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Covariances between regression coefficient estimates in a single mediator model

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Abstract

This study presents formulae for the covariances between parameter estimates in a single mediator model. These covariances are necessary to build confidence intervals (CI) for effect size measures in mediation studies. We first analytically derived the covariances between the parameter estimates in a single mediator model. Using the derived covariances, we computed the multivariate-delta standard errors, and built the 95% CIs for the effect size measures. A simulation study evaluated the accuracy of the standard errors as well as the Type I error, power, and coverage of the CIs using various parameter values and sample sizes. Finally, we presented a numerical example and a SAS MACRO that calculates the CIs for the effect size measures.

1. Introduction

Mediation models have been the focus of much substantive and statistical research (Holland, 1988; Kenny, Kashy, & Bolger, 1998; MacKinnon, Lockwood, Hoffman, West, & Sheets, 2002). Numerous examples of mediation studies can be found across different areas of social science such as social psychology (Ajzen & Fishbein, 1980), clinical psychology (Kendall & Treadwell, 2007), prevention studies (MacKinnon, 1994; Orlando, Ellickson, McCaffrey, & Longshore, 2005), and organizational research (Frese, Garst, & Fay, 2007).

One of the important issues in mediation studies is to build confidence intervals (CI) and test hypotheses regarding various effects in a single mediator model (MacKinnon & Dwyer, 1993; MacKinnon *et al.*, 2002). These effects include measures of effect sizes, criteria for the mediator to exist, and tests of relations among different parameters in a single mediator model. To test a hypothesis and build a CI for an effect in a single mediator model, standard errors of the effects are needed. One of the common methods to obtain standard errors of the effect size measures in a single mediator model is to use the multivariate delta method and covariances among the parameter estimates (Rao, 1973, pp. 486–500). Although some of the covariances have been proposed in the literature (Clogg, Petkova, & Haritou, 1995; MacKinnon, 2008, chap. 4; McGuigan & Langholz, 1988; Sobel, 1982, 1986), no study to date has addressed deriving all of the covariances in a single mediator model.

The purpose of this study was first to derive all of the covariances between the parameter estimates in a single mediator model. Next, we used the derived covariances to calculate the multivariate-delta standard errors of the several effect size measures in a single mediator model. In addition, a simulation study assessed the accuracy of the multivariate-delta standard errors and evaluated the Type I error, power, and coverage of the CIs. A numerical example and a

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SAS MACRO that computes the CIs for the effect size measures are presented. Finally, the manuscript concludes with limitations of this study and future research directions.

1.1. Single mediator model

The focus of this study is a single mediator model where an independent variable is hypothesized to change a mediator which, in turn, changes a dependent variable. Three equations used to assess quantities in the single mediator model are shown below. In these equations upper-case roman letters (e.g. X) and bold Greek letters (e.g. ϵ_1) denote vectors or matrices. Lower-case roman letters (e.g. a), and the hat sign (i.e. $\hat{a}$) is used to denote a parameter estimate (e.g. $\hat{a}$).

$$Y=cX+\epsilon_1 \quad (1)$$

$$Y=bM+c'X+\epsilon_2 \quad (2)$$

$$M=aX+\epsilon_3 \quad (3)$$

where Y is the dependent variable, X is the independent variable, and M is the mediator. Equation (1) states a model where the independent variable (X) predicts the dependent variable (Y). Equation (2) represents the model where the independent variable (X) and mediator (M) together predict the dependent variable (Y), and Equation (3) represents the hypothesized model where the independent variable (X) predicts the mediator (M). ϵ_1 , ϵ_2 , and ϵ_3 are residual vectors. The mediated or indirect effect of the independent variable X on the outcome variable Y is measured by ab (Alwin & Hauser, 1975; Bollen, 1987; Fox, 1980; Sobel, 1982, 1986) or $c - c'$ (Freedman & Schatzkin, 1992; MacKinnon, 2008, chap. 4; McGuigan & Langholz, 1988). The total effect of the independent variable on the outcome variable is c or $ab + c'$ which is the sum of the direct (c') and indirect effect (ab) (MacKinnon & Dwyer, 1993). The parameters in a single mediator model are estimated using the least square or maximum likelihood estimation methods.

1.2. Effect size measures

Effect size measures provide a meaningful way of comparing the mediated effect across mediation studies regardless of sample sizes (MacKinnon, 2008). One of the most common measures of effect size to gauge the mediated effect is the ratio of the mediated effect to the total effect (Alwin & Hauser, 1975; MacKinnon, Warsi, & Dwyer, 1995). There are three algebraically equivalent expressions of the ratio of the mediated effect: $(ab)/(ab + c')$; $(ab)/c$; and $1 - (c'/c)$. The ratio of mediated effect to the total effect gauges the proportion of the total effect of the independent variable (X) on the dependent variable (Y) transmitted through the mediator variable (M). For example, the ratio of the mediated effect to the total effect of 0.40 means that 40% of the total effect is transmitted through the mediator variable.

Another related measure of the mediated effect is the ratio of the mediated effect to the non-mediated effect $(ab)/c'$ (Sobel, 1982). For example, if this measure is equal to 0.33, a researcher can conclude that the size of the mediated effect is one third of the non-mediated effect. Another useful measure to identify the surrogate (mediator) end-point is the ratio of the total effect to the effect of X on $M(c/a)$ (Buyse & Molenberghs, 1998). Surrogate end-points are the variables used instead of an ultimate outcome when the ultimate outcome has a low occurrence frequency

or takes a long time to occur (MacKinnon, 2008). For a variable (M) to be considered as a surrogate end-point for the ultimate dependent variable (Y), the ratio of the total effect (c) to the effect of X on $M(a)$ is expected to be one, because the magnitude of the relation between the mediator and the independent variable should be equal to the magnitude of the relation between the dependent and the independent variable.

Although the point estimates of the effect size measures provide useful information on the relative size of the mediated effect, they do not give any indication of the statistical significance or sampling errors of the estimated measures (MacKinnon *et al.*, 2002). Therefore, CIs are recommended as they incorporate both point estimates and sampling errors, which are measured by standard errors; CIs also can be utilized for significance testing (Harlow, Mulaik, & Steiger, 1997). Standard errors of the effects are a function of the variances and covariances between the parameter estimates because the effect size measures are a function of the parameter estimates in a single mediator model (i.e. $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{c}'$). Obtaining covariances between the parameter estimates is a prerequisite to deriving standard errors of the effect size measures.¹ It also should be noted that deriving standard errors of the effect size measures requires approximate methods such as the multivariate delta method (Rao, 1973, pp. 486–500), which will be discussed later in this manuscript.

1.3. Variance–covariance among parameter estimates

There are 10 variances and covariances among the parameter estimates ($\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{c}'$) in a single mediator. $\text{Var}(\hat{a})$, $\text{Var}(\hat{b})$, $\text{Var}(\hat{c})$, $\text{Var}(\hat{c}')$, and $\text{Cov}(\hat{b}, \hat{c}')$ can be obtained by the least squares and maximum likelihood estimation methods of regression analysis (Goldberger, 1964). Because separate regression equations represent a single mediator model (i.e. Equations (1–3)), deriving covariances between the parameter estimates from different equations (i.e. $\text{Cov}(\hat{a}, \hat{b})$, $\text{Cov}(\hat{a}, \hat{c})$, $\text{Cov}(\hat{a}, \hat{c}')$, $\text{Cov}(\hat{b}, \hat{c})$, and $\text{Cov}(\hat{c}, \hat{c}')$) is not straightforward. Previous research addressed two of the covariances between the parameter estimates. Sobel (1982, 1986) showed that the covariance between $\hat{a}$ and $\hat{b}$ was zero. Clogg *et al.* (1995), MacKinnon (2008, chap. 4), and McGuigan and Langholz (1988) derived the covariance between $\hat{c}$ and $\hat{c}'$. To our knowledge, no study has addressed the remaining covariances between the parameter estimates (i.e. $\text{Cov}(\hat{a}, \hat{c})$, $\text{Cov}(\hat{a}, \hat{c}')$, $\text{Cov}(\hat{b}, \hat{c})$). In the next section, we derive the covariances between all of the parameter estimates in a single mediator model, and calculate the standard errors of the select effect size measures (i.e. $(ab)/(ab + c')$, $(ab)/c$, $1 - (c'/c)$, $(ab)/c'$, and c/a) using the derived covariances and the multivariate delta method.

2. Method

2.1. Analytical background

Before proceeding further, it is necessary to provide some analytical background regarding the single mediator model. Note that Equations (2) and (3) are the main equations and Equation (1) does not provide any ‘independent’ information by itself. That is, as shown in Equations (10–13), Equation (1) is a linear combination of Equations (2) and (3). Assumptions regarding Equations (2) and (3) are that the error terms are random normal variables that are identically and independently distributed from each other. Moreover, predictors are assumed to be constant, and thus independent from respective random error vectors in each regression equation. Finally, the normality assumption renders both maximum likelihood and least squares

¹It should be noted that if the covariance term is zero (e.g. $\text{Cov}(\hat{a}, \hat{b})$), it will not appear in the standard error formula. For example, using the multivariate delta method, the standard error for the mediated effect is

$$SE(\widehat{ab}) \approx \sqrt{b^2 V(\widehat{a}) + a^2 V(\widehat{b}) + 2abcov(\widehat{a}, \widehat{b})} = \sqrt{b^2 V(\widehat{a}) + a^2 V(\widehat{b})}.$$

estimates to be identical (Goldberger, 1964). Having said that, the rest of our argument is based on the least squares method.

$$\varepsilon_2 \sim \text{MVN}(\mathbf{0}, \sigma_2^2 \mathbf{I}) \quad (4)$$

$$\varepsilon_3 \sim \text{MVN}(\mathbf{0}, \sigma_3^2 \mathbf{I}) \quad (5)$$

$$\text{Cov}(X, \varepsilon_2) = \mathbf{0} \quad (6)$$

$$\text{Cov}(M, \varepsilon_2) = \mathbf{0} \quad (7)$$

$$\text{Cov}(X, \varepsilon_3) = \mathbf{0} \quad (8)$$

$$\text{Cov}(\varepsilon_2, \varepsilon_3) = \mathbf{0} \quad (9)$$

The error term in Equation (1) is comprised of the error terms in Equations (2) and (3). That is, substituting Equation (3) into Equation (2) yields:

$$Y = (ab + c')X + (b\varepsilon_3 + \varepsilon_2) \quad (10)$$

Comparing with Equation (1), this leads to the following results:

$$c = ab + c' \quad (11)$$

$$\varepsilon_1 = b\varepsilon_3 + \varepsilon_2 \quad (12)$$

$$\sigma_1^2 = b^2 \sigma_3^2 + \sigma_2^2 \quad (13)$$

$$\text{Cov}(\varepsilon_1, \varepsilon_2) = \text{Cov}(b\varepsilon_3 + \varepsilon_2, \varepsilon_2) = \sigma_2^2 \mathbf{I} \quad (14)$$

$$\text{Cov}(\varepsilon_1, \varepsilon_3) = \text{Cov}(b\varepsilon_3 + \varepsilon_2, \varepsilon_3) = b\sigma_3^2 \mathbf{I} \quad (15)$$

To derive covariances between the three regression coefficients, we used two methods. The two methods are mathematically equivalent, and thus are expected to yield the same results. The first method used the least squares estimates of regression equations, and derived the covariance between the coefficients based on the least squares assumptions and relations between predictors and random error terms using matrix algebra (Rao, 1973). The second method used covariance algebra without relying on the least squares estimates of regression coefficients. Throughout the paper, we refer to the first and second method as the matrix algebra method and the univariate algebra method, respectively. We presented the details of these calculations in the Appendix.

The least squares estimates of Equations (1–3) are unbiased (Goldberger, 1964). That is,

$$\widehat{a} = (X^T X)^{-1} X^T M \quad (16)$$

$$E(\widehat{a}) = a \quad (17)$$

$$V(\widehat{a}) = \sigma_3^2 (X^T X)^{-1} \quad (18)$$

$$\widehat{b} = \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} Y \quad (19)$$

$$V(\widehat{b}) = \frac{(X^T X)}{(X^T X M^T M - X^T M M^T X)} \sigma_2^2 \quad (20)$$

$$\widehat{c}' = \frac{(M^T M X^T - X^T M M^T)}{(M^T M X^T X - X^T M M^T X)} Y \quad (21)$$

$$E(\widehat{c}') = c' \quad (22)$$

$$V(\widehat{c}') = \frac{(M^T M)}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2 \quad (23)$$

$$\widehat{c} = (X^T X)^{-1} X^T Y \quad (24)$$

$$E(\widehat{c})=c \quad (25)$$

$$V(\widehat{c})=\sigma_1^2(X^T X)^{-1} \quad (26)$$

Below are the results from the matrix algebra method. A summary of the matrix algebra method derived covariances is shown in Table 1.

$$\text{Cov}(\widehat{a}, \widehat{c})=b\sigma_3^2(X^T X)^{-1}=b\text{Var}(\widehat{a}) \quad (27)$$

$$\text{Cov}(\widehat{a}, \widehat{c}')=0 \quad (28)$$

$$\text{Cov}(\widehat{a}, \widehat{b})=0 \quad (29)$$

$$\text{Cov}(\widehat{b}, \widehat{c})=0 \quad (30)$$

$$\text{Cov}(\widehat{b}, \widehat{c}')=-\frac{M^T X}{(M^T M X^T X - X^T M M^T X)}\sigma_2^2 \quad (31)$$

$$\text{Cov}(\widehat{c}, \widehat{c}')=\sigma_2^2(X^T X)^{-1} \quad (32)$$

The $\text{Cov}(\widehat{c}, \widehat{c}')$ derived in Equation (32) is the same as the one suggested in the literature (MacKinnon, 2008, chap. 4; McGuigan & Langholz, 1998).

In the univariate algebra method, as mentioned before, we used univariate covariance algebra between the least squares estimators without relying on matrix algebra. For example, we used the following formulae.

$$\begin{aligned} \text{Cov}(\widehat{a}, \widehat{c}) &= \text{Cov}(\widehat{a}, \widehat{ab} + \widehat{c}') = \text{Cov}(\widehat{a}, \widehat{ab}) + \text{Cov}(\widehat{a}, \widehat{c}') = \text{Cov}(\widehat{a}, \widehat{ab}) \\ \text{Cov}(\widehat{a}, \widehat{c}) &= b\text{Var}(\widehat{a}) = b\sigma_3^2(X^T X)^{-1} \end{aligned} \quad (33)$$

$$\text{Cov}(\widehat{c}, \widehat{c}') = \text{Var}(\widehat{c}) - b^2 \text{Var}(\widehat{a}) \quad (34)$$

To check this result with the matrix algebra method, we first substituted Equations (18) and (26) into Equation (34):

$$\begin{aligned}\text{Cov}(\widehat{c}, \widehat{c}') &= \text{Var}(\widehat{c}) - b^2 \text{Var}(\widehat{a}) = \sigma_1^2 (X^T X)^{-1} - b^2 \sigma_3^2 (X^T X)^{-1} = (\sigma_1^2 - b^2 \sigma_3^2) (X^T X)^{-1} \\ &= \sigma_2^2 (X^T X)^{-1}\end{aligned}$$

As expected, the matrix algebra method and the univariate algebra method produced the same results.

$$\text{Cov}(\widehat{c}, \widehat{c}') = \text{Var}(\widehat{c}) - b^2 \text{Var}(\widehat{a}) = \sigma_2^2 (X^T X)^{-1} \quad (35)$$

$$\text{Cov}(\widehat{b}, \widehat{c}) = \text{Cov}(\widehat{b}, \widehat{c}') + a \text{Var}(\widehat{b}) = 0 \quad (36)$$

$$\text{Cov}(\widehat{a}, \widehat{c}') = 0 \quad (37)$$

2.2. Standard errors of the effect size measures

In this section, we discuss the multivariate delta method (Rao, 1973, pp. 486–500) and its application to derive the standard errors of the five effect sizes measures. As mentioned earlier, although the point estimates of the effect size measures provide useful information, CIs are much more informative because they take into account the sampling error of the point estimates. To build CIs for the effect size measures it is essential to obtain the standard error of the measures. Because the effect size measures are non-linear functions of the parameters in a single mediator model, approximate methods such as the multivariate delta method, a multivariate extension of the first-order Taylor series of the effect size measures, were used to derive the standard errors.

More formally, suppose that statistic L is a function of a vector of t parameters, $P^T = (p_1, \dots, p_t)$ such that $L = f(P) = f(p_1, \dots, p_t)$. Note that superscript T stands for the vector transpose operation. Also, suppose that $\widehat{P}^T = (\widehat{p}_1, \dots, \widehat{p}_t)$ is an estimate of the parameters in vector P with a multivariate normal distribution, $\widehat{P} = \text{MVN}(P, \Sigma)$, where P is the vector of the expected values of parameters (e.g. $E(\widehat{P}) = P$), and Σ is the covariance matrix of the parameters estimated by the sample covariance matrix $\mathbf{S}$. For example, because the ratio of the indirect effect to the direct effect is a function of parameters a , b , and c' , $L = f(a, b, c') = (ab)/c'$, we have $P^T = (a, b, c')$, $\widehat{P}^T = (\widehat{a}, \widehat{b}, \widehat{c}')$.

The multivariate delta method approximates the distribution of a non-linear function of random variables distributed as multivariate normal. More specifically, as the sample size increases, we have

$$L \sim N(f(P), \mathbf{D} \Sigma \mathbf{D}^T),$$

where $\mathbf{D} = (\partial L)/(\partial P)$ is the row vector of the first partial derivatives of L with regard to the parameters in $L = f(p_1, \dots, p_t)$. For $(\widehat{a} \widehat{b})/\widehat{c}'$ we have $\widehat{\mathbf{D}} = (\partial \widehat{L})/(\partial \widehat{P}) = ((\partial \widehat{L})/(\partial \widehat{p}_1), \dots, (\partial \widehat{L})/(\partial \widehat{p}_t))$ and $\widehat{\mathbf{D}} = (\partial \widehat{L})/(\partial \widehat{P}) = ((\widehat{b}/\widehat{c}'), (\widehat{a}/\widehat{c}'), (-\widehat{a}\widehat{b}/\widehat{c}'^2))$.

More conveniently, general formulae can be obtained for the standard errors of $(\hat{a}\hat{b})/(\hat{a}\hat{b} + \hat{c}')$, $(\hat{a}\hat{b})/\hat{c}$, $1 - (\hat{c}'/\hat{c})$, $(\hat{a}\hat{b})/\hat{c}'$, and $\hat{c}/\hat{a}$ using the multivariate delta method. After some algebraic simplification, the approximate standard errors of the measures were as follows:

$$SE\left(\frac{\widehat{ab}}{\widehat{ab} + \widehat{c}'}\right) \approx \sqrt{\left(\frac{\widehat{bc'}}{(\widehat{ab} + \widehat{c}')^2}\right)^2 V(\widehat{a}) + \left(\frac{\widehat{ac'}}{(\widehat{ab} + \widehat{c}')^2}\right)^2 V(\widehat{b}) + \left(\frac{\widehat{ab}}{(\widehat{ab} + \widehat{c}')^2}\right)^2 V(\widehat{c}') - 2\left(\frac{\widehat{a^2bc'}}{(\widehat{ab} + \widehat{c}')^4}\right) \text{Cov}(\widehat{b}, \widehat{c}')} \quad (38)$$

$$SE\left(\frac{\widehat{ab}}{\widehat{c}}\right) \approx \sqrt{\left(\frac{\widehat{b}}{\widehat{c}}\right)^2 V(\widehat{a}) + \left(\frac{\widehat{a}}{\widehat{c}}\right)^2 V(\widehat{b}) + \left(\frac{\widehat{ab}}{\widehat{c}^2}\right)^2 V(\widehat{c}) - 2\left(\frac{\widehat{ab^2}}{\widehat{c}^3}\right) \text{Cov}(\widehat{a}, \widehat{c})} \quad (39)$$

$$SE\left(1 - \frac{\widehat{c}'}{\widehat{c}}\right) \approx \sqrt{\left(\frac{\widehat{c}'}{\widehat{c}^2}\right)^2 V(\widehat{c}) + \left(\frac{1}{\widehat{c}}\right)^2 V(\widehat{c}') - 2\left(\frac{\widehat{c}'}{\widehat{c}^3}\right) \text{Cov}(\widehat{c}, \widehat{c}')} \quad (40)$$

$$SE\left(\frac{\widehat{ab}}{\widehat{c}'}\right) \approx \sqrt{\left(\frac{\widehat{b}}{\widehat{c}'}\right)^2 V(\widehat{a}) + \left(\frac{\widehat{a}}{\widehat{c}'}\right)^2 V(\widehat{b}) + \left(\frac{\widehat{ab}}{\widehat{c}'^2}\right)^2 V(\widehat{c}') - 2\left(\frac{\widehat{a^2b}}{\widehat{c}'^3}\right) \text{Cov}(\widehat{b}, \widehat{c}')} \quad (41)$$

$$SE\left(\frac{\widehat{c}}{\widehat{a}}\right) \approx \sqrt{\left(\frac{\widehat{c}}{\widehat{a}^2}\right)^2 V(\widehat{a}) + \left(\frac{1}{\widehat{a}}\right)^2 V(\widehat{c}) - 2\left(\frac{\widehat{c}}{\widehat{a}^3}\right) \text{Cov}(\widehat{a}, \widehat{c})} \quad (42)$$

The derived standard errors or $(\hat{a}\hat{b})/(\hat{a}\hat{b} + \hat{c}')$ and $\hat{c}/\hat{a}$ match the results from MacKinnon *et al.* (1995). The contribution of our paper is that the formulas for additional effect size measures are provided, and the analytical formulas for all the covariances are given. In MacKinnon *et al.*'s (1995) paper, the covariances between the estimates were taken from the computer program output, and general formulas were not produced as they are in this paper.

2.3. Simulation

The purpose of the simulation study was first to assess the accuracy of the derived standard errors. In addition, we evaluated the Type I error, power, and coverage of the 95% CIs of the effect size measures under various conditions. Three factors were varied in the simulation study: sample size, population parameters for a and b (a and b were set to be equal), and c' . Based on Equations (2) and (3), the variables X , M , and Y were generated from continuous normal distribution using the SAS 9.1.3 RANNOR function and the computer clock as a seed. Commonly used samples in mediation study in social science were used in the present study: $N = 50, 100, 200, 500$, and $1,000$ (Mackinnon *et al.*, 2002). The parameter values for a and b took on the values 0.14, 0.39, and 0.59, while c' was 0, 0.14, 0.39, and 0.59. The parameter values were chosen so that they represented zero, small, medium, and large effect sizes of the regression models in Equations (2) and (3). Small ($a = b = 0.14$), medium ($a = b = 0.39$), and large ($a = b = 0.59$) effect sizes explained 2%, 13%, and 26% of the variance in the dependent variable (Y), respectively (Cohen, 1988, pp. 412–414).

Within each condition, 1,000 replications were conducted. For each replication, the covariances of the regression coefficients and the standard errors of the effect size measures were calculated using Equations (38–42). True values of the standard errors were estimated by computing the standard deviation of 1,000 estimates for each effect size measure. To assess the accuracy of the derived standard errors, we calculated the root mean square error (RMSE) and relative bias for standard errors using the following formulae:

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{1,000} (\hat{w}_i - w)^2}{1,000}}$$

$$\text{Relative bias} = \frac{(\bar{w} - w)}{w},$$

where w was the true value of the standard error, $\hat{w}_i$ was the derived standard error of the effect size measure for the replication i , and $\bar{w}$ was the average of the 1,000 estimates of the standard errors within each condition.

To clarify further, we briefly describe the simulation process for one of the measures, the proportion mediated effect, $(ab)/c$. For notational simplicity, let m and $\hat{m}_i$ be the population value and the sample estimate of the proportion mediated effect for the replication i , respectively. For each condition, using the population values of the parameters a , b , and c' , we can readily calculate the population value of the proportion mediated effect, $m = (ab)/c$. For instance, for the condition where $a = b = c' = 0.14$, the population value of the proportion mediated effect was $m = 0.1228$. For the replication i , the sample value of the proportion mediated effect was $\hat{m}_i = (\hat{a}\hat{b})/\hat{c}$ where $\hat{a}$, $\hat{b}$, and $\hat{c}$ were the least squares estimates. In addition, for each $\hat{m}_i$, we calculated the multivariate-delta standard error of the proportion mediated effect, $\hat{w}_i = SE(\hat{a}\hat{b}/\hat{c})$, using Equation (39).

For this condition ($m = 0.1228$), we need to summarize the simulation results to obtain the estimate of the outcomes of interest. One of the outcomes was the estimate of multivariate-delta standard error across 1,000 replications, $\hat{w}$. That is, $\hat{w} = \Sigma \hat{w}_i / 1,000$. To assess the accuracy of the empirical estimate of the multivariate-delta standard error, one needs to obtain the true value of the standard error. Within each condition, we estimated the true value of the standard error of the proportion mediated effect by computing the standard deviation of 1,000 sample

estimates the measure ($\hat{m}_i$). That is, $w = \sqrt{\sum_{i=1}^{1,000} (\hat{m}_i - \bar{m})^2 / 999}$, where $\bar{m}$ is the mean of the 1,000 $\hat{m}_i$'s.

Further, we examined the Type I error and the power of 95% CIs for the effect size measures. The Type I error for a CI was the rejection rate when the population (true) value of an effect size measure was zero. For example, when both a and b were equal to zero, the Type I error was the number of the times out of 1,000 replications that the CI (i.e. $(\hat{a}\hat{b}/\hat{c}) \pm 1.96 \times SE(\hat{a}\hat{b}/\hat{c})$) did not contain zero. Similarly, the power of a CI of an effect size measure was the rejection rate for the conditions where the population value of the effect size was not zero. For instance, when $a = b = 0.14$, the power was the proportion of the times that the CI of the proportion mediated effect did not include zero.

Finally, the coverage rate of a CI was the frequency with which the CI contained the population (true) value of an effect size measure. For example, the population value of the proportion mediated effect for the condition where $a = b = c' = 0.14$ was 0.1228. For this condition, if the 95% CI in a replication contained the value of 0.1228, the coverage was set to one; otherwise, the coverage equalled zero. The simulation results are shown in the next section.

2.4. Results

Because the measures of RMSE and relative bias were virtually the same for the three measures of the proportion mediated effects (i.e. $(\hat{a}\hat{b})/(\hat{a}\hat{b} + \hat{c}')$, $(\hat{a}\hat{b})/\hat{c}$, $1 - (\hat{c}'/\hat{c})$) to save space, only the results for $(\hat{a}\hat{b})/(\hat{a}\hat{b} + \hat{c}')$ were presented.

As shown in Table 2, several trends can be identified to understand the behaviour of the standard error for the proportion mediated effect. When the direct effect (c') was zero or small (0.14), both larger sample and effect sizes were needed to obtain a reasonably accurate (i.e. less than 10% relative bias with small RMSE) estimates of the standard error of the proportion mediated effect. When the direct effect was zero ($c' = 0$), the minimum sample sizes of 500 and 200 were required for the medium (i.e. $a = b = 0.39$) and large ($a = b = 0.59$) effect sizes, respectively. For the small direct effect (i.e. $c' = 0.14$), the minimum sample sizes of 200 and 100 were required when the effect sizes were medium and large, respectively. Moreover, when both direct effect and effect sizes were small (i.e. $a = b = c' = 0.14$), a large sample size (e.g. 1,000 or higher) was necessary to obtain a reasonably accurate standard error for the proportion mediated effect.

The simulation results of the ratio of the indirect to direct effect ($(\hat{a}\hat{b})/\hat{c}'$) are shown in Table 3. As the size of the direct effect increased, the minimum sample size to achieve a reasonably accurate standard error decreased. For example, for the small direct effect ($c' = 0.14$), when the effect size was small (i.e. $a = b = 0.14$), a sample size equal or greater than 1,000 was necessary to obtain a reasonably accurate standard error for the measure. However, when the effect size was medium and large, a sample size greater than 1,000 would be required. For the medium direct effect ($c' = 0.39$), the minimum sample size to achieve a reasonably accurate standard error was 200 regardless of the magnitude of the coefficients a and b . While for the large direct effect ($c' = 0.59$) and the small effect size ($a = b = 0.14$), the minimum sample size of 50 was required, for the medium and large effect sizes, a sample size of at least 100 was satisfactory.

The simulation results for the measure (c/a) are shown in Table 4. As can be seen in Table 4, the effect size (i.e. a and b) was the most influential factor while the magnitude of the direct effect (c') had a trivial impact. As the effect size increased, the sample size required to obtain a reasonably accurate standard error of the measure decreased. For example, when the effect was small ($a = b = 0.14$) and $c' = 0.14$ the minimum sample size to obtain an accurate standard error was 500. For the medium effect ($a = b = 0.39$) and $c' = 0.14$, the minimum sample size to achieve a reasonable accuracy of the standard errors was 50. Finally, when the effect was large ($a = b = 0.59$) and $c' = 0, 0.14$, and 0.39 , the sample size of 50 provided reasonably accurate standard errors. To assist readers to locate the minimum sample sizes to achieve a reasonable accuracy in the standard errors of the measures, we summarized the findings of the simulation in Table 5.

Tables 6–8 contain the results from the simulation study of the Type I error and power of the 95% CIs of the effect size measures. In addition, one can find the minimum sample sizes to achieve a certain level of power in Table 9. For three equivalent measures of the proportion mediated effect, the results regarding the power were virtually the same. Therefore, in the interest of space, we presented the results for the measure $ab/(ab + c')$. However, the Type I error rate for the measure $(1 - c'/c)$ differed from the other two measures of the proportion mediated effect (i.e. ab/c and $ab/(ab + c')$). We will discuss the Type I error rates for $1 - c'/c$ without tabulating the results.

As shown in Table 6, the Type I error rates for CIs of the two effect size measures ab/c and $ab/(ab + c')$ were virtually zero. This means that the CIs based on the multivariate-delta standard errors were 'conservative' in rejecting the null hypothesis (i.e. $H_0: ab/c = 0$) when there was

no mediated effect. However, the Type I error rate for the measure $1 - c'/c$, not shown in Table 6, varied from 0.11 to 0.18. This means that the CI for measure $1 - c'/c$ was at least twice and at most three times more likely than the nominal value of 0.05 to falsely reject the null hypothesis, and conclude that the proportion mediated effect was significant.

Regarding the power of the proportion mediated effect, several trends are evident in Table 6. In general, for a fixed sample size, the power of the CI for the proportion mediated effect increased as the magnitude of the effects (i.e. $a = b$) and c' increased. In other words, as the magnitudes of the effects and c' increased, the minimum sample size to achieve a certain level of power (e.g. 0.80) decreased. In addition, for the small effect ($a = b = 0.14$), the minimum sample size to achieve 0.90 power was 1,000. That is, when the effects of the coefficients in a single mediator model were small according to Cohen's benchmark, a researcher needs a relatively large sample size to achieve a minimum level of power of 0.90. For the medium and large effects, the minimum sample size to obtain a high level of power (e.g. 0.90) was less than 500. To facilitate finding the required minimum sample sizes to obtain indicated levels of power, we summarized the simulation results in Table 9.

As can be seen in Table 7, the Type I error rates of the CI for the ratio of the indirect to direct effect (ab/c') were also low. That is, when there was no mediation effect ($ab = 0$), the probability that the CI contained zero was almost equal to one. Regarding the power, when the direct effect was zero ($c' = 0$), the CI had the lowest power. In fact, for $c' = 0$, the highest level of power to detect a significant ratio (ab/c') was 0.06 regardless of the magnitude of the effect ($a = b$). For a direct effect (c') greater than zero, as the magnitude of the effect ($a = b$) increased, the power also increased. However, for the small direct effect ($c' = 0.14$), one still needs a sample size between 500 and 1,000 to achieve power greater than 0.80. For the medium and large direct effects and at least the medium effect (i.e. $a = b \geq 0.39$), a minimum sample size of 200 produced power of at least 0.99. The minimum sample sizes of the power study for the measure ab/c' are shown in Table 9.

The Type I error and power of the CI for the measure c/a are shown in Table 8. Note that the $H_0: c/a = 0$. The multivariate-delta CIs for the measure c/a had a near zero Type I error rate. Concerning power, for the zero effect ($a = b = 0$), power remained very low regardless of the sample size. For the small effect ($a = b = 0.14$), when the direct effect was greater than zero, the CI with a minimum sample size of 500 produced power of at least 0.76. The minimum sample sizes for other combination of effects and direct effects are summarized in Table 9.

We summarized the results of the coverage rate of 95% CIs of the effect size measures in Tables 10–12. For the proportion mediated effect, the coverage rates of the CIs of the three equivalent measures were virtually the same. Thus, we only presented the results for the measure $ab/(ab + c')$ in Table 10. In addition, we excluded the coverage rate for the condition $a = b = 0$ because the coverage rate was the Type I error (see Table 6). In general, the coverage rate of the CI of the proportion mediated effect was above 0.83, except when $c' = 0$ and $a = b = 0.14$. For this condition, the coverage rate ranged from 0.46 (sample size = 50) to 0.73 (sample size = 1,000).

Table 11 contains the coverage rates of the CI of the measure ab/c' . Again, we excluded the coverage rates corresponding to zero effect ($a = b = 0$) as they represented the Type I error rates. The coverage rate of the CI varied from 0.78 to 0.96. The minimum coverage rate occurred when the effect was large ($a = b = 0.59$), the direct effect was small ($c' = 0.14$), and the sample size was 50. As shown in Table 12, the coverage rate of the measure c/a ranged from 0.83 to 1.0. In general, it appears that the sample size and the magnitudes of the coefficients ($a = b$) in a single mediator model had a trivial impact on the coverage rate of the 95% CI for the measure c/a .

2.5. Example

In this section, using the derived covariances and the multivariate-delta standard errors, we illustrate a numerical example that builds 95% CIs for the estimates of the five effect size measures (i.e. $(ab)/(ab + c')$, $(ab)/c$, $1 - (c'/c)$, $(ab)/c'$, and c/a). The example utilizes a SAS MACRO shown in the Appendix to compute the CIs for the effect size measures. The data in the example were from a hypothetical study designed to improve the norms of eating a healthy diet (M), which in turn would improve health outcomes (Y). Five hundred participants in the study were randomized to receive varying amounts of exposure to a nutrition programme. The population parameter values for a , b , and c' corresponded to medium (0.39), medium (0.39), and large (0.59) effects, respectively. The values of X followed the standard normal distribution, and were produced using the RANNOR function in SAS 9.1.3. Values for Y and M were generated based on Equations (2) and (3), respectively. Parameter estimates were $\hat{a} = 0.44$, $\hat{b} = 0.38$, $\hat{c} = 0.72$, and $\hat{c}' = 0.56$. Based on the formulae provided for standard errors (i.e. Equations (38–42)) and the SAS MACRO code in the Appendix, the 95% CIs (e.g. $(\hat{ab}/\hat{c}) \pm 1.96 \times SE(\hat{ab}/\hat{c})$) were calculated. The results are shown in Table 13.

None of the CIs of the effect sizes contained zero indicating that the measures were statistically significant from zero. In addition, all of the CIs included the population values of the respected effect size measures. The proportion of the total effect of exposure to the nutrition programme on the diet that was mediated through the diet norms equalled 0.23, and the mediated effect was 0.30 times the size of the direct effect. The total effect of the programme on the diet ($\hat{c}$) was 1.67 times the effect of the programme on the diet norms ($\hat{a}$).

3. Discussion

Computing the standard errors of the effect size measures in a single mediator model is essential to build CIs and to test hypotheses regarding the measures. One method to obtain the standard errors of the measures is to use the multivariate delta method, which requires the covariances among the parameter estimates. To our knowledge, no study has derived all of the covariances in a single mediator model. First, we derived the covariances between the least squares parameter estimates using matrix algebra. Note that the derived covariances were exact and were not based on a large sample assumption. Using the derived covariances and the multivariate delta method, the standard errors of the five effect size measures ($(\hat{ab})/(\hat{ab} + \hat{c}')$, $(\hat{ab})/\hat{c}$, $1 - (\hat{c}'/\hat{c})$, $(\hat{ab})/\hat{c}'$, and $\hat{c}/\hat{a}$) were calculated. The same approach could be used to obtain standard errors of other quantities in a single mediator model such as the difference between the mediated effect and direct effect, $ab - c'$. The covariances between the parameter estimates described in this paper are required to calculate the standard errors of the quantities.

Because the multivariate delta method produces approximate standard errors, the derived standard errors were not exact. Thus, we conducted a simulation study to assess the accuracy of the standard errors in terms of relative bias and RMSE. Table 5 presents the minimum sample sizes required to achieve a reasonable accuracy (i.e. less than 10% relative bias with small RMSE) for different combination of parameter sizes. In addition, we examined the Type I error, power, and coverage of the 95% CIs for the effect size measures (e.g. $(\hat{ab}/\hat{c}) \pm 1.96 \times SE(\hat{ab}/\hat{c})$). Except for the measure $1 - c'/c$, all of the Type I error rates were virtually zero. The Type I error for $1 - c'/c$ ranged from 0.11 to 0.18. In general, the power increased as the coefficients values and sample sizes increased. Table 9 contains the minimum sample sizes needed to achieve the indicated levels of power.

Further, the coverage rate of the 95% CI of the proportion mediated effect was above 0.83, except when the direct effect was zero ($c' = 0$) and the effect size was small ($a = b = 0.14$). For this condition, the coverage rate ranged from 0.46 ($N = 50$) to 0.73 ($N = 1,000$). The coverage rate for 95% CIs of the measures ab/c' and c/a ranged from 0.78 to 0.96 and from 0.83 to 1.0,

respectively. Finally, we illustrated a hypothetical numerical example that used the SAS MACRO in the Appendix to estimate 95% CIs for the effect size measures.

One limitation of the derived covariances was that independent variables were considered fixed. That is, X in Equation (2) and both X and M in Equation (3) were considered fixed when estimating the model. This assumption may not be true in applied research and these results may not be applicable in such situations. Another limitation is that in estimating the standard errors we assumed that the residual terms in Equations (2) and (3) were normally distributed. Although the violation of normality assumption may not affect the accuracy of the approximate standard errors in larger samples, it is likely that the standard errors may become less accurate in smaller samples. One remedy for this is to use bootstrap CIs when sample sizes are small (MacKinnon, Lockwood, & Williams, 2004; Shrout & Bolger, 2002). Future simulation studies are needed to investigate the impact of the violation of normality on the accuracy of the derived standard errors and the coverage of the CIs. Finally, in building the CIs, we assumed that the distribution of the effect size measures were symmetrical around the point estimates. Previous research suggested that such an assumption may not be true, especially for smaller sample sizes (MacKinnon *et al.*, 2004; Stone & Sobel, 1990). Asymmetrical CIs can be built using empirical methods such as bootstrap (Bollen & Stine, 1990; MacKinnon *et al.*, 2004; Shrout & Bolger, 2002).

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Appendix

Derivation of covariances between $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{c}'$

Matrix algebra method

Based on the least squares assumptions stated in Equations (1–9) in a single mediator model, we can estimate the parameters in a single mediation model (i.e. Equations (1–3)):

$$\hat{a} = (X^T X)^{-1} X^T M \quad (\text{A1})$$

$$E(\hat{a}) = a \quad (\text{A2})$$

$$V(\widehat{a}) = \sigma_3^2 (X^T X)^{-1} \quad (\text{A3})$$

$$\widehat{b} = \frac{(X^T X M^T - M^T X X^T) Y}{(X^T X M^T M - X^T M M^T X)} \quad (\text{A4})$$

$$\begin{aligned} V(\widehat{b}) &= \frac{(X^T X M^T - M^T X X^T)^T}{(X^T X M^T M - X^T M M^T X)} (\sigma_2^2 \mathbf{I}) \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} \\ &= \frac{(X^T X M^T - M^T X X^T)(X^T X M^T - M^T X X^T)^T}{(X^T X M^T M - X^T M M^T X)^2} \sigma_2^2 \\ &= \frac{(X^T X M^T M - X^T M M^T X)^2}{(X^T X M^T M - X^T M M^T X)(M X^T X - X X^T M)} \sigma_2^2 \\ &= \frac{(X^T X M^T M - X^T M M^T X)^2}{(X^T X)(M^T M X^T X - M^T X X^T M)} \sigma_2^2 \\ &= \frac{(X^T X M^T M - X^T M M^T X)^2}{(X^T X)} \sigma_2^2 \\ &= \frac{(X^T X)}{(X^T X M^T M - X^T M M^T X)} \sigma_2^2 \\ V(\widehat{b}) &= \frac{(X^T X)}{(X^T X M^T M - X^T M M^T X)} \sigma_2^2 \quad (\text{A5}) \end{aligned}$$

$$\widehat{c}' = \frac{(M^T M X^T - X^T M M^T) Y}{(M^T M X^T X - X^T M M^T X)} \quad (\text{A6})$$

$$E(\widehat{c}') = c' \quad (\text{A7})$$

$$\begin{aligned} V(\widehat{c}') &= \frac{(M^T M X^T - X^T M M^T)(M^T M X^T - X^T M M^T)^T}{(M^T M X^T X - X^T M M^T X)^2} \sigma_2^2 \\ &= \frac{M^T M}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2 \\ V(\widehat{c}') &= \frac{M^T M}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2 \quad (\text{A8}) \end{aligned}$$

$$\widehat{c} = (X^T X)^{-1} X^T Y \quad (\text{A9})$$

$$E(\widehat{c}) = c \quad (\text{A10})$$

$$V(\widehat{c}) = \sigma_1^2 (X^T X)^{-1} \quad (\text{A11})$$

We derive covariance between the regression parameters. The following formulae are used:

$$\widehat{a} - E(\widehat{a}) = (X^T X)^{-1} X^T \epsilon_3 \quad (\text{A12})$$

$$\widehat{b} - E(\widehat{b}) = \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} \varepsilon_2 \quad (\text{A13})$$

$$\widehat{c} - E(\widehat{c}) = (X^T X)^{-1} X^T \varepsilon_1 \quad (\text{A14})$$

$$\widehat{c}' - E(\widehat{c}') = \frac{(M^T M X^T - X^T M M^T)}{(M^T M X^T X - X^T M M^T X)} \varepsilon_2 \quad (\text{A15})$$

Now, we derive the covariances between the regression coefficients:

$$\begin{aligned} \text{Cov}(\widehat{a}, \widehat{c}) &= E\{(\widehat{a} - E(\widehat{a}))(\widehat{c} - E(\widehat{c}))\} = E\{(X^T X)^{-1} X^T \varepsilon_1 \times (X^T X)^{-1} X^T \varepsilon_3\} = \\ \text{Cov}\{(X^T X)^{-1} X^T \varepsilon_1, (X^T X)^{-1} X^T \varepsilon_3\} &= (X^T X)^{-1} X^T \text{Cov}(\varepsilon_1, \varepsilon_3) X (X^T X)^{-1} = b \sigma_3^2 (X^T X)^{-1} \end{aligned}$$

Based on Equation (A3) we have:

$$\begin{aligned} \text{Cov}(\widehat{a}, \widehat{c}) &= b \sigma_3^2 (X^T X)^{-1} = b \text{Var}(\widehat{a}) \\ \text{Cov}(\widehat{a}, \widehat{c}') &= E\{(\widehat{a} - E(\widehat{a}))(\widehat{c}' - E(\widehat{c}'))\} = \\ \frac{(M^T M X^T - X^T M M^T)}{(M^T M X^T X - X^T M M^T X)} \text{Cov}(\varepsilon_2, \varepsilon_3) X (X^T X)^{-1} &= 0 \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} \text{Cov}(\widehat{a}, \widehat{c}') &= 0 \\ \text{Cov}(\widehat{a}, \widehat{b}) &= E\{(\widehat{a} - E(\widehat{a}))(\widehat{b} - E(\widehat{b}))\} = \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} \times \\ \text{Cov}(\varepsilon_2, \varepsilon_3) X (X^T X)^{-1} &= 0 \end{aligned} \quad (\text{A17})$$

$$\begin{aligned} \text{Cov}(\widehat{a}, \widehat{b}) &= 0 \\ \text{Cov}(\widehat{b}, \widehat{c}) &= E\{(\widehat{b} - b)(\widehat{c} - c)\} = \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} \text{Cov}(\varepsilon_2, \varepsilon_1) X (X^T X)^{-1} = \\ \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} (\sigma_2^2 \mathbf{I}) X (X^T X)^{-1} &= \\ \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} X (X^T X)^{-1} \sigma_2^2 &= \frac{(X^T X M^T X - M^T X X^T X)}{(X^T X M^T M - X^T M M^T X)} (X^T X)^{-1} \sigma_2^2 = \\ \frac{(X^T X M^T X - X^T X M^T X)}{(X^T X M^T M - X^T M M^T X)} (X^T X)^{-1} \sigma_2^2 &= \frac{0}{(X^T X M^T M - X^T M M^T X)} (X^T X)^{-1} \sigma_2^2 = 0 \end{aligned} \quad (\text{A18})$$

$$\begin{aligned} \text{Cov}(\widehat{b}, \widehat{c}) &= 0 \\ \text{Cov}(\widehat{b}, \widehat{c}') &= E\{(\widehat{b} - b)(\widehat{c}' - \widehat{c}')\} = \text{Cov}\left(\frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} \varepsilon_2, \frac{(M^T M X^T - X^T M M^T)}{(M^T M X^T X - X^T M M^T X)} \varepsilon_2\right) = \\ \frac{(X^T X M^T - M^T X X^T)}{(X^T X M^T M - X^T M M^T X)} \text{Var}(\varepsilon_2) \frac{(M^T M X^T - X^T M M^T)}{(M^T M X^T X - X^T M M^T X)} &= \\ \frac{(X^T X M^T M^T - X^T X M^T M M^T X - M^T X X^T X M^T M + M^T X X^T M M^T X)}{(X^T X M^T M - X^T M M^T X)^2} \sigma_2^2 &= \\ \frac{(-M^T X X^T X M^T M + M^T X X^T M M^T X)}{(X^T X M^T M - X^T M M^T X)^2} \sigma_2^2 &= \frac{(M^T M X^T X - X^T M M^T X)}{(M^T M X^T X - X^T M M^T X)^2} \sigma_2^2 = \\ -\frac{M^T X}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2 & \end{aligned} \quad (\text{A19})$$

$$\begin{aligned} \text{Cov}(\widehat{b}, \widehat{c}') &= -\frac{M^T X}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2 \\ \text{Cov}(\widehat{c}, \widehat{c}') &= E\{(\widehat{c}' - E(\widehat{c}'))(\widehat{c} - E(\widehat{c}))\} = \text{Cov}\left(\frac{(M^T M X^T X - X^T M M^T X)}{(M^T M X^T X - X^T M M^T X)} \boldsymbol{\varepsilon}_2, (X^T X)^{-1} X^T \boldsymbol{\varepsilon}_1\right) = \\ &= \frac{(M^T M X^T X - X^T M M^T X)}{(M^T M X^T X - X^T M M^T X)} \text{Cov}(\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_1) X (X^T X)^{-1} = \frac{(M^T M X^T X - X^T M M^T X)}{(M^T M X^T X - X^T M M^T X)} (\sigma_2^2 \mathbf{I}) X (X^T X)^{-1} = \\ &= \frac{(M^T M X^T X - X^T M M^T X)}{(M^T M X^T X - X^T M M^T X)} (X^T X)^{-1} \sigma_2^2 = \sigma_2^2 (X^T X)^{-1} \end{aligned} \quad (\text{A20})$$

$$\text{Cov}(\widehat{c}, \widehat{c}') = \sigma_2^2 (X^T X)^{-1} \quad (\text{A21})$$

Univariate algebra method

As mentioned before, in the univariate algebra method we used covariance algebra to derive covariances between the least squares estimates of coefficients in the single mediation model. The following is an example of the covariance algebra:

$$\text{Cov}(\widehat{a}, \widehat{c}) = \text{Cov}(\widehat{a}, \widehat{ab} + \widehat{c}') = \text{Cov}(\widehat{a}, \widehat{ab}) + \text{Cov}(\widehat{a}, \widehat{c}') = \text{Cov}(\widehat{a}, \widehat{ab})$$

Before proceeding further, it should be noted that following statements are true:

$$\begin{aligned} \text{Cov}(\widehat{a}, \widehat{c}') &= 0 \text{ and } \text{Cov}(\widehat{a}, \widehat{b}) = 0 \text{ and as a result, } E(\widehat{ab}) = E(\widehat{a})E(\widehat{b}) \text{ and} \\ E(\widehat{a^2b}) &= E(\widehat{a^2})E(\widehat{b}). \\ \text{Cov}(\widehat{a}, \widehat{ab}) &= E\{(\widehat{a} - a)(\widehat{ab} - E(\widehat{ab}))\} = E\{(\widehat{a} - a)(\widehat{ab} - ab)\} = E\{\widehat{a^2b} - \widehat{aab} - a\widehat{ab} + a^2b\} = \\ &= E(\widehat{a^2b}) - abE(\widehat{a}) - aE(\widehat{ab}) + a^2b = b\{E(\widehat{a^2}) - a^2\} = b\text{Var}(\widehat{a}) \end{aligned}$$

$$\begin{aligned} \text{Cov}(\widehat{a}, \widehat{c}) &= b\text{Var}(\widehat{a}) = b\sigma_3^2 (X^T X)^{-1} \\ \text{Cov}(\widehat{c}, \widehat{c}') &= \text{Cov}(\widehat{c}, \widehat{c} - \widehat{ab}) = \text{Var}(\widehat{c}) - \text{Cov}(\widehat{c}, \widehat{ab}) = \text{Var}(\widehat{c}) - b\text{Cov}(\widehat{a}, \widehat{c}) = \text{Var}(\widehat{c}) - b^2\text{Var}(\widehat{a}) \end{aligned} \quad (\text{A22})$$

$$\text{Cov}(\widehat{c}, \widehat{c}') = \text{Var}(\widehat{c}) - b^2\text{Var}(\widehat{a}) \quad (\text{A23})$$

To check this result with the first method, we first substituted Equations (A3) and (A11) into Equation (A23):

$$\begin{aligned} \text{Cov}(\widehat{c}, \widehat{c}') &= \text{Var}(\widehat{c}) - b^2\text{Var}(\widehat{a}) = \sigma_1^2 (X^T X)^{-1} - b^2 \sigma_3^2 (X^T X)^{-1} = (\sigma_1^2 - b^2 \sigma_3^2) (X^T X)^{-1} \\ &= \sigma_2^2 (X^T X)^{-1} \\ \text{Cov}(\widehat{c}, \widehat{c}') &= \text{Var}(\widehat{c}) - b^2\text{Var}(\widehat{a}) = \sigma_2^2 (X^T X)^{-1} \\ \text{Cov}(\widehat{b}, \widehat{c}) &= \text{Cov}(\widehat{b}, \widehat{ab} + \widehat{c}') = \text{Cov}(\widehat{b}, \widehat{c}') + a\text{Var}(\widehat{b}) \end{aligned} \quad (\text{A24})$$

If we substitute Equations (A5) and (A20) into the above equation, we have:

$$\text{Cov}(\widehat{b}, \widehat{c}) = \frac{M^T X}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2 + a \frac{(X^T X)}{(X^T X M^T M - X^T M M^T X)} \sigma_2^2 = \frac{\sigma_2^2}{(X^T X M^T M - X^T M M^T X)} (a X^T X - M^T X)$$

Now, if we substitute a with $\widehat{a}$ (Equation (A1)), we have

$$\begin{aligned} \frac{\sigma_2^2}{(X^T X M^T M - X^T M M^T X)} (a X^T X - M^T X) &= \frac{\sigma_2^2}{(X^T X M^T M - X^T M M^T X)} \{(X^T X)^{-1} X^T M X^T X - M^T X\} = \\ \frac{\sigma_2^2}{(X^T X M^T M - X^T M M^T X)} \{X^T M (X^T X)^{-1} X^T X - M^T X\} &= \frac{\sigma_2^2}{(X^T X M^T M - X^T M M^T X)} \{X^T M - M^T X\} = 0 \end{aligned}$$

$$\begin{aligned} \text{Cov}(\widehat{b}, \widehat{c}) &= \text{Cov}(\widehat{b}, \widehat{c}') + a \text{Var}(\widehat{b}) = 0 \\ \text{Cov}(\widehat{a}, \widehat{c}') &= \text{Cov}(\widehat{a}, \widehat{c} - \widehat{a}\widehat{b}) = \text{Cov}(\widehat{a}, \widehat{c}) - \text{Cov}(\widehat{a}, \widehat{a}\widehat{b}) = b \text{Var}(\widehat{a}) - b \text{Var}(\widehat{a}) = 0 \end{aligned} \quad (\text{A25})$$

$$\text{Cov}(\widehat{a}, \widehat{c}') = 0 \quad (\text{A26})$$

A SAS macro to calculate the confidence intervals for the effect size measures

Name specifies the name of the SAS data set; X , M , and Y are the variables in the data set. For example if the SAS data set name is 'example' with $x1$, $m1$, $y1$ as independent, mediator, and dependent variables, respectively, the following SAS code runs the MACRO: %confidence (Name = example, X = x1, M = m1, Y = y1); run;.

%Macro Confidence (Name,X,M,Y);

*EQUATION Y EQUALS X OR TOTAL EFFECT;

Data SIM;

set &Name;

X=&X;M=&M;Y=&Y;

X2=X*X;

M2=M*M;

XM=X*M;

XY=X*Y;

MY=M*Y;

Keep X--MY;

run;

PROC REG data=sim OUTEST=FILE COVOUT noprint; MODEL Y=X/;

```

DATA B; SET FILE;

IF UPCASE(_TYPE_)='PARMS'; C=X;MSE1=_RMSE*_RMSE;

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X Y;

KEEP C MSE1;

DATA C; SET FILE;IF UPCASE(_NAME_)='X'; SEC=SQRT(X);

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X Y;

KEEP SEC;

DATA MODEL1; MERGE B C;run;

*EQUATION Y EQUALS X AND M;

PROC REG DATA=SIM OUTEST=FILE COVOUT noprint;

MODEL Y=X M/;

DATA B; SET FILE;

IF UPCASE(_TYPE_)='PARMS'; CP=X;

BM2=X; MSE2=_RMSE*_RMSE;

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X Y M;

KEEP MSE2 BM2 CP;

DATA C; SET FILE;

IF UPCASE(_NAME_)='X'; SEBM2=SQRT(X); SECP=SQRT(X);

KEEP SEBM2 SECP;

DATA D; SET FILE; IF UPCASE(_NAME_)='M'; SEB=SQRT(M);

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X Y M;

KEEP SEB;

DATA E; SET FILE; B=M; IF UPCASE(_TYPE_)='PARMS';

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X Y M;

KEEP B;

DATA F; SET FILE; IF UPCASE(_NAME_)='M'; CBC=X;

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X Y M;

KEEP CBC;

```

```

DATA MODEL2; MERGE B C D E F;run;

*EQUATION M EQUALS X REGRESSION MODEL;

PROC REG DATA=SIM OUTEST=FILE COVOUT noprint;

MODEL M=X;

DATA B; SET FILE; A=X; IF UPCASE(_TYPE_)='PARMS';

MSE=_RMSE*_RMSE;

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X M;

KEEP A MSE;

DATA C; SET FILE; IF UPCASE(_NAME_)='X'; SEA=SQRT(X);

DROP _MODEL_ _NAME_ _TYPE_ _DEPVAR_ _RMSE_ INTERCEPT X M;

KEEP SEA;

DATA MODEL3; MERGE B C;run;

*THIS SECTION SAVES THE VALUE OF X SQUARED;

PROC MEANS DATA=SIM SUM NOPRINT; VAR X2 M2 XM XY MY;

OUTPUT OUT=OUT SUM=SUMX SUMM SUMXM SUMXY SUMMY;

PROC CORR NOPRINT COV DATA=SIM OUTP=COV;

VAR X M Y;

DATA A; SET COV; IF UPCASE(_TYPE_)='COV'; IF UPCASE(_NAME_)='X';

VARX=X; COVXM=M; COVXY=Y; KEEP VARX COVXM COVXY;

DATA B; SET COV; IF UPCASE(_TYPE_)='COV'; IF UPCASE(_NAME_)='M';

VARM=M; COVMY=Y; KEEP VARM COVMY;

DATA C; SET COV; IF UPCASE(_TYPE_)='COV'; IF UPCASE(_NAME_)='Y';

VARY=Y; KEEP VARY;

DATA VAR; MERGE A B C;

*THE DATA SET ALL CONTAINS THE ESTIMATES FROM THE REPLICATION;

DATA ALL; MERGE OUT VAR MODEL1 MODEL2 MODEL3;

RUN;

*****;

```

```

DATA TEST;

SET ALL;

DOF=_FREQ_ - 2;

NOBS=_FREQ_;

covbcp=cbc;

S=SUMX*SUMM-SUMXM*SUMXM;T=1/sumx;

*Method 1;

COVAB1=0;

COVACP1=0;

COVAC1=b*MSE/SUMX;

COVBCP1=-SUMXM*mse2/S;

COVBC1=0;

COVCCP1=mse2/SUMX;

*standard error of the effects;

*ab/ab + cp;

c4=(a*b + cp)**4;

var_prop_med1=(b*cp)**2/c4*SEA**2+

(a*cp)**2/c4*SEB**2 + (a*b)**2/c4*SECP**2-2*(a*a*b*cp/c4)*covbcp1;

*ab/c;

var_prop_med2=(b/c)**2*SEA**2 + (a/c)**2*SEB**2+(a*b/c**2)

**2*SECP**2 -

2*(a*b*b/c**3)*covac1;

*1-cp/c;

var_prop_med3=(cp/c**2)**2*SEC**2+(1/c)**2*SECP**2 -

2*(cp/c**3)*covccp1;

*ab/cp;

var_ind_dir=(b/cp)**2*SEA**2 + (a/cp)**2*SEB**2 + (a*b/cp**2)

**2*SECP**2 -

```

```

2*(a*a*b/cp**3)*covbcpl;

*c/a;

var_prop_med4=(c/a**2)**2*SEA**2+(1/a**2)*SEC**2 - 2
*(c/a**3)*covac1;

se_prop_med1=sqrt(var_prop_med1);

se_prop_med2=sqrt(var_prop_med2);

se_prop_med3=sqrt(var_prop_med3);

se_ind_dir=sqrt(var_ind_dir);

se_prop_med4=sqrt(var_prop_med4);

cl=a*b/(a*b + cp)-1.96*se_prop_med1;effect=1;cl_i=1;output;

cl=a*b/(a*b + cp); effect=1; cl_i=2; output;

cl=a*b/(a*b + cp) + 1.96*se_prop_med1; effect=1; cl_i=3; output;

cl=a*b/c - 1.96*se_prop_med2; effect=2; cl_i=1; output;

cl=a*b/c; effect=2; cl_i=2; output;

cl=a*b/c + 1.96*se_prop_med2; effect=2; cl_i=3; output;

cl=1 - (cp/c) - 1.96*se_prop_med3; effect=3;cl_i=1; output;

cl=1 - (cp/c); effect=3;cl_i=2; output;

cl=1 - (cp/c) + 1.96*se_prop_med3; effect=3;cl_i=3; output;

cl=a*b/cp - 1.96*se_ind_dir; effect=4; cl_i=1; output;

cl=a*b/cp; effect=4; cl_i=2; output;

cl=a*b/cp + 1.96*se_ind_dir; effect=4; cl_i=3; output;

cl=c/a - 1.96*se_prop_med4; effect=5; cl_i=1; output;

cl=c/a; effect=5; cl_i=2; output;

cl=c/a + 1.96*se_prop_med4; effect=5; cl_i=3; output;

keep cl effect cl_i a b c cp;

run;

proc format;

value effect

```

```

1='Proportion mediated: ab/(ab + cp)'

2='Proportion mediated: ab/c'

3='Proportion mediated: 1 - cp/c'

4='Ratio of indirect to direct: ab/cp'

5='Ratio of total effect to a(M on X)'

;

value cli 1='Lower CI Limit'

2='Point Estimate'

3='Upper CI Limit';

run;

proc tabulate noseps order=data data=test;

where effect GT 0;

class effect cl_i;

var cl;

table effect, cl="*(mean="*F=10.4)*cl_i=" /condense indent=3

RTS=40;

format cl_i cli.

effect effect.;

title '95%Symmetric Confidence Interval Based on Multivariate Delta

Method';

run;

data temp;

set test (obs=1);

t=1; parm=a; output;

t=2; parm=b; output;

t=3; parm=cp; output;

t=4; parm=c; output;

t=5; parm=a*b; output;

```

```
keep t parm;

run;

proc format;

value t

    1='a'

    2='b'

    3='Indirect Effect (cp)'

    4='Total Effect (c)'

    5='Mediated Effect(ab)';

run;

proc tabulate data=temp;

class t;

var parm;

table t='parameter estimate', parm="*mean=";

format t t.;

title 'Parameter Estimates';

run;

%mend;
```


Table 1

Derived covariances between the coefficients in a single mediator model

$\hat{a}$	$\hat{b}$	$\hat{c}'$	$\hat{c}$
$\hat{a}$			
$\sigma_3^2(X^T X)^{-1}$			
$\hat{b}$	0		
	$\frac{(X^T X)}{(X^T X M^T M - X^T M M^T X)} \sigma_2^2$		
$\hat{c}'$	0		
	$\frac{-(M^T X)}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2$	$\frac{(M^T M)}{(M^T M X^T X - X^T M M^T X)} \sigma_2^2$	
$\hat{c}$	0		
$b \sigma_3^2(X^T X)^{-1}$		$\sigma_2^2(X^T X)^{-1}$	$\sigma_1^2(X^T X)^{-1}$

Note. X , M , and Y are the vectors representing independent, mediator, and dependent variables, respectively. b is the mean for the parameter estimate $\hat{b}$. σ_1^2 , σ_2^2 , and σ_3^2 are the residual variances for the regression Equations (1–3), respectively.

Table 2

Root mean square error and relative bias of the standard error of $ab/(ab + c')$

Population parameters	Sample size			
	50	100	200	500
$c' = 0$				
RMSE				
Small effect: $a = b = 0.14$	30,469.44	323,409.96	454,858.22	74,740.74
Medium effect: $a = b = 0.39$	1,095.14	2,662.02	4,519.19	0.74
Large effect: $a = b = 0.59$	330,969.79	2.07	0.16	0.04
Relative bias				
Small effect: $a = b = 0.14$	45.99	200.74	303.12	92.90
Medium effect: $a = b = 0.39$	8.30	10.36	7.43	-0.10
Large effect: $a = b = 0.59$	85.75	-0.12	-0.04	0.00
$c' = 0.14$				
RMSE				
Small effect: $a = b = 0.14$	1,106.84	75,595.08	120.52	0.34
Medium effect: $a = b = 0.39$	102.15	26.16	0.18	0.03
Large effect: $a = b = 0.59$	90.88	0.11	0.04	0.01
Relative bias				
Small effect: $a = b = 0.14$	8.85	82.39	3.63	-0.17
Medium effect: $a = b = 0.39$	2.67	0.15	-0.04	-0.02
Large effect: $a = b = 0.59$	0.41	-0.03	0.01	0.00
$c' = 0.39$				
RMSE				
Small effect: $a = b = 0.14$	3.20	0.03	0.01	0.00
Medium effect: $a = b = 0.39$	1.75	0.03	0.01	0.00
Large effect: $a = b = 0.59$	13.03	0.02	0.01	0.00
Relative bias				
Small effect: $a = b = 0.14$	0.34	-0.03	-0.06	-0.02
Medium effect: $a = b = 0.39$	-0.30	-0.01	0.00	0.01
Large effect: $a = b = 0.59$	0.07	0.00	0.01	-0.01

Population parameters	Sample size			
	50	100	200	500
$c' = 0.59$				
RMSE				
Small effect: $a = b = 0.14$	0.04	0.02	0.01	0.00
Medium effect: $a = b = 0.39$	0.04	0.02	0.01	0.00
Large effect: $a = b = 0.59$	0.03	0.01	0.00	0.00
Relative bias				
Small effect: $a = b = 0.14$	0.02	0.00	-0.02	0.01
Medium effect: $a = b = 0.39$	-0.01	-0.06	0.01	-0.03
Large effect: $a = b = 0.59$	-0.01	0.00	-0.01	-0.03

Note. RMSE is the root mean square error.

Table 3

Root mean square error and relative bias of the standard error of ab/c'

Population parameters	Sample size			
	50	100	200	500
$c' = 0.14$				1,000
RMSE				
Small effect: $a = b = 0.14$	7,377.92	98,658.01	72.80	1.70
Medium effect: $a = b = 0.39$	18,401.54	8,566.52	3,584.50	5.96
Large effect: $a = b = 0.59$	3,252.35	7,308.14	22,379.44	2,266.45
Relative bias				
Small effect: $a = b = 0.14$	31.33	193.54	2.73	-0.10
Medium effect: $a = b = 0.39$	27.39	16.15	6.76	-0.16
Large effect: $a = b = 0.59$	8.84	10.48	15.82	2.47
$c' = 0.39$				
RMSE				
Small effect: $a = b = 0.14$	30.45	0.07	0.02	0.01
Medium effect: $a = b = 0.39$	53.37	0.43	0.08	0.02
Large effect: $a = b = 0.59$	147.48	1,641.80	0.19	0.06
Relative bias				
Small effect: $a = b = 0.14$	0.95	-0.10	-0.09	-0.02
Medium effect: $a = b = 0.39$	0.89	-0.11	-0.04	0.00
Large effect: $a = b = 0.59$	0.86	2.23	-0.05	-0.02
$c' = 0.59$				
RMSE				
Small effect: $a = b = 0.14$	0.07	0.02	0.01	0.00
Medium effect: $a = b = 0.39$	0.40	0.06	0.02	0.00
Large effect: $a = b = 0.59$	1.36	0.13	0.05	0.01
Relative bias				
Small effect: $a = b = 0.14$	-0.06	-0.04	-0.03	-0.01
Medium effect: $a = b = 0.39$	-0.18	-0.09	0.01	0.01
Large effect: $a = b = 0.59$	-0.21	-0.06	-0.03	-0.03

Note. RMSE is the root mean square error.

Table 4

Root mean square error and relative bias of the standard error of c/a

Population parameters	Sample size				
	50	100	200	500	1,000
$c' = 0$					
RMSE					
Small effect: $a = b = 0.14$	5,379.13	10,767.67	310.50	34.95	0.08
Medium effect: $a = b = 0.39$	2.08	0.15	0.04	0.02	0.01
Large effect: $a = b = 0.59$	0.13	0.04	0.02	0.01	0.00
Relative bias					
Small effect: $a = b = 0.14$	18.07	17.64	2.99	0.41	-0.01
Medium effect: $a = b = 0.39$	0.07	0.02	-0.03	0.05	0.01
Large effect: $a = b = 0.59$	-0.04	0.02	0.00	-0.02	0.00
$c' = 0.14$					
RMSE					
Small effect: $a = b = 0.14$	101,866.58	7,523.00	3,128.60	1.49	0.26
Medium effect: $a = b = 0.39$	3.85	0.17	0.06	0.02	0.01
Large effect: $a = b = 0.59$	0.13	0.05	0.02	0.01	0.00
Relative bias					
Small effect: $a = b = 0.14$	37.82	15.10	9.21	-0.05	-0.06
Medium effect: $a = b = 0.39$	0.07	0.03	0.01	0.04	-0.05
Large effect: $a = b = 0.59$	0.02	-0.04	0.01	0.01	-0.01
$c' = 0.39$					
RMSE					
Small effect: $a = b = 0.14$	2,576,573.31	20,744.73	27,008.17	811.32	0.54
Medium effect: $a = b = 0.39$	13,782.56	1.63	0.10	0.03	0.02
Large effect: $a = b = 0.59$	0.24	0.07	0.03	0.01	0.00
Relative bias					
Small effect: $a = b = 0.14$	210.12	18.54	13.56	0.78	-0.04
Medium effect: $a = b = 0.39$	9.83	-0.21	0.00	0.01	0.03
Large effect: $a = b = 0.59$	-0.04	-0.02	-0.01	-0.03	-0.02

Population parameters	Sample size			
	50	100	200	500
$c' = 0.59$				
RMSE				1,000
Small effect: $a = b = 0.14$	26,941.75	6,657.41	59,789.47	301.04
Medium effect: $a = b = 0.39$	163.61	22.82	0.17	0.05
Large effect: $a = b = 0.59$	0.79	0.10	0.04	0.01
Relative bias				
Small effect: $a = b = 0.14$	16.94	8.15	13.72	0.13
Medium effect: $a = b = 0.39$	0.59	-0.40	-0.05	0.01
Large effect: $a = b = 0.59$	-0.15	-0.04	-0.02	0.01

Note. RMSE is the root mean square error.

Table 5

Minimum sample sizes for the standard errors of the effect size measures

Population parameters	Effect size measures			
	$ab/(ab + c')$	ab/c	$1 - c'/c$	ab/c'
$c' = 0$				c/a
Small effect: $a = b = 0.14$	–	–	–	1,000
Medium effect: $a = b = 0.39$	500	500	500	50
Large effect: $a = b = 0.59$	200	200	200	50
$c' = 0.14$				
Small effect: $a = b = 0.14$	1,000	1,000	1,000	500
Medium effect: $a = b = 0.39$	200	200	200	50
Large effect: $a = b = 0.59$	100	100	100	50
$c' = 0.39$				
Small effect: $a = b = 0.14$	100	100	100	1,000
Medium effect: $a = b = 0.39$	100	100	100	200
Large effect: $a = b = 0.59$	100	100	100	200
$c' = 0.59$				
Small effect: $a = b = 0.14$	50	50	50	50
Medium effect: $a = b = 0.39$	50	50	50	100
Large effect: $a = b = 0.59$	50	50	50	100

Note. Dashes indicate that a sample size greater than 1,000 may be required that was not included in the simulation.

Table 6
Type I error and power for the 95% CI of the proportion mediated effect: $ab/(ab + c')$

Population parameters	Sample size				
	50	100	200	500	1,000
Type I error					
$c' = 0$					
Zero effect: $a = b = 0$	0	0	0	0	0
$c' = 0.14$					
Zero effect: $a = b = 0$	0	0	0	0	0
$c' = 0.39$					
Zero effect: $a = b = 0$	0	0	0	0	0
$c' = 0.59$					
Zero effect: $a = b = 0$	0	0	0	0	0
Power					
$c' = 0$					
Small effect: $a = b = 0.14$	0	0	0	0.01	0.04
Medium effect: $a = b = 0.39$	0.05	0.20	0.49	0.91	0.99
Large effect: $a = b = 0.59$	0.49	0.87	0.99	1	1
$c' = 0.14$					
Small effect: $a = b = 0.14$	0	0	0.01	0.36	0.94
Medium effect: $a = b = 0.39$	0.14	0.58	0.96	1	1
Large effect: $a = b = 0.59$	0.74	0.98	1	1	1
$c' = 0.39$					
Small effect: $a = b = 0.14$	0.01	0.02	0.10	0.57	0.97
Medium effect: $a = b = 0.39$	0.32	0.86	1	1	1
Large effect: $a = b = 0.59$	0.92	1	1	1	1
$c' = 0.59$					
Small effect: $a = b = 0.14$	0	0.02	0.11	0.58	0.97
Medium effect: $a = b = 0.39$	0.38	0.88	1	1	1
Large effect: $a = b = 0.59$	0.93	1	1	1	1

Table 7

Type I error and power for the 95% CI of the ratio of indirect to direct effect: ab/c'

Population parameters		Sample size				
		50	100	200	500	1,000
Type I error						
$c' = 0$						
	Zero effect: $a = b = 0$	0	0	0	0	0
	$c' = 0.14$					
$c' = 0.14$						
	Zero effect: $a = b = 0$	0	0	0	0	0
	$c' = 0.39$					
$c' = 0.39$						
	Zero effect: $a = b = 0$	0	0	0	0	0
	$c' = 0.59$					
$c' = 0.59$						
	Zero effect: $a = b = 0$	0	0	0	0	0
	Power					
$c' = 0$						
	$c' = 0$					
	Small effect: $a = b = 0.14$	0	0	0	0	0.01
$c' = 0.14$						
	Medium effect: $a = b = 0.39$	0.01	0.01	0.03	0.03	0.05
	Large effect: $a = b = 0.59$	0.03	0.02	0.04	0.05	0.06
$c' = 0.14$						
	$c' = 0.14$					
	Small effect: $a = b = 0.14$	0	0	0	0.11	0.85
$c' = 0.39$						
	Medium effect: $a = b = 0.39$	0	0.01	0.18	0.72	0.97
	Large effect: $a = b = 0.59$	0.01	0.06	0.22	0.70	0.96
$c' = 0.39$						
	$c' = 0.39$					
	Small effect: $a = b = 0.14$	0	0	0.03	0.49	0.97
$c' = 0.59$						
	Medium effect: $a = b = 0.39$	0.01	0.33	0.99	1	1
	Large effect: $a = b = 0.59$	0.05	0.71	1	1	1
$c' = 0.59$						
	$c' = 0.59$					
	Small effect: $a = b = 0.14$	0	0	0.06	0.54	0.97
$c' = 0.59$						
	Medium effect: $a = b = 0.39$	0.04	0.67	1	1	1
	Large effect: $a = b = 0.59$	0.31	0.98	1	1	1

Table 8

Type I error and power for 95% CI of the ratio of total effect to a: c/a

Population parameters		Sample size				
		50	100	200	500	1,000
Type I error						
$c' = 0$	Zero effect: $a = b = 0$	0	0	0	0	0
	Power					
	$c' = 0$					
$c' = 0.14$	Small effect: $a = b = 0.14$	0	0	0.01	0.04	0.08
	Medium effect: $a = b = 0.39$	0.16	0.33	0.50	0.92	0.99
	Large effect: $a = b = 0.59$	0.64	0.89	0.99	1	1
$c' = 0.39$	Zero effect: $a = b = 0$	0	0	0.01	0.01	0.03
	Small effect: $a = b = 0.14$	0.02	0.05	0.19	0.76	0.99
	Medium effect: $a = b = 0.39$	0.39	0.76	0.97	1	1
$c' = 0.59$	Large effect: $a = b = 0.59$	0.88	0.99	1	1	1
	Zero effect: $a = b = 0$	0.01	0.03	0.03	0.05	0.04
	Small effect: $a = b = 0.14$	0.09	0.25	0.50	0.88	0.99
$c' = 0.59$	Medium effect: $a = b = 0.39$	0.73	0.97	1	1	1
	Large effect: $a = b = 0.59$	0.98	1	1	1	1
	Zero effect: $a = b = 0$	0.03	0.04	0.05	0.05	0.07
$c' = 0.59$	Small effect: $a = b = 0.14$	0.13	0.27	0.52	0.87	0.99
	Medium effect: $a = b = 0.39$	0.81	0.98	1	1	1
	Large effect: $a = b = 0.59$	0.99	1	1	1	1

Table 9

The minimum sample sizes to achieve power for the 95% confidence intervals

Population parameters	Effect size measures		
	$ab/(ab + c')$	ab/c'	c/a
$c' = 0$			
Small effect: $a = b = 0.14$	–	–	–
Medium effect: $a = b = 0.39$	500 (0.91)	–	500 (0.92)
Large effect: $a = b = 0.59$	100 (0.87)	–	100 (0.89)
$c' = 0.14$			
Zero effect: $a = b = 0$			–
Small effect: $a = b = 0.14$	1,000 (0.94)	1,000 (0.85)	1,000 (0.99)
Medium effect: $a = b = 0.39$	200 (0.96)	1,000 (0.97)	200 (0.97)
Large effect: $a = b = 0.59$	100 (0.98)	1,000 (0.96)	50 (0.88)
$c' = 0.39$			
Zero effect: $a = b = 0$			–
Small effect: $a = b = 0.14$	1,000 (0.97)	1,000 (0.97)	500 (0.88)
Medium effect: $a = b = 0.39$	100 (0.86)	200 (0.99)	100 (0.97)
Large effect: $a = b = 0.59$	50 (0.92)	200 (1.0)	50 (0.98)
$c' = 0.59$			
Zero effect: $a = b = 0$			–
Small effect: $a = b = 0.14$	1,000 (0.97)	1,000 (0.97)	500 (0.87)
Medium effect: $a = b = 0.39$	100 (0.88)	200 (1.0)	50 (0.81)
Large effect: $a = b = 0.59$	50 (0.93)	100 (0.98)	50 (0.99)

Note. Dashes indicate that a sample size greater than 1,000 may be required. The blank cells mean the power calculation was not applicable. The numbers in parentheses show the power values corresponding to the sample sizes. Because the simulation study included a limited range of sample sizes, the values shown in the table represent the minimum sample size to achieve the indicated power.

Table 10

Coverage rate for the 95% confidence interval of $ab/(ab + c')$

Population parameters	Coverage rate				
	50	100	200	500	1,000
$c' = 0$					
Small effect: $a = b = 0.14$	0.46	0.51	0.54	0.64	0.73
Medium effect: $a = b = 0.39$	0.83	0.86	0.91	0.91	0.94
Large effect: $a = b = 0.59$	0.91	0.92	0.94	0.96	0.95
$c' = 0.14$					
Small effect: $a = b = 0.14$	0.94	0.91	0.90	0.93	0.93
Medium effect: $a = b = 0.39$	0.93	0.93	0.94	0.97	0.94
Large effect: $a = b = 0.59$	0.93	0.94	0.95	0.94	0.95
$c' = 0.39$					
Small effect: $a = b = 0.14$	0.95	0.91	0.91	0.94	0.94
Medium effect: $a = b = 0.39$	0.93	0.93	0.95	0.96	0.96
Large effect: $a = b = 0.59$	0.96	0.94	0.95	0.95	0.94
$c' = 0.59$					
Small effect: $a = b = 0.14$	0.94	0.91	0.90	0.94	0.93
Medium effect: $a = b = 0.39$	0.92	0.94	0.94	0.95	0.96
Large effect: $a = b = 0.59$	0.94	0.93	0.95	0.94	0.95

Table 11

Coverage rate for the 95% confidence interval of ab/c'

Population parameters	Coverage rate				
	50	100	200	500	1,000
$c' = 0.14$					
Small effect: $a = b = 0.14$	0.91	0.89	0.88	0.90	0.92
Medium effect: $a = b = 0.39$	0.82	0.86	0.87	0.92	0.92
Large effect: $a = b = 0.59$	0.78	0.83	0.89	0.91	0.93
$c' = 0.39$					
Small effect: $a = b = 0.14$	0.95	0.90	0.91	0.94	0.93
Medium effect: $a = b = 0.39$	0.88	0.92	0.93	0.95	0.96
Large effect: $a = b = 0.59$	0.90	0.91	0.94	0.94	0.95
$c' = 0.59$					
Small effect: $a = b = 0.14$	0.94	0.90	0.90	0.94	0.93
Medium effect: $a = b = 0.39$	0.91	0.93	0.93	0.95	0.96
Large effect: $a = b = 0.59$	0.91	0.92	0.94	0.94	0.95

Table 12

Coverage rate for the 95% confidence interval of c/a

Population parameters	Coverage rate				
	50	100	200	500	1,000
$c' = 0$					
Small effect: $a = b = 0.14$	1.00	1.00	1.00	0.99	0.97
Medium effect: $a = b = 0.39$	0.99	0.98	0.97	0.95	0.95
Large effect: $a = b = 0.59$	0.97	0.97	0.95	0.95	0.94
$c' = 0.14$					
Small effect: $a = b = 0.14$	0.92	0.91	0.94	0.93	0.95
Medium effect: $a = b = 0.39$	0.97	0.98	0.96	0.96	0.95
Large effect: $a = b = 0.59$	0.96	0.97	0.95	0.95	0.94
$c' = 0.39$					
Small effect: $a = b = 0.14$	0.85	0.86	0.88	0.92	0.93
Medium effect: $a = b = 0.39$	0.94	0.94	0.95	0.96	0.95
Large effect: $a = b = 0.59$	0.95	0.96	0.96	0.96	0.95
$c' = 0.59$					
Small effect: $a = b = 0.14$	0.83	0.87	0.88	0.93	0.93
Medium effect: $a = b = 0.39$	0.92	0.94	0.94	0.96	0.95
Large effect: $a = b = 0.59$	0.94	0.95	0.96	0.95	0.96

Table 13

95% confidence intervals and point estimates for example data

Effect size measures	Population value	Point estimate	Lower limit	Upper limit
Proportion mediated: $ab/(ab + c')$, ab/c , $1 - c'/c$	0.20	0.23	0.16	0.30
Ratio of indirect to direct: ab/c'	0.26	0.30	0.18	0.41
Ratio of total effect to a : c/a	1.90	1.67	1.35	1.99