, we show the reconstructed ROI images within the trajectory plane at $z=0$ mm (column 1), the transverse slice at $z=27$ mm (column 2), the coronal slice at $x=0$ mm (column 3), and the sagittal slice at $y=-27$ mm (column 4), respectively, obtained from the generated data containing trunca-

FIG. 9. ROI images reconstructed from full-scan, truncated data by use of the existing (upper row) and the proposed (lower row) BPF algorithms within the trajectory plane at $z=0$ mm (column 1), a transverse slice at $z=27$ mm (column 2), a coronal slice at $x=0$ mm (column 3), and a sagittal slice at $y=-27$ mm (column 4). The display gray scale is [1.0, 1.04].

FIG. 10. Profiles of the true image (solid line) and the images obtained from full-scan, truncated data by use of the existing (dotted line) and proposed (dashed line) BPF algorithms along lines specified by (a) $y=20.2$ mm and $z=0$ mm and by (b) $x=0$ mm and $y=-27$ mm.

tions by using the existing (upper row) and proposed (lower row) BPF algorithms, respectively. The profiles along lines in the reconstructed images, specified by $y=20.2$ mm and $z=0$ mm and by $x=0$ mm and $y=-27$ mm are displayed in Fig. 10. Agreement between the reconstructed and true images, as shown in Fig. 10(a), demonstrates that both the existing and proposed BPF algorithms can reconstruct exact ROI images from the truncated data within the trajectory plane. However, the two algorithms yield different ROI images within nontrajectory planes because they exploit differently the approximate redundant information. Comparison shown in Fig. 10(b) suggests that the proposed BPF algorithm seems to yield an image with slightly less intensity-drop artifacts than does the existing BPF algorithm for the truncated data.

III.C. Quantitative study of spatial resolution properties

In addition to reconstruction-accuracy studies described above, we have also performed numerical studies to evaluate quantitatively the properties of the proposed algorithm. The modulation transfer function (MTF) has been widely used for characterizing spatial resolution property of a shift-invariant system. However, the resolution property in an image reconstructed by use of an algorithm in its discrete form is generally shift-variant, and thus an MTF cannot be meaningfully computed from a reconstructed image. Therefore, other indices such as the modulus of the Fourier transform (MFT) and the average modulus of the Fourier transform (AMFT) have been used for evaluating the resolution property in reconstructed images.²¹ The MFT is defined as the modulus of the discrete Fourier transform of a local impulse function reconstructed by use of an algorithm in its discrete form. The AMFT is the average MFTs over polar angles in the Fourier space. Therefore, these two indices characterize the spatially varying resolution property through displaying the frequency contents at each individual spatial location in a reconstructed image.

In the resolution study, we used the same scanning configuration as that in Sec. III A to generate cone-beam data at 300 views uniformly distributed over 2π . A pointlike sphere phantom of 0.6 mm in diameter, which is smaller than the detector element (2.2 mm) or the x-ray source (2.2 mm), was used in this study. The attenuation coefficient of this phantom is chosen to be 20 mm^{-1} . In an attempt to mimic the

FIG. 11. Upper row: MFTs of the reconstructed images for the pointlike phantom within the transverse plane obtained by use of the existing (dashed curve) and proposed (solid curve) BPF algorithms at positions (a) (0.5 mm, 0.0 mm, 0.0 mm), (b) (50.0 mm, 0.0 mm, 0.0 mm), and (c) (50.0 mm, 0.0 mm, 50.0 mm), respectively. The units of x and y axes are mm^{-1} . Lower row: the corresponding AMFTs obtained by use of the existing (“ $\diamond$ ”) and proposed (“+”) BPF algorithms. The unit of x axis is mm^{-1} .

effect of the finite sizes of x-ray source and detector element, we subdivided uniformly the source and detector element into 5×5 subelements. Therefore, for each detector element, there exist 625 straight lines that are used for computing projection data. For each detector element, we computed data for image reconstruction by taking the logarithm of the average of the negative exponential of the integration of the object function over these lines connecting the source and detector subelements. In the study, we have investigated the reconstruction resolution properties by computing the MFTs and AMFTs at three locations: $(x, y, z) = (0.5 \text{ mm}, 0 \text{ mm}, 0 \text{ mm})$ (i.e., the center of the field of view), $(50 \text{ mm}, 0 \text{ mm}, 0 \text{ mm})$, and $(50 \text{ mm}, 0 \text{ mm}, 50 \text{ mm})$, respectively. Using the existing and proposed BPF algorithms, we reconstructed impulse response images at these locations from their corresponding data, followed by the computation of the MFTs and AMFTs for evaluation of the in-plane resolution at these locations, and the results are displayed in Fig. 11. It can be observed that the MFTs and AMFTs obtained with both BPF algorithms are virtually identical, suggesting that the proposed BPF algorithm without spatially varying weighting factor retains the resolution properties of the existing BPF algorithm. Moreover, it can also be seen that the MFTs for these three distinct locations are different, which indicates that the resolution properties in the reconstructed images are spatially varying.

III.D. Quantitative study of noise properties

We have also performed computer-simulation studies to evaluate quantitatively the noise properties of the proposed BPF algorithm in comparison with those of the existing BPF algorithm. In the studies, the responses of these algorithms to two different noise models, an object-independent Gaussian model and an object-dependent Poisson model, were investigated. With the modified Shepp–Logan phantom and scanning configuration used in Sec. III A, we first generated noiseless data at 300 views uniformly distributed over 2π .

FIG. 12. Empirical image variances within (a) a transverse plane at $z=0 \text{ mm}$, (b) a coronal plane at $x=0 \text{ mm}$, and (c) a sagittal plane at $y=0 \text{ mm}$, respectively, obtained from the data containing Gaussian noise by use of the existing (upper row) and proposed (lower row) BPF algorithms. The display window is $[3 \times 10^{-5}, 8 \times 10^{-5}]$.

For each noise model, 3000 sets of noisy data were created by use of the noiseless data as the means. The standard deviation of Gaussian noise is 0.16% of the maximum value in the noiseless data, whereas the standard deviation for the Poisson noise is the noiseless data scaled to yield a total photon count of 5×10^5 for each detector bin. From the 3000 noisy data sets containing Gaussian or Poisson noises, we reconstructed 3000 images by using the existing and proposed BPF algorithms. Based on the reconstructed noisy images, we then computed empirical image variances. In Fig. 12, we show the empirical image variances within a transverse plane at $z=0.0 \text{ mm}$, a coronal plane at $x=0.0 \text{ mm}$, and a sagittal plane at $y=0.0 \text{ mm}$, respectively, for the data containing Gaussian noise. For a quantitative comparison, we also show in Fig. 13 the empirical variance profiles obtained with the existing (dashed curve) and proposed (solid curve) BPF algorithms along three lines specified by (a) $y=0 \text{ mm}$ and $z=0 \text{ mm}$, (b) $x=0 \text{ mm}$ and $y=-96 \text{ mm}$, and (c) $x=-96 \text{ mm}$ and $y=0 \text{ mm}$, respectively. The results in Figs. 12 and 13 clearly demonstrate that the proposed BPF algorithm yields images with generally lower variances than the existing BPF algorithm, implying that the former is less susceptible to data noises than the latter. Moreover, we display in Fig. 14 the empirical image variances within a transverse plane at $z=0.0 \text{ mm}$, a coronal plane at $x=0.0 \text{ mm}$, and

FIG. 13. Profiles of empirical image variances along three lines specified by (a) $y=0 \text{ mm}$ and $z=0 \text{ mm}$, (b) $x=0 \text{ mm}$ and $y=-96 \text{ mm}$, and (c) $x=-96 \text{ mm}$ and $y=0 \text{ mm}$, respectively, obtained from the data containing Gaussian noise by use of the existing (dashed curve) and proposed (solid curve) BPF algorithms. The unit of y axis is 10^{-5} .

FIG. 14. Empirical image variances within (a) a transverse plane at $z=0$ mm, (b) a coronal plane at $x=0$ mm, and (c) a sagittal plane at $y=0$ mm, respectively, obtained from the data containing Poisson noise by use of the existing (upper row) and proposed (lower row) BPF algorithms. The display window is $[2 \times 10^{-6}, 1.2 \times 10^{-5}]$.

a sagittal plane at $y=0.0$ mm, respectively, for data containing Poisson noise. The quantitative results of the Poisson noise study are shown in Fig. 15, which also demonstrate that the proposed BPF algorithm yields images with generally lower variances than the existing BPF algorithm.

IV. CONCLUSION

In this work, we have proposed a BPF algorithm that, unlike the existing BPF algorithm, involves no spatially varying weighting factor in its backprojection step for image reconstruction from circular cone-beam data. We have also carried out numerical studies to validate and evaluate the proposed BPF algorithm using nontruncated and truncated circular cone-beam data. The proposed BPF algorithm retains the desirable property of the existing BPF algorithm, which can reconstruct exactly the ROI image from truncated data for a midplane in a circular scan. Quantitative studies of the proposed BPF algorithm demonstrate that, while the proposed BPF algorithm yields image resolution comparable to that produced by the existing BPF algorithm, the former has improved noise properties over the latter in terms of overall noise level and uniformity in the reconstructed images. The proposed BPF algorithm may be applied to image reconstruction from the data acquired by use of configurations with short focal lengths and/or large fan angles, which may

FIG. 15. Profiles of empirical image variances along three lines specified by (a) $y=0$ mm and $z=0$ mm, (b) $x=0$ mm and $y=-96$ mm, and (c) $x=-96$ mm and $y=0$ mm, respectively, obtained from the data containing Poisson noise by use of the existing (dashed curve) and proposed (solid curve) BPF algorithms. The unit of y axis is 10^{-6} .

be encountered in compact micro-CT (Ref. 27) and CT applications in image-guided radiation therapy.²⁸ The proposed BPF algorithm can be generalized to reconstructing images from the data acquired with nonplanar detectors such as curved detector and spherical detector.^{29,30}

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APPENDIX: THE WEIGHTED BACKPROJECTION

We show below that Eqs. (10) and (16) are (approximately) equivalent. Let

$$\gamma(\lambda) = \arctan \frac{\vec{r}' \cdot \hat{e}_u(\lambda)}{R - \vec{r}' \cdot \hat{e}_w(\lambda)} = \arctan \frac{u}{S}. \quad (\text{A1})$$

When $\lambda = -\pi$, one can easily show that $\lambda' = -2\gamma$. Substituting Eq. (17) into Eq. (10), one can obtain that

$$\begin{aligned} g(x'_c, \lambda_a, \lambda_b, z_0) &= \int_{\lambda_a}^{\lambda_b} \frac{d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} \\ &= \int_{\lambda_a}^{\lambda_b} \frac{d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} [\omega(\lambda, \vec{r}') - \omega(\lambda', \vec{r}')] \\ &\quad \times \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} \\ &= \int_{\lambda_a}^{\lambda_b} \frac{\omega(\lambda, \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} \\ &\quad - \int_{\lambda_a}^{-2\gamma} \frac{\omega(\lambda', \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}} \\ &\quad - \int_{-2\gamma}^{\lambda_b} \frac{\omega(\lambda', \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u, v, \lambda)}{d\lambda} \right|_{\hat{\beta}}. \quad (\text{A2}) \end{aligned}$$

Using Eq. (15), we can rewrite the second term in Eq. (A2) as

$$\begin{aligned} T_2 &= \int_{\lambda_a}^{-2\gamma} \frac{\omega(\lambda', \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u', v, \lambda')}{d\lambda'} \right|_{\hat{\beta}} \quad \text{for } v = 0, \\ T_2 &\approx \int_{\lambda_a}^{-2\gamma} \frac{\omega(\lambda', \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \left. \frac{dP(u', v, \lambda')}{d\lambda'} \right|_{\hat{\beta}} \quad \text{for } v \neq 0. \quad (\text{A3}) \end{aligned}$$

As Eq. (A5) in Appendix A of Ref. 17 shows,

$$\frac{d\lambda}{d\lambda'} = \frac{|\vec{r}' - \vec{r}_0(\lambda)|}{|\vec{r}' - \vec{r}_0(\lambda')|}. \quad (\text{A4})$$

Using Eq. (A4) in Eq. (A3) yields

$$\begin{aligned}
T_2 &= \int_{\lambda_b}^{\pi} \frac{\omega(\lambda', \vec{r}') d\lambda'}{|\vec{r}' - \vec{r}_0(\lambda')|} \frac{dP(u', v, \lambda')}{d\lambda'} \bigg|_{\hat{\beta}} \\
&= \int_{\lambda_b}^{\pi} \frac{\omega(\lambda, \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} \quad \text{for } v = 0, \\
T_2 &\approx \int_{\lambda_b}^{\pi} \frac{\omega(\lambda', \vec{r}') d\lambda'}{|\vec{r}' - \vec{r}_0(\lambda')|} \frac{dP(u', v, \lambda')}{d\lambda'} \bigg|_{\hat{\beta}} \\
&= \int_{\lambda_b}^{\pi} \frac{\omega(\lambda, \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} \quad \text{for } v \neq 0. \quad (\text{A5})
\end{aligned}$$

It should be noted that the integral range $[\lambda_a, -2\gamma]$ in Eq. (A3) has been changed to $[\lambda_b, \pi]$. Similarly, the third term in Eq. (A2) can be written as

$$\begin{aligned}
T_3 &= \int_{-\pi}^{\lambda_a} \frac{\omega(\lambda, \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} \quad \text{for } v = 0, \\
T_3 &\approx \int_{-\pi}^{\lambda_a} \frac{\omega(\lambda, \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} \quad \text{for } v \neq 0. \quad (\text{A6})
\end{aligned}$$

Substituting Eqs. (A4) and (A6) into Eq. (A2), we obtain

$$\begin{aligned}
g(x'_c, \lambda_a, \lambda_b, z_0) &= \int_{-\pi}^{\pi} \frac{\omega(\lambda, \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} \\
&= g_w(x'_c, \lambda_a, \lambda_b, z_0) \quad \text{for } z_0 = 0, \\
g(x'_c, \lambda_a, \lambda_b, z_0) &\approx \int_{-\pi}^{\pi} \frac{\omega(\lambda, \vec{r}') d\lambda}{|\vec{r}' - \vec{r}_0(\lambda)|} \frac{dP(u, v, \lambda)}{d\lambda} \bigg|_{\hat{\beta}} \\
&= g_w(x'_c, \lambda_a, \lambda_b, z_0) \quad \text{for } z_0 \neq 0,
\end{aligned}$$

which are the results in Eq. (18).

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