Using hyperbolic Lagrangian coherent structures to investigate vortices in bioinspired fluid flows

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We use direct Lyapunov exponents to identify Lagrangian coherent structures (LCSs) in a bioinspired fluid flow: the wakes of rigid pitching panels with a trapezoidal planform geometry chosen to model idealized fish caudal fins. When compared with commonly used Eulerian criteria, the Lagrangian method has previously exhibited the ability to define structure boundaries without relying on a preselected threshold. In addition, qualitative changes in the LCS have previously been shown to correspond to physical changes in the vortex structure. For this paper, digital particle image velocimetry experiments were performed to obtain the time-resolved velocity fields for Strouhal numbers of 0.17 and 0.27. A classic reverse von Kármán vortex street pattern was observed along the midspan of the near wake at low Strouhal number, but at higher Strouhal number the complexity of the wake increased downstream of the trailing edge. The spanwise vortices spread transversely across the wake and lose coherence, and this event was shown to correspond to a qualitative change in the LCS at the same time and location. © 2010 American Institute of Physics. [doi:10.1063/1.3270045]

In the field of fluid dynamics, there does not yet exist an objective, quantitative definition of a vortex. Most criteria proposed in the past succeed at locating vortex cores, but an arbitrary threshold is necessary to draw the boundary between a vortex and the freestream flow, or between different vortices. Here, we use direct Lyapunov exponents (DLEs) to identify Lagrangian coherent structures (LCSs) in fluid flows. This method has previously exhibited the ability to define structure boundaries, and qualitative changes in the LCS have been shown to correspond to physical coherent structure events and phenomena. We first demonstrate the benefits of a LCS analysis by reporting new results related to previously published work using both positive- and negative-time LCSs as well as hyperbolicity criteria. We then investigate a bioinspired fluid flow: the wakes of rigid pitching panels with a trapezoidal planform geometry chosen to model idealized fish caudal fins. We demonstrate that not only do LCSs capture the vortices that dominate the flow downstream of the pitching panel, but also that a qualitative change in the LCS corresponds to a transition to a more complicated wake organization observed in flow visualization experiments. It is expected that LCS will continue to be useful in understanding how these bioinspired flows form and evolve, and how the vortex organization influences propulsive performance.

I. INTRODUCTION

Here, we calculate the DLE field to identify LCSs in three-dimensional (3D) unsteady flows. Previous work on flow structure identification has been primarily Eulerian,

which evaluates the spatial structure of quantities derived from the instantaneous velocity field and its gradient. Examples of Eulerian criteria include the Q criterion of Hunt *et al.*,¹ the swirl strength (λ_{ci}^2) of Zhou *et al.*,² the Δ criterion of Chong *et al.*,³ and the λ_2 criterion of Jeong and Hussein.⁴ These criteria, among others, have been widely used to study vortex dominated fluid flows, although none has emerged as a definitive tool of choice. In practice, they yield similar results when used to study the same flows but they share several disadvantages. For example, although they are invariant with respect to Galilean transformations, they are not invariant to time-dependent rotations, and thus are not objective (frame independent).⁵ Furthermore, Eulerian criteria require a user-defined threshold to indicate the regions where a coherent structure exists.

In contrast, more recent Lagrangian methods, such as LCS analysis or the M_z criterion of Haller,⁵ identify coherent structures based on the properties along fluid particle trajectories. An immediate advantage of these methods is their objectivity: they remain invariant with respect to rotation of the reference frame. A further advantage of Lagrangian methods is their insensitivity to short-term anomalies in the velocity field. DLE fields calculated from discrete data have been shown to be robust and relatively insensitive to imperfect velocity data as long as the errors remain small in a special time-weighted norm.⁶ For this reason, LCS analyses perform well for experimental data in addition to analytically defined velocity fields and finely resolved computational cases.

Because LCS analysis has only recently been used as a vortex identification tool, it has been primarily used in the area of incompressible flows, and most often in two-dimensional flow fields. Lekien and Leonard⁷ performed a LCS analysis to find coherent structures in the currents of

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Monterey Bay in California based on radar data, and used the results to assist in glider trajectory planning. Shadden *et al.*^{8,9} employed DLE to identify the structure of a piston-generated vortex ring and to capture the vortex ring wake structure of jellyfish swimming using two-dimensional digital particle image velocimetry (DPIV) data. Their results led to insights on the energetics of jellyfish swimming and a new metric to measure fluid mixing. DLE has also been used as a structure identification tool in two-dimensional quasigeostrophic turbulence,¹⁰ the stratospheric polar vortex,¹¹ freely decaying two-dimensional turbulence,¹² a magnetically forced two-dimensional conducting fluid experiment,¹³ and a vortex-shedding two-dimensional airfoil.¹⁴ In three dimensions, DLE has been computed by Haller⁵ on two established analytic flow solutions and Shadden and Taylor¹⁵ performed a LCS analysis on 3D computational solutions of the blood flow through abdominal aortic aneurysms. Green *et al.*¹⁶ described the turbulent structure composition and evolution of an isolated hairpin vortex and a fully turbulent channel flow.

Here, we use LCS to investigate the coherent structure composition of three different flow fields: the analytic solution of Hill's spherical vortex, the direct numerical simulation of a hairpin vortex in a turbulent channel, and the experimentally measured velocity fields in the wake of a pitching trapezoidal panel. Although LCS analysis of the spherical and hairpin vortices has been reported previously,¹⁶ in Sec. II we present additional results that yield more information about the boundaries of the vortices in these two cases. The main results, presented in Sec. III, use the LCS analysis to investigate the wake downstream of a simple unsteady propulsor used to model fish caudal fin swimming.

The locomotion of fish and aquatic animals is achieved by the oscillation of their fins and flukes, which creates highly 3D, unsteady flow fields that are dominated by coherent vortices shed by the trailing edge. Previous two-dimensional experiments have shown that the unsteady foils generate two counter-rotating spanwise vortices at the trailing edge during each flapping cycle, referred to as a 2S wake.^{17,18} In a thrust-producing scenario, these vortices are arranged in a reverse von Kármán vortex street. The vortices are aligned such that induced jets between subsequent structures in the wake are directed downstream, adding momentum to the flow.

One of the governing nondimensional parameters that are used in these unsteady flows is the Strouhal number,

$$St = \frac{fA}{U}, \quad (1)$$

where f is the frequency of oscillation, A is the width of the wake, and U is the freestream velocity. The peak-to-peak amplitude of the trailing edge is commonly used as an approximation for A . Triantafyllou *et al.*¹⁹ used a linear stability analysis on the wake of an oscillating foil to predict the St for optimal propulsive efficiency and found that it lies in the range of $0.25 \leq St \leq 0.35$. This was confirmed with a two-dimensional flapping foil experiment. Furthermore, the analysis of a variety of fish species locomotion has shown that many marine animals swim in this Strouhal number range.

FIG. 1. (Color) White light flow visualization of the wake of a rectangular pitching panel. From Buchholz and Smits. (a) $St=0.23$ and (b) $St=0.43$ (Ref. 29). [Reprinted with permission from J. H. J. Buchholz and A. J. Smits, *Phys. Fluids* **17**, 091102 (2005). Copyright 2005, American Institute of Physics.]

Experiments and computations on pitching and flapping bodies of finite aspect ratio (AR) have revealed that the 3D wake structure is considerably more complex than the two-dimensional case, and that it is a strong function of both Strouhal number and AR ($=S/c$, where S is the span and c is the chord).^{20–27} Buchholz²⁸ performed flow visualization experiments on a series of low AR propulsors and found that with increasing St , the wake transitions from the 2S configuration to a 2P configuration, where two pairs of oppositely signed vortices are shed each pitching cycle. Each pair shifts transversely with their induced velocity, effectively splitting the wake. See Fig. 1.

Here we use a rigid flat plate of trapezoidal planform geometry in purely pitching motion to approximate a simple fish caudal tail motion and geometry. We perform both Eulerian and Lagrangian analyses of the flow field to investigate the evolution of coherent structures downstream of the panel trailing edge and its dependence on St .

II. LAGRANGIAN ANALYSIS

The Lagrangian analysis presented in this paper uses the DLE,³⁰ which is also referred to in the literature as the finite-time Lyapunov exponent. This is a scalar quantity calculated at each point in space that is a measure of the maximum rate

of separation among neighboring particle trajectories initialized near that point. More precisely, if we denote the position of a particle at time t , which began at position $\mathbf{x}_0$ at time t_0 as $\mathbf{x}(t, \mathbf{x}_0, t_0)$, we can define a coefficient of expansion after a time T as the square of the largest singular value of the deformation gradient $\partial \mathbf{x}(t_0 + T, \mathbf{x}_0, t_0) / \partial \mathbf{x}_0$,

$$\sigma_T(\mathbf{x}_0, t_0) = \lambda_{\max} \left(\left[\frac{\partial \mathbf{x}(t_0 + T, \mathbf{x}_0, t_0)}{\partial \mathbf{x}_0} \right]^T \left[\frac{\partial \mathbf{x}(t_0 + T, \mathbf{x}_0, t_0)}{\partial \mathbf{x}_0} \right] \right). \quad (2)$$

The DLE field is then defined as

$$\text{DLE}_T(\mathbf{x}_0, t_0) = \frac{1}{2T} \log \sigma_T(\mathbf{x}_0, t_0). \quad (3)$$

Large values of the DLE field may indicate locally maximal stretching among particle trajectories, and ridges in the DLE field are referred to as LCSs. The LCS are nearly material lines that move with the local convective velocity with finite, but small, flux across them.³¹ Trajectories that stretch relative to each other when integrated in negative (backward) time converge in positive (forward) time. Therefore, by integrating particle trajectories in both positive and negative times we can calculate positive- and negative-time DLE fields (pDLE and nDLE, respectively), from which we extract the positive- and negative-time LCS (pLCS and nLCS, respectively). These structures are candidate repelling and attracting material lines in the flow, and can delineate the boundaries between qualitatively different regions in a flow.⁸

An analysis of hyperbolicity along LCS yields additional information about the evolution of coherent structures. The local separation among nearby trajectories captured by LCS can be caused by both exponential repulsion and local strain. For vortex identification, we are interested in the regions of exponential repulsion and attraction in the flow that separate trajectories inside coherent structures from those that convect with the freestream flow. To implement a hyperbolicity criterion on the LCS such that we retain only the hyperbolically attracting and repelling structures, we use Theorem 3 of Haller.⁶ Specifically, we compute the rate of strain normal to the surface of the LCS given by $\langle \mathbf{n}, \mathbf{S}\mathbf{n} \rangle$, where $\mathbf{n}$ is the unit normal to the LCS and $\mathbf{S}$ is the rate of strain tensor given by

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (4)$$

We confirm a ridge to be an attracting material line if the strain rate normal to the nLCS surface is negative. Similarly, we confirm a ridge to be a repelling material line if the strain rate normal to the pLCS surface is positive.

In previous work using LCS,^{30,31} the definition of a LCS included any maximizing ridge of the DLE field, but it was assumed that the only ridges of interest were those that satisfied some version of a hyperbolicity criterion. Here, we refer to all ridges as LCS and to those that satisfy the hyperbolicity criterion as hyperbolic LCS, or as attracting and repelling material lines. We do this to enable a discussion of the hyperbolicity characteristics along the LCS.

The LCS plots in this paper are visualized as ridges of the DLE field above a threshold chosen to reveal the com-

plete structure boundary, and those regions in which the DLE fields do not satisfy the hyperbolicity criteria are left blank. Thresholds were chosen so that the representations of coherent structures were as clear and complete as possible. Location, shape, and size of coherent structures are unaffected by the threshold value and do not affect the conclusions drawn from these results.

In this paper, we first apply the complete LCS analysis on two previously investigated fluid flows. The Hill's spherical vortex in Sec. II A is an analytic solution of the Euler equations that produce an axisymmetric spherical vortex in uniform flow, which bears resemblance to experimentally generated vortex rings, as shown in the work of Shadden *et al.*⁸ In Sec. II B, we review the results of Green *et al.*,¹⁶ who used direct numerical simulation to generate data of both an isolated hairpin vortex and fully turbulent channel flow. Here, a new insight is gained by adding the hyperbolic pLCS to the analysis. Finally, in Sec. III we use the concepts of LCS and hyperbolicity to examine the generation and evolution of the wake of a simple bioinspired propulsor.

A. Hill's spherical vortex

To illustrate the relevance of hyperbolicity and how both the positive-time and negative-time LCSs are used to locate the boundaries of coherent structures, we investigate a classic vortex flow: the steady Hill's spherical vortex. This case is an analytic solution of the Euler equations that yield an axisymmetric spherical vortex in a uniform flow. Newton³² gives the stream function in cylindrical coordinates (r, φ, z) inside and outside the sphere of radius a by

$$\Psi_{\text{in}} = \frac{\alpha}{10} r^2 (a^2 - z^2 - r^2), \quad z^2 + r^2 < a^2, \quad (5)$$

$$\Psi_{\text{out}} = -\frac{\alpha}{15} a^2 r^2 \left(1 - \frac{a^3}{(z^2 + r^2)^{(3/2)}} \right), \quad z^2 + r^2 > a^2.$$

The velocity field is then given by

$$u_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad u_\varphi = 0, \quad u_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}. \quad (6)$$

For reference, Fig. 2(a) shows the streamlines in a two-dimensional cut through the middle of the spherical vortex. Figures 2(b) and 2(c) show the negative-time DLE field in a cross section of Hill's spherical vortex calculated using two different integration times. Ridges of the nDLE field are plotted as red, and ridges of the pDLE field are plotted as blue, a convention adopted for all figures presented here. The circular boundary of the spherical vortex at $r=1$ is clearly represented by a maximizing ridge of the nDLE field. In Fig. 2(b), the complete boundary is not seen, but as the integration time is increased, the whole boundary emerges and the LCS becomes sharper and clearer. The integration time can be increased or decreased depending on the amount of detail desired from the calculation, but the location of the ridge indicating the boundary of the vortex does not change.

The results of the positive-time DLE calculation are shown in Figs. 2(d) and 2(e). For a sufficiently long integration time, the circular boundary of the steady spherical vor-

FIG. 2. (Color) Hill's spherical vortex and its negative- and positive-time DLE fields with red and blue showing maximum values. (a) Streamlines with a green circle indicating the vortex boundary at $r=1$; (b) nDLE, short integration time; (c) nDLE, long integration time; (d) pDLE, short integration time; and (e) pDLE, long integration time.

tex can also be obtained using the positive-time calculation. However, when the appropriate hyperbolicity criteria are applied to the LCS, as in Fig. 3, it can be seen that both pLCS and nLCS are necessary to draw the complete hyperbolic boundary.

The upstream half of the structure is bounded by the nLCS while the pLCS provides the downstream boundary, and the two do not overlap. There also exist ridges at the upstream and downstream ends of the structure which are aligned with the vortex axis of symmetry. At the upstream end of the structure $((r,z)=(0,1))$, which is one of two stagnation points in the flow field, the hyperbolic pLCS aligned with the symmetry axis intersects the hyperbolic nLCS that bounds the spherical vortex. The intersection of the hyper-

FIG. 3. (Color) Hyperbolic pLCS (blue) and nLCS (red) field for Hill's spherical vortex. Black circles highlight saddle points.

bolic pLCS and nLCS reveals a saddle point at this location, and the pLCS and nLCS represent the associated stable and unstable manifolds. A similar saddle point is revealed at the downstream stagnation point of the Hill's spherical vortex $((r,z)=(0,-1))$. Because the saddle points are located at the same location as the stagnation points in the flow, it is believed that they are physically significant structural elements in the boundaries of individual vortices as depicted by the LCS. Similar hyperbolic points have previously been identified in two-dimensional aperiodic fluid flows by Haller.³³ In Fig. 3 the nLCS and pLCS meet at three other points on the vortex boundary: $(r,z)=(\pm 1,0)$ and $(r,z)=(0,0)$. At these points, the LCSs meet tangentially and do not intersect, as the hyperbolicity criterion cannot be simultaneously satisfied for both the pLCS and nLCS.

B. Isolated hairpin vortex

Another example in which a LCS analysis has proven useful in the investigation of coherent structures is the single hairpin vortex, a structure commonly found in turbulent wall-bounded flows.¹⁶ The method used to generate a single hairpin vortex simulation was introduced by Zhou *et al.*² From a Direct Numerical Simulation (DNS) database of fully turbulent channel data, linear stochastic estimation was used to find the statistically most probable flow field for the creation of a single hairpin. The resulting most probable flow field is then used as an initial condition for the DNS solver to study the evolution of the structure.

Figure 4 shows plots of the hairpin vortex using both a Eulerian vortex criterion and nDLE fields (from Green *et al.*). In Fig. 4, an isosurface of the swirl criterion (10% max value) is plotted. Figures 4(b)–4(d) show the nDLE fields at the three two-dimensional cross sections of the structure, which are indicated by the black planes plotted in Fig. 4(a).

While much information about the development of these structures was obtained through the use of the nDLE plots, more information can be revealed when the positive-time LCS is included in the analysis. Figure 5(a) shows the two-dimensional plane normal to the channel wall that cuts through the hairpin head, as in Fig. 4(d). Figure 5(b) shows the plane parallel to the wall that cuts through the counter-rotating hairpin legs, as in Fig. 4(c). Saddle points, represented as intersections of the hyperbolic pLCS and the nLCS, are again present along the vortex core boundaries and are located at the upstream and downstream ends of the hairpin head in Fig. 5(a) and of the hairpin legs in Fig. 5(b). It is interesting to note that these structurally stable saddle points are similar to those observed in the LCS plots of the steady Hill's spherical vortex in Sec. II A.

If the same analysis is performed on a fully turbulent channel simulation, similar patterns of hyperbolic pLCS and nLCS are apparent. In Fig. 6, one such structure is highlighted with a black box. This structure is bounded by alternating pLCS and nLCS, with time-dependent saddle points located both upstream and downstream of the vortex core piercing through the plane. It is postulated that this is a cross section of the head of a hairpin vortex in this fully turbulent

FIG. 4. Two-dimensional nDLE plots of the isolated hairpin: (a) 10% max λ_{ci}^2 superimposed on location of the three planes, (b) constant-streamwise cut, (c) constant wall-normal cut, and (d) constant-spanwise cut (Ref. 16). [Reprinted with permission from M. A. Green, C. W. Rowley, and G. Haller, J. Fluid Mech. 572, 111 (2007). Copyright 2007, Cambridge University Press.]

flow. The locations of these intersections are easy to locate in a quantitative sense and may be useful for future structure identification and tracking in complicated flows.

Using an evaluation of hyperbolicity along the nLCS only, Green *et al.* showed that the formation of a secondary hairpin vortex upstream of the primary hairpin corresponds to a loss of hyperbolicity along the nLCS. If hyperbolic pLCS is also plotted, as in Fig. 7, it is clear that saddle points are created upstream and downstream of this new vortex core at the same time the nLCS loses hyperbolicity. Figure 7(a) shows the hyperbolic nLCS and pLCS before the birth of the secondary hairpin at the location along the structure at which it will be formed. Figure 7(b) shows the hyperbolic nLCS and pLCS at the location of the new hairpin after its formation, and the two new saddle points are highlighted using a black box.

III. ANALYSIS OF UNSTEADY WAKES

A. Experimental setup

To investigate the vortex wake of biologically inspired rigid pitching panels, digital particle image velocimetry (DPIV) experiments were conducted to obtain the two-component velocity fields at the midspan of the wake. These experiments were conducted in a recirculating water channel with a test section that has a width of 0.46 m and a maximum depth of 0.29 m. An acrylic plate, 12 mm thick and 1.22 m long, was used to cover the free surface and prevent the

formation of surface waves which would both influence the flow physics in the test section and distort flow visualizations and DPIV image acquisition from above.

The rigid trapezoidal panel was mounted at its leading edge to a 4.76 mm diameter pitching shaft. To support the pitching apparatus, the shaft was attached to the trailing edge of a symmetric fairing based on a NACA 0012-64 airfoil, as described by Buchholz and Smits³⁴ and shown in Fig. 8 (for a rectangular panel). The fairing had a chord length of 50.8 mm, and the trailing edge was truncated to allow the attachment of the pitching shaft. The panel and the fairing were mounted vertically in the water channel test section, as shown in Fig. 9(a). The actuation of the panel used the same mechanism as the previous work by Buchholz,²⁸ which achieved the pitching motion using a four-bar linkage.

The trapezoidal panel was made of 2 mm thick acrylic and had swept edges that were set at an angle (θ) of 45° from the streamwise direction. A schematic of the trapezoidal panel geometrical parameters is shown in Fig. 9(b). The panel had a chord (c) of 70 mm and a trailing edge span (S) of 175 mm. Experiments were conducted with a panel trailing edge amplitude (A) of 10 mm, pitching frequency of 1 Hz (period $T_p = 1$ s), and freestream velocities of 36 and 60 mm/s. This resulted in Strouhal numbers of 0.27 and 0.17, respectively.

The two-dimensional planes in which images were captured were oriented parallel to the streamwise flow and normal to the panel surface. The particles used to seed the flow

FIG. 5. (Color) Hyperbolic pLCS (blue) and nLCS (red) of the isolated hairpin head in a two-dimensional cross section of the hairpin vortex. (a) Constant-spanwise (x - y) plane, plotted as regions of $DLE > 50\%$ maximum value that satisfy the corresponding hyperbolicity criteria, and (b) constant wall-normal (x - z) plane, plotted as regions of $DLE > 43\%$ maximum value that satisfy the corresponding hyperbolicity criteria. Time-dependent saddle points are highlighted using a black box.

were $13\ \mu\text{m}$ hollow silvered spheres and were illuminated using a Spectra Physics argon-ion continuous-wave laser. The laser beam passed through a fiber optic cable, collimator, and Powell lens which emitted a 2 mm thick laser sheet. A Redlake HG-LE camera was used to acquire the images and was mounted vertically above the water channel. It was externally triggered by a Stanford Research Systems four channel digital delay/pulse generator (Model DG535), which controlled the timing of the experiment.

FIG. 6. (Color) Hyperbolic pLCS (blue) and nLCS (red) of the isolated hairpin head in a two-dimensional slice in a constant-spanwise (x - y) plane through the fully turbulent channel, plotted as regions of $DLE > 41\%$ maximum value that satisfies the corresponding hyperbolicity criteria. A possible hairpin vortex is highlighted by a white box.

FIG. 7. (Color) Hyperbolic pLCS (blue) and nLCS (red) of the secondary hairpin head (a) before formation and (b) after formation LCS is plotted as regions of $DLE > 60\%$ maximum value that satisfies the corresponding hyperbolicity criteria. Time-dependent saddle points are highlighted with a black box.

Each two-dimensional velocity field was obtained at 25 discrete phases in the pitching cycle. In each plane, 20 image pairs (Δt between each image: $0.02T_p$) were acquired at each of the 25 phases in the panel motion, and the resulting 20 velocity fields at each phase were phase averaged. This yielded 25 phase-averaged velocity data sets per pitching cycle with a time resolution of $0.04T_p$.

Each DPIV data set spans 0.10 m ($1.4c$) in the streamwise direction and 0.15 m ($2.1c$) in the transverse direction. Full data sets were taken at three overlapping streamwise locations for a total streamwise data length of 0.28 m ($2.7c$), where the upstream edge of the data is located on the panel chord at $x=0.45c$. The Redlake camera was operated at its full resolution of 1128×752 pixels, which resulted in streamwise and transverse grid spacings of 2.2 mm. The DPIV analysis in each plane was performed using software developed by Jiménez.³⁵

FIG. 8. Fairing and panel assembly (Ref. 28). (From J. H. J. Buchholz, Ph.D. thesis, Princeton University, 2006. Reprinted with permission.)

B. Eulerian analysis

In Fig. 10, the spanwise vorticity (ω_z) distributions along the midspan of the trapezoidal pitching panel wake at $St=0.17$ are plotted at four phases of the panel motion. Anomalies in the vorticity field near $x=0.7c$ and $x=2c$ are caused by small discontinuities in the velocity field, as these are the regions in which data acquired at different down-

FIG. 9. (a) Panel and fairing position in water channel test section (Ref. 28). (From J. H. J. Buchholz, Ph.D. thesis, Princeton University, 2006. Reprinted with permission.) (b) Diagram of trapezoid dimensions.

stream stations overlap. The effect is exacerbated when the derivatives of the field are taken to compute the vorticity.

The spanwise vorticity is organized into coherent structures with a clear 2S pattern. One vortex is shed each half cycle and consecutive vortices have alternating sign. However, the wake of the trapezoidal panel pitching with $St=0.27$ is more complicated. As shown in Fig. 11, the spanwise structures spread in the transverse direction, lose vorticity magnitude, and lose coherence approximately one chord length downstream of the trailing edge. This may indicate a possible breakdown or transition to a 2P configuration of the midspan wake. However, without a clearer understanding of the boundaries of the individual vortices, it is not possible to accurately describe the vortex dynamics at this location.

C. LCS analysis

To more precisely describe the vortex interactions caused or induced by the wake transition, a LCS analysis was conducted using the two-component velocity fields obtained by the DPIV experiments. The spanwise velocity (w) was assumed to be zero, which is a questionable assumption in these highly 3D flows. However, the large-scale structures in this wake are expected to be associated primarily with the spanwise vorticity, which is normal to the velocity plane. In addition, the LCS results shown here are at the midspan of the wake, around which the flow is expected to be symmetric.

Particles that leave the domain were assumed to continue in the streamwise direction with the freestream velocity. For those trajectories that are advected upstream of the field of view during the negative-time calculation, a uniform freestream assumption was made, as this is the boundary condition upstream of the pitching apparatus. Downstream of the measurement volume, however, the loss of information inherent in this assumption causes a lack of sharp ridges in the positive-time DLE field. In many of the figures, the pLCSs often are not revealed in the downstream third of the data domain for this reason.

Spanwise vorticity is plotted simultaneously with the nLCS and pLCS at four phases of the panel wake for $St=0.17$ in Fig. 12. At this Strouhal number, all DLE calculations were done using an integration time of two pitching periods. It is apparent that both the Eulerian and Lagrangian techniques reveal the same large scale structures, but the LCS provides a transverse boundary of the wake and additional detail of the dynamics among the vortices. As expected, the LCS is also relatively insensitive to the velocity field discontinuity that was highlighted by the vorticity field.

To more closely describe the structure of the wake at the midspan, the hyperbolicity criteria were applied to the pLCS and nLCS and the resulting structures are shown in Fig. 13. The boundaries of the individual spanwise vortex cores consist of time-dependent saddle points at each transverse end of the structures. Within each vortex core, an alternating scroll pattern of hyperbolic pLCS and nLCS is observed. This pattern is only apparent for approximately two thirds of the streamwise domain due to the loss of information during the pDLE calculation.

FIG. 10. Two-dimensional planes of spanwise vorticity in the midspan plane at four phases of the panel motion for $St=0.17$: (a) $\phi=0^\circ$, (b) $\phi=90^\circ$, (c) $\phi=180^\circ$, and (d) $\phi=270^\circ$.

In Fig. 14, LCS is superimposed on plots of spanwise vorticity at the midspan for the panel pitching at $St=0.27$. For this Strouhal number, an integration time of four pitching cycles was used. Again, both the Eulerian and Lagrangian methods track the spanwise structure size and location. At the same streamwise location at which we observe a wake transition in the vorticity plots, we observe a qualitative change in the pattern of the LCS as the width of the wake expands in the transverse direction and there are no longer clear overlapping swirls of the LCS ridges.

To inspect this LCS pattern change that occurs approximately one chord length downstream of the trailing edge, hyperbolicity criteria were applied and images that depict the evolution of the hyperbolic LCS are shown in Fig. 15 at four phases of the panel pitching motion. A qualitative change in the structure of the LCS is seen as the structures move downstream. In Fig. 15(a), black boxes highlight two time-dependent saddle points that are part of the boundary of two

distinct vortices as they convect downstream. The distance between the saddles decreases [Fig. 15(b)] until they seemingly merge into each other after one pitching cycle [Fig. 15(c)]. The qualitative change in the LCS that defines the boundaries of consecutive vortices coincides with the loss of coherence of the individual vortices. The hyperbolic pLCS persists through this point [Fig. 15(d)], providing the boundary between the two branches of the split wake downstream of the transition.

IV. SUMMARY

A LCS analysis was used to examine the organization of vorticity in the wake of a purely pitching trapezoidal panel. In those regions where the wake consisted of a 2S vortex street, a characteristic pattern was seen in the hyperbolic pLCS and nLCS plots including quantitatively identifiable time-dependent saddle points on the vortex boundaries and

FIG. 11. Two-dimensional planes of spanwise vorticity in the midspan plane at four phases of the panel motion for $St=0.27$: (a) $\phi=0^\circ$, (b) $\phi=90^\circ$, (c) $\phi=180^\circ$, and (d) $\phi=270^\circ$.

FIG. 12. (Color) nLCS (red) and pLCS (blue), plotted as regions of $DLE > 24\%$ maximum value, superimposed on two-dimensional planes of spanwise vorticity at the midspan for $St=0.17$: (a) $\phi=0^\circ$, (b) $\phi=90^\circ$, (c) $\phi=270^\circ$, and (d) $\phi=360^\circ$.

FIG. 13. (Color) Hyperbolic nLCS (red) and pLCS (blue), plotted as regions of $DLE > 35\%$ maximum value that satisfies the corresponding hyperbolicity criteria, in the midspan of the panel pitching at $St=0.17$ with $\phi=0^\circ$. Black boxes highlight saddle points that are part of individual vortex core boundaries.

an alternating scroll pattern within the vortex cores. In the wake of the panel pitching at $St=0.27$, plots of spanwise vorticity showed a breakdown of the spanwise structures and possible transition into a 2P organization approximately one chord length downstream of the trailing edge. At the same location in the wake, the LCS results exhibited a qualitative change in which two nearby saddles, bounding subsequent spanwise vortices, seemingly merged.

That the changes in LCS structure correspond to physically significant changes in the composition and organization

FIG. 14. (Color) nLCS (red) and pLCS (blue), plotted as regions of $DLE > 31\%$ maximum value, superimposed on two-dimensional planes of spanwise vorticity at the midspan for $St=0.27$: (a) $\phi=0^\circ$, (b) $\phi=90^\circ$, (c) $\phi=270^\circ$, and (d) $\phi=360^\circ$.

FIG. 15. (Color) Hyperbolic pLCS and nLCS, plotted as regions of $DLE > 47\%$ maximum value that satisfies the corresponding hyperbolicity criteria, at the midspan of the panel pitching at $St=0.27$. Black boxes highlight the merging of two nearby saddle points: (a) $\phi=180^\circ$, (b) $\phi=360^\circ$, (c) $\phi=540^\circ$, and (d) $\phi=720^\circ$.

of vortices had been shown previously. For example, in the case of the isolated hairpin head, a loss of hyperbolicity along a nLCS and the appearance of nearby saddle points upstream of the primary vortex indicated the birth of a secondary vortex. For the transitioning panel wake studied here, time-dependent saddle points observed in the midspan cross section of the wake were part of the boundaries of distinct vortices. Two saddle points were observed to merge at the same time and place that individual vortices interacted and lost coherence, and the wake split.

More information is necessary for a fuller understanding of the complex vortex dynamics at this transition. Additional data downstream would enable a more complete calculation of the pLCS, and a more refined LCS extraction method, such as those described in Malhotra and Wiggins³⁶ or Shadden *et al.*,³¹ should be implemented to obtain sharper LCS

plots. Also, a fully 3D, three-component velocity field would allow us to plot the complete two-dimensional LCS surfaces in the 3D flow fields.

Additionally, the LCS could be used to guide a momentum analysis to examine the locomotive forces generated by the pitching panel. Peng *et al.*³⁷ had some success performing a similar analysis in the wake of a bluegill sunfish pectoral fin, even though their DPIV plane was normal to the freestream and they could not follow particle trajectories out of plane. At the midspan of the pitching trapezoidal panel presented here, out of plane velocities are assumed to be small, so a similar force analysis has a good potential to provide further information about the performance.

However, even considering only two-dimensional data in the plane of symmetry, hyperbolic LCS has yielded a more complete picture of the shape and size of these structures than that previously available from Eulerian analysis. They have also provided quantitative hints to the underlying dynamics that results in the creation and destruction of the coherent structures that dominate the fluid flows presented here.

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