Generalized shot noise model for time-reversal in multiple-scattering media allowing for arbitrary inputs and windowing

Kevin J. Haworth

Department of Radiology and the Applied Physics Program, University of Michigan, Ann Arbor, Michigan 48109

J. Brian Fowlkes and Paul L. Carson

Department of Radiology, Department of Biomedical Engineering, and the Applied Physics Program, University of Michigan, Ann Arbor, Michigan 48109

Oliver D. Kripfgans^{a)}

Department of Radiology and the Applied Physics Program, University of Michigan, Ann Arbor, Michigan 48109

(Received 21 May 2008; revised 26 January 2009; accepted 25 February 2009)

A theoretical shot noise model to describe the output of a time-reversal experiment in a multiple-scattering medium is developed. This (non-wave equation based) model describes the following process. An arbitrary waveform is transmitted through a high-order multiple-scattering environment and recorded. The recorded signal is arbitrarily windowed and then time-reversed. The processed signal is retransmitted into the environment and the resulting signal recorded. The temporal and spatial signal and noise of this process is predicted statistically. It is found that the time when the noise is largest depends on the arbitrary windowing and this noise peak can occur at times outside the main lobe. To determine further trends, a common set of parameters is applied to the general result. It is seen that as the duration of the input function increases, the signal-to-noise ratio (SNR) decreases (independent of signal bandwidth). It is also seen that longer persisting impulse responses result in increased main lobe amplitudes and SNR. Assumptions underpinning the generalized shot noise model are compared to an experimental realization of a multiple-scattering medium (a time-reversal chaotic cavity). Results from the model are compared to random number numerical simulation. © 2009 Acoustical Society of America. [DOI: 10.1121/1.3106133]

PACS number(s): 43.60.Tj, 43.60.Cg [DRD]

Pages: 3129–3140

I. INTRODUCTION

Parvulescu and Clay¹ performed the first time-reversal acoustics (TRA) experiments in 1965. In the following two decades, however, relatively little work was done in TRA. Beginning approximately 15 years ago, however, there has been a burst of activity, led in large part by Professor Mathias Fink. Over this time, nearly all of the experiments can be described by the general process of sending a short pulse into a medium and recording the resulting wave $s(t)$. The recorded signal is time-reversed $s(t) \rightarrow s(-t)$ (and possibly processed) and then retransmitted into the medium. The resulting time-reversal focused signal is recorded and analyzed.

One area of TRA that has shown both surprising and fruitful results is the time-reversal of waves that have traveled through random high-order multiple-scattering environments. Traditionally, random multiple-scattering environments display chaotic behaviors that prohibit focusing through them. Propagating waves through a “forest-of-needles” to a point receiver and then time-reversing this signal and showing that it would refocus at its origin was the first demonstration of the ability to focus through this type of

medium.² Surprisingly, it was found that the focused signals were even more well defined after going through the multiple-scattering medium than if focusing was performed in a homogenous medium!² These results were extended to other high-order multiple-scattering environments such as a chaotic cavity, where the boundaries of a reverberant material scattered the sound.³ Various approaches have been taken to describe the unexpected results of TRA in multiple-scattering media. This paper is concerned with generalizing the approach first developed by Derode *et al.*⁴ to explain this phenomenon. This approach treats the multiple-scattering events via a shot noise model. The shot noise approach allowed Derode *et al.*⁴ to successfully model 1-bit time-reversal. It has also been used to determine the impact of windowing a signal before performing time-reversal.⁵ Others have also extended the shot noise model to incorporate additional phenomena, such as scattering dependencies in forest-of-needle experiments.⁶ In addition to the shot noise approach other approaches have used scattering theory,^{7–9} eigenmode decomposition,³ and Green’s functions.¹⁰ Each of which makes different assumptions and elucidates different effects (e.g., coherent backscatter,⁹ noise emission in time-reversal,¹¹ etc.).

The primary motivation for the development of this generalized model is to predict the signal-to-noise ratio (SNR) for future time-reversal chaotic cavity (TRCC) experiments

^{a)} Author to whom correspondence should be addressed. Electronic mail: oliver.kripfgans@umich.edu

FIG. 1. (a) Experimentally obtained IR for a TRCC. (b) R^2 values indicating that the amplitudes in each 50- μ s time-interval of the IR are well modeled as normal random variables.

(though other applications may be found, as illustrated below). In particular, for cases where one is interested in transmitting signals besides delta functions, the long tone-bursts are associated with high-intensity focused ultrasound for heating or acoustic radiation force experiments. TRCCs were first introduced by Fink and co-workers.^{3,12,13} TRCC experiments work by having a transducer transmit an acoustic/elastic pulse into a solid (typically a metal). The sound reverberates within the cavity, reflecting off of the solid's walls. The acoustic signal at any point within the cavity can quickly become a diffuse wave. If the diffuse wave is recorded, time-reversed, and retransmitted, the waves will approximately retrace their paths and focus at the transducer that originally transmitted the pulse. The initial experiments by Draeger and Fink measured elastic waves in a two dimensional silicon wafer. Quieffin *et al.*¹⁴ showed that this concept could be extended to three dimensional (3D) solids and more importantly that if the solid were put into contact with a water bath, signal would leak out of the solid and could be recorded with a hydrophone in the water. Then using spatial reciprocity, they showed that if the signal recorded by the hydrophone was time-reversed and retransmitted by the original transducer, a pulse would focus on the hydrophone's location outside of the cavity. The location of the hydrophone could be varied and thus it was found that TRCCs allowed focusing throughout a 3D volume with as few as one ultrasound transducer. Additional work by Fink and co-workers has led to prototypes for imaging devices¹⁵ and high amplitude ultrasound therapy devices¹⁶ among other applications. Since the initial development of TRCCs, Sarvazyan and co-workers^{17–19} have also made significant progress in understanding and utilizing TRCCs.

Modeling of high-order multiple-scattering TRA has found other applications. These include biomedical engineering,²⁰ non-destructive testing and evaluation,^{21,22} geophysics,^{23–25} underwater acoustics,²⁶ imaging,^{27,28} and (wireless) communication.^{29–31} In many of these applications, one may be interested in sending not just a short pulse through the multiple-scattering medium but an extended pulse that could be used to contain extensive information or induce an effect. In this paper, the model initiated by Derode *et al.*⁴ is generalized to account for arbitrary input functions and arbitrary windowing. The goal of the model will be to compute the expectation value and variance of a time-reversal focused signal through a multiple-scattering medium.

This article is organized as follows. Section II outlines the problem in greater detail (Sec. II B), derives the expectation value (Sec. II C), variance (Sec. II D), and directivity pattern (Sec. II F), and provides some physical (Sec. II A) and numerical support (Sec. II E) for the model. In Sec. III a set of common parameters is then applied to these general results, allowing the expectation value (Sec. III A) and variance (Sec. III B) to be simplified. The SNR under these conditions is also derived and discussed (Sec. III C).

II. GENERAL THEORY

Derode *et al.*⁴ provided an explanation for using a shot noise model to describe multiple-scattering events that is briefly repeated here for completeness. For any high-order random multiple-scattering process, the majority of the signal is composed of a diffuse wave that is the result of interference of a large number of multiple-scattering paths. Since the scattering is random (e.g., random variations in the impedance of the medium) the arrival time from each path can be described as a series of Poisson impulses. For an impulse sent through the medium, the output can be described by the convolution of the acousto-electrical impulse response (IR) with the Poisson impulses corresponding to every path. By definition, this is a shot noise process.³² Since the number of paths is large, one can assume that the density of Poisson impulses per acousto-electrical IR is large. This then allows the shot noise process to be approximated as a normal random process with mean zero (assuming that the input signal is also mean zero) and variance $\sigma^2(t)$.³²

A. Experimental validation of shot noise assumptions

To further motivate the use of a shot noise model, the IR was experimentally obtained from a TRCC (Imasonic SAS, Besançon, France) constructed to be the same as the one used by Montaldo *et al.*¹⁶ The IR was obtained by driving one of the elements on the TRCC with a step function generated from an HP33120A function generator (Agilent, Palo Alto, CA) and amplified by an ENI A-300 rf power amplifier (Rochester, NY). The face opposite the elements was placed in a water tank, and a hydrophone (PVF₂, Raytheon Co., Waltham, MA) recorded the signal transmitted into the water. The IR [Fig. 1(a)] of the cavity was recorded and broken into 50 μ s time-intervals. If the IR can be modeled as a normal random variable, then the amplitudes within each time-

interval should have a normal distribution. To determine if this is true, the amplitudes in each interval were plotted in a histogram. The histogram was then fitted to a Gaussian function $Ae^{(x-\mu)^2/2\sigma^2}$ and the R -squared value (i.e., square of the Pearson product-moment correlation coefficient) computed to determine the goodness-of-fit. Figure 1(b) shows that the computed R -squared values are indeed all close to 1, indicating a good fit.

While it has been demonstrated that the assumptions used for the shot noise model are valid for a specific case, the derivations that follow should hold for any high-order multiple-scattering process that satisfy the assumptions made above. In fact, the derivation is not necessarily specific to acoustic waves and could be applied to any wave phenomena where phase coherent detectors with reversible signals exist or are developed (e.g., radio frequency electromagnetic waves). The results of the derivations will be compared to a random number numerical simulation, rather than a physical experiment or a numerical simulation based on the wave equation. Others have shown that the shot noise model does describe physical results of time-reversal focusing under various particular conditions.^{5,6}

B. Time-reversal focused signal

The generic time-reversal experiment that will be modeled occurs as follows. An input signal $g(t)$ is transmitted into the scattering medium, which has an IR $h(t)$. The resulting output signal $g(t) \otimes h(t)$ is recorded (where $\otimes_t$ is a convolution over time t). A window function $W(t)$ is applied to the output signal, which is then normalized by its maximum value $\mathbf{M}$:

$$\frac{1}{\mathbf{M}} W(t) \cdot (g(t) \otimes h(t)). \quad (1)$$

The window function selects the desired portion of the total output signal that will be used in the time-reversal experiment. The windowed function is then time-reversed ($t \rightarrow -t$) with a temporal shift of T to ensure causality, yielding

$$\frac{1}{\mathbf{M}} W(T-t) \cdot (g(-t) \otimes h(T-t)). \quad (2)$$

The time-reversed function is now retransmitted into the medium yielding an output

$$r(t) = \left(\frac{1}{\mathbf{M}} W(T-t) \cdot (g(-t) \otimes h(T-t)) \right) \otimes_t h(t). \quad (3)$$

The expectation value and variance of this output will be computed.

C. Expectation value

As Derode *et al.*⁴ showed, even though $\mathbf{M}$ is a random variable, because of its origins in $g(t) \otimes h(t)$, it is approximately constant and thus can be pulled out of the expectation integral. As such it will now be denoted as M . Haworth³³ provided an approximation for M . To compute the expectation value of Eq. (3), the convolutions are written as inte-

grals. The integration variables for the convolutions are θ_t and τ_θ . The functional variables associated with each convolution are t and θ , respectively,

$$E\{r(t)\} \approx \frac{1}{M} E \left\{ \int_{\theta_t=-\infty}^{\infty} \left(W(T-\theta_t) \times \int_{\tau_\theta=-\infty}^{\infty} g(-\tau_\theta) \times h(T-(\theta_t-\tau_\theta)) d\tau_\theta \right) h(t-\theta_t) d\theta_t \right\}. \quad (4)$$

Since $h(t)$ defines a stochastic process with a normal probability distribution function $f(h(t))$, the expectation value of $r(t)$ can be computed as

$$E\{r(t)\} = \int_{h(t)=-\infty}^{\infty} [r(t; h(t)) \cdot f(h(t))] dh(t). \quad (5)$$

Using this notation, Eq. (4) is first rewritten expressing the expectation value as an integral. The integrals are then reordered, making the integral over $h(t)$ the innermost integral, and finally rewriting that integral in the $E\{\cdot\}$ notation

$$E\{r(t)\} \approx \frac{1}{M} \int_{\tau_\theta=-\infty}^{\infty} \int_{\theta_t=-\infty}^{\infty} g(-\tau_\theta) W(T-\theta_t) \times E\{h(T-(\theta_t-\tau_\theta))h(t-\theta_t)\} d\theta_t d\tau_\theta. \quad (6)$$

Now all constants with respect to the stochastic process $h(t)$ have been removed from the expectation value. As was described above, $h(t)$ is a normal random variable with mean zero and variance $\sigma^2(t)$. The expectation value of two mean zero, jointly normal random variables multiplied together is³²

$$E\{\mathbf{x}(t_1) \cdot \mathbf{y}(t_2)\} = \rho(t_1-t_2) \cdot \sigma_x(t_1) \cdot \sigma_y(t_2), \quad (7)$$

where $\rho(t_1-t_2) = \rho(t_2-t_1)$ is the correlation coefficient of the normal random variables $\mathbf{x}(t_1)$ and $\mathbf{y}(t_2)$, and $\sigma_x(t_1)$ and $\sigma_y(t_2)$ are the standard deviations of $\mathbf{x}(t_1)$ and $\mathbf{y}(t_2)$, respectively. Applying this and rewriting the integrals in traditional convolution notation yields

$$E\{r(t)\} \approx \frac{1}{M} g(-t) \otimes_t [\rho(T-t)((W(T-t) \cdot \sigma(T-t)) \otimes \sigma(t))]. \quad (8)$$

Equation (8) is the statistical approximation for the expectation value of the time-reversal focused signal. Note that it is non-zero only for a duration approximately as long as the input function $g(t)$. This time with non-zero amplitude will be referred to as the main lobe of the signal and all times outside this as side lobes. Looking at the square bracket term, one sees that a rapidly oscillating function $\rho(T-t)$ is multiplied by a relatively slow changing function $(W(T-t) \cdot \sigma(T-t)) \otimes_t \sigma(t)$ (assuming that the window function is not rapidly changing). As a result, the term in the square brackets will look approximately like a scaled version of $\rho(T-t)$. Therefore, how large this term is, and thus how large $E\{r(t)\}$ is, depends directly on the amplitude and decay of the envelope and how large the window is. Therefore the result found by others that the expected signal increases as the amplitude of the IR (or time-reversed signal in the case of 1-bit time-reversal) increases is confirmed.^{4,5,34} Equation (8)

is easily shown to simplify to the corresponding result of Derode *et al.*⁴ [term 1 of Eq. A2] by letting the window extend to positive and negative infinities [i.e., $W(t)=1 \forall t$], setting $T=0$, and letting $g(t) \rightarrow \delta(t)$.

D. Variance

Since the variance can be computed from

$$\text{VAR}\{r(t)\} = E\{r^2(t)\} - E^2\{r(t)\},$$

the process is similar to what was done in Sec. III C. However, the mathematics are more tedious due to the squaring of $r(t)$.³³ The result is

$$\begin{aligned} & \left(\frac{1}{M^2} \int_{\theta_1=-\infty}^{\infty} \int_{\theta_2=-\infty}^{\infty} W(T-\theta_1)W(T-\theta_2)\sigma^2(t-\theta_1)\sigma^2(t-\theta_2) \right. \\ & \quad \times [g(-\theta_1) \otimes_{\theta_1} \rho(T-\theta_1-(t-\theta_2))] \\ & \quad \times [g(-\theta_2) \otimes_{\theta_2} \rho(T-\theta_2-(t-\theta_1))] d\theta_2 d\theta_1 \Big) \\ & + \left(\frac{1}{M^2} \int_{\theta_1=-\infty}^{\infty} W(T-\theta_1)\sigma(t-\theta_1) \right. \\ & \quad \times [\sigma(t) \otimes_{\theta_1} \{W(T-t)\rho(t-\theta_1) \\ & \quad \times [g(-\theta_1) \otimes_{\theta_1} \{\sigma^2(T-\theta_1)[g(-t) \otimes_{\theta_1} \rho(t-\theta_1)]\}] d\theta_1 \Big). \quad (9) \end{aligned}$$

Each of these two terms can now be related to the physical processes occurring in multiple-scattering. The nature of the symmetry of the square brackets in the first term indicates that this term only contributes at the main lobe and is zero outside of it. This term comes from variations in the total energy contained in the time-reversed signal that is transmitted into the medium. That is, it originates from the variance in the total energy in the time-reversed signal over different realizations of multiple-scattering processes (i.e., a reordering of the scatterers or placement of the sound source). Note that on average the energy will be the same for a particular interval, and conservation of energy dictates that the total energy in $h(t)$ over all time will be constant. However, over different realizations of the multiple-scattering environment, the distribution of energy will change, even for the same time-interval. Due to this, the first term will be referred to as the coherent or correlated variance. To minimize this variance, one might ensure that the wave is completely diffuse or try to include a larger portion of the signal when time-reversing. Further changes that can be made to reduce this term are outlined in Sec. III.

The second term is due to the interference of the time-reversed signal and $h(t)$ at all times when the two waveforms are uncorrelated. Thus one can refer to this term as being the incoherent or uncorrelated variance. To maximize the main lobe to side lobe ratio (a possible SNR definition) one would be interested in minimizing the second term while maximizing the expectation value. Alternatively, for a more consistent main lobe amplitude, the first term should be minimized.

E. Comparison of derived model with numerical simulation

In addition to verifying the equations by showing their reduction to previously obtained results, the equations were also compared to ensembles of simulated data, as was done by Derode *et al.*⁴ The simulated data were created from a normally distributed random array [created using the `randn` function in MATLAB (The Mathworks, Inc., Natick, MA), which is based on the Ziggurat algorithm³⁵ that has a period of approximately 2^{64}] convolved with a normalized 3.5-cycle sine-wave to simulate $h(t)$, where the 3.5-cycle sine-wave models an acousto-electrical IR. $h(t)$ was then convolved with $g(t)$ (chosen as a delta function for this simulation), windowed, and time-reversed. The result was then convolved with the original $h(t)$ to give a simulated realization of $r(t)$. The mean and standard deviation of an ensemble of 500 simulated $r(t)$ [each with a unique $h(t)$] were plotted against the analytical solution for $E\{r(t)\}$ [Eq. (8)] and the square root of the $\text{VAR}\{r(t)\}$ [Eq. (9)] (Fig. 2). The figure demonstrates that the model accurately predicts the numerical simulation. R^2 values quantifying the goodness-of-fit are shown for each plot.

Based on the equations derived for the expectation value and variance and Fig. 2, one can draw conclusions for how various parameters will impact the time-reversal focusing. Initially, if one focuses on Figs. 2(b) and 2(c) it is possible to see the impact of the decay constant of the envelope of the IR. As one would physically expect, both the variance and the expectation value increase as the decay constant increases. This is due to the fact that there is more energy in the time-reversed signal. Also, one can see that the side lobes fall off more slowly as the time constant increases. The extrapolation of this result has been seen for 1-bit time-reversal, where the side lobes are approximately constant in amplitude⁴ and is consistent with the physical explanation of the incoherent variance given earlier. Next, the impact of shifting the window to later times can also be seen [Figs. 2(c) and 2(d)]. Looking at the expectation value, one sees that its maximum amplitude has decreased for Fig. 2(d). This is expected for two physical reasons. First, since the signal is windowed, the signal that is retransmitted has less total energy. Additionally, this signal correlates with the IR at a later time when the IR has decreased in amplitude. Thus the amplitude of the correlation of the signals (which is essentially what the time-reversal process does) is smaller. Analogous reasoning also leads to explaining why the coherent variance is seen to be smaller. For Figs. 2(b)–2(d), the shape of the incoherent portion of the side lobes initially increases in all cases as more of the time-reversed signal is transmitted and incoherently interferes with the IR. Once the entire time-reversed signal has been transmitted and no more energy is being injected into the system, the incoherent interference decreases as the magnitude of the IR falls off with time. Thus for a delayed window, such as Fig. 2(d), the incoherent interference peaks and begins to fall off before the time-reversed signal has lined up and coherently interferes with the IR. Hence, the variance peaked around 600 μs while the expectation value peak did not occur until 1050 μs . This

FIG. 2. (a) Sample of a numerically simulated IR with the gray section corresponding to the windowed portion used for (d). (b)–(d) compare the expectation/mean value (top plots) and standard deviation (i.e., square root of the variance) (lower plots) for the statistical model (black line) and numerical simulation (gray line). (b) Time-reversal focusing of the full IR shown in (a). (c) Time-reversal focusing of a full IR with a slower decay than the one shown in (a). (d) Time-reversal focusing of the windowed (gray) portion of the IR in (a). R^2 values are given for each plot demonstrating that the model is a good fit to the numerical simulation.

should always be seen whenever the window zeros out the initial part of the recorded signal $g(t) \otimes h(t)$. Note that the shape of the side lobes after the peak should fall off like $\sigma(t)$, while the shape before the peak will be more complicated and depend on both $g(t)$ and $\sigma(t)$. Further trends for changing parameters will be investigated more closely in Sec. III.

F. Directivity pattern

To estimate the directivity pattern, the same process as above will be used; however, the IR will be specific to a particular location x_o . As a result, Eq. (2) is written as

$$\frac{1}{M} W(T-t) \cdot (g(-t) \otimes h(x_o, T-t)). \quad (10)$$

This windowed and time-reversed signal is then retransmitted into the cavity and recorded at a different location x_1 . Thus Eq. (10) is convolved with an IR $h(x_1, t)$ from a different location x_1 . Assuming that the change in distance is large enough for $h(x_o, t)$ and $h(x_1, t)$ to decorrelate, which can be determined from the van Cittert–Zernike theorem,^{4,36} the expectation value and variance are easy to compute. The expectation value goes to zero since $\rho(t_1 - t_2, x_o - x_1)$ in Eq. (7) goes to zero as $x_o - x_1$ increases. Similarly, the coherent variance component will also go to zero, leaving only the incoherent variance component. Assuming that each of the paths has the same scattering statistics [i.e., $h(x_o, t)$ and $h(x_1, t)$ both decay as $\sigma(t)$] then, the total variance is

$$\begin{aligned} \text{VAR}\{r(x_1, t; x_o)\} = & \left(\frac{1}{M^2} \int_{\theta_1=-\infty}^{\infty} W(T-\theta_1) \sigma(t-\theta_1) \right. \\ & \times [\sigma(t) \otimes \rho(t-\theta_1, x_1-x_o)] \\ & \times [g(-\theta_1) \otimes \rho(t-\theta_1, x_o-x_o)] \Big) d\theta_1 \end{aligned}$$

$$\times [g(-t) \otimes \rho(t-\theta_1, x_o-x_o)] \Big) d\theta_1 \Big). \quad (11)$$

Just as Derode *et al.*⁴ found previously in the simplified case, it is found here that the -6 dB width of the directivity pattern is a measure of the correlation length of the scattered waves. For applications in imaging, this term determines the background noise, above which all signals must be observed. Also note that as the amplitude of $g(t)$ increases, the noise floor will increase.

If $h(t)$ and $h'(t)$ decay with different decay envelopes $[\sigma(t)$ and $\sigma'(t)$ respectively], then the result becomes

$$\begin{aligned} \text{VAR}\{r(x_1, t; x_o)\} = & \left(\frac{1}{M^2} \int_{\theta_1=-\infty}^{\infty} W(T-\theta_1) \sigma'(t-\theta_1) \right. \\ & \times [\sigma'(t) \otimes \rho(t-\theta_1, x_1-x_o)] \\ & \times [g(-\theta_1) \otimes \rho(t-\theta_1, x_o-x_o)] \Big) d\theta_1 \Big). \end{aligned} \quad (12)$$

III. APPLICATION TO A COMMON SET OF PARAMETERS

While it was possible to ascertain some qualitative physical insights from the above equations, they do not lend themselves to easily determining the quantitative impact of various parameters (such as changing the window placement or input function). In this section certain conditions for the envelope $\sigma(t)$, window function $W(t)$, and input function that are commonly seen in experimental work will be assumed

[Eq. (13)]. This will make it possible to simplify $E\{r(t)\}$ [Eq. (8)] and $\text{VAR}\{r(t)\}$ [Eq. (9)] to the point where trends can be surmised. This will result in a better understanding of the above equations and show the results for commonly seen experimental parameters. If any of these assumptions are violated, one can always return to the original equations from Sec. II.

As others have noted, for high-order multiple-scattering events, the envelope of the scattered signal, which is proportional to the standard deviation, often decays exponentially^{4,5,37} [e.g., Fig. 1(a)]. Therefore it will be assumed that $\sigma(t)=u(t)e^{-\alpha t}$, where $u(t)$ is the Heaviside function. Next, a rect-window will be used for windowing the time-reversed signal. It will also be assumed that the duration of $g(t)$, t_g , is small compared to the envelope decay time constant ($\tau_\sigma=1/\alpha$). While there is interest in choosing long input functions, this assumption is necessary to simplify the expectation value and variance. As will be seen, as the assumption $t_g \ll \tau_\sigma$ is initially violated (i.e., $t_g < \tau_\sigma$ holds but $t_g \ll \tau_\sigma$ does not hold), the trends found for the expectation value and variance below will still hold approximately, though the exact equations will not. This is because this condition is merely used to assure that the envelopes of functions do not change over time-intervals specified below. As $\sigma(t)$ does change, it is slow and smooth so the impact is not dramatic. Of course when $t_g \ll \tau_\sigma$ is strongly violated (i.e., $t_g \geq \tau_\sigma$), one must return to the equations derived in Sec. II. Finally, it will be assumed that the duration of the acousto-electric IR $\rho(t)$, t_ρ , is small compared to the t_g . Summarizing these assumptions,

$$\sigma(t) = u(t)e^{-\alpha t}, \quad (13a)$$

$$W(t) = \begin{cases} 1 & \text{if } t_{\text{on}} \leq t \leq t_{\text{off}} \\ 0 & \text{otherwise,} \end{cases} \quad (13b)$$

$$t_\rho \ll t_g \ll \frac{1}{\alpha} = \tau_\sigma. \quad (13c)$$

A. Expectation value

Recalling the expectation value [Eq. (8)],

$$E\{r(t)\} \approx \frac{1}{M} g(-t) \otimes_i [\rho(T-t)((W(T-t)\sigma(T-t)) \otimes \sigma(t))].$$

Applying the assumptions outlined in Eq. (3) and beginning with the innermost portion,

$$\begin{aligned} & (W(T-t) \cdot \sigma(T-t)) \otimes \sigma(t) \\ &= \int_{-\infty}^{\infty} W(T-\tau) \sigma(T-\tau) \sigma(t-\tau) d\tau \end{aligned} \quad (14)$$

$$\text{VAR}_{\text{coherent}} \approx \frac{\kappa_1}{4\alpha M^2} \begin{cases} 0 & \text{if } t < T - t_g \text{ or } T < t \\ e^{4\alpha(T-t)} e^{-4\alpha t_{\text{on}}} (e^{-4\alpha(T-t_{\text{on}}-t)} - e^{-4\alpha\Delta t}) & \text{if } T - t_g \leq t \leq T - t_{\text{on}} \\ e^{4\alpha(T-t)} e^{-4\alpha t_{\text{on}}} (1 - e^{-4\alpha\Delta t}) & \text{if } T - t_{\text{on}} \leq t < T, \end{cases} \quad (20)$$

$$= \int_{T-t_{\text{off}}}^{T-t_{\text{on}}} u(T-\tau) e^{-\alpha(T-\tau)} u(t-\tau) e^{-\alpha(t-\tau)} d\tau \quad (15)$$

$$= \frac{e^{-\alpha t} e^{\alpha T}}{2\alpha} e^{-2\alpha t_{\text{on}}} (1 - e^{-2\alpha\Delta t}), \quad (16)$$

where $\Delta t = t_{\text{off}} - t_{\text{on}}$ is the window width. Equation (16) is then multiplied by $\rho(T-t)$, which is non-zero only for $t \approx T$. Additionally, since Eq. (16) is a slowly changing function (on the order of $1/\alpha$), the following approximation can be made:

$$\begin{aligned} & \rho(T-t)((W(T-t) \cdot \sigma(T-t)) \otimes \sigma(t)) \\ & \approx \delta(T-t) \left(\frac{e^{-\alpha t} e^{\alpha T}}{2\alpha} e^{-2\alpha t_{\text{on}}} (1 - e^{-2\alpha\Delta t}) \right) \end{aligned} \quad (17)$$

$$= \frac{1}{2\alpha} e^{-2\alpha t_{\text{on}}} (1 - e^{-2\alpha\Delta t}) \delta(T-t). \quad (18)$$

The application of the assumptions [Eq. (13)] to the expectation value [Eq. (8)] is³³

$$E\{r(t)\} \approx \frac{1}{2M\alpha} e^{-2\alpha t_{\text{on}}} (1 - e^{-2\alpha\Delta t}) \cdot g(T-t). \quad (19)$$

First, it is seen that the expectation value is a scaled version of the input function and does not increase in amplitude as t_g increases (as will be the case for the variance). Second, as the window shifts to later times (i.e., t_{on} increases), the peak amplitude of the expectation value drops off exponentially. This is a result of the following. First, the waveform transmitted into the system, $(1/M)W(T-t)(g(T-t) \otimes h(-t))$, is always the same magnitude since it is normalized by M . It is also always the same shape since the exponential decay function is self-similar. Second, a signal is not obtained until this waveform is correlated with the portion of $h(t)$ from which it came, the magnitude of this portion being $e^{-\alpha t_{\text{on}}}$. Therefore, one would expect the convolution of $(1/M)W(T-t)(g(T-t) \otimes h(-t))$ and $h(t)$ to scale as $(1/M)e^{-\alpha t_{\text{on}}}$. As the window width increases, the peak amplitude of the expectation value grows. This being due to the fact that more signal is included in the pulse-compression that occurs during time-reversal. Since the IR falls off exponentially, the contribution naturally saturates. Finally, as the decay time increases, the peak increases due to more energy being included for a given window width. These dependences can be seen in Fig. 3.

B. Variance

1. Coherent variance

The simplification of the coherent variance based on the assumptions in Eq. (13) for $t_g > t_{\text{on}}$ is³³

$$\begin{aligned} & \text{if } t < T - t_g \text{ or } T < t \\ & \text{if } T - t_g \leq t \leq T - t_{\text{on}} \\ & \text{if } T - t_{\text{on}} \leq t < T, \end{aligned} \quad (20)$$

FIG. 3. The dependence of the expectation value on (a) window width Δt and placement t_{on} (choosing a decay time of $t_{\sigma}=1500 \mu\text{s}$), (b) window width Δt and decay constant t_{σ} (choosing a window placement of $t_{\text{on}}=1000 \mu\text{s}$), and (c) window placement t_{on} and decay constant t_{σ} (choosing a window width of $\Delta t=1000 \mu\text{s}$).

where $\kappa_1 = \int_{-t_g/2}^{t_g/2} g(T-(t+\tau))g(T-(t-\tau))d\tau$. If the duration of the input function is less than the window turn on time ($t_g \leq t_{\text{on}}$), then³³

$$\text{VAR}_{\text{coherent}} \approx \frac{\kappa_1}{4\alpha M^2} \begin{cases} 0 & \text{if } t < T - t_g \text{ or } T < t \\ e^{4\alpha(T-t)} e^{-4\alpha t_{\text{on}}} (1 - e^{-4\alpha \Delta t}) & \text{if } T - t_g \leq t < T. \end{cases} \quad (21)$$

From this result, it can clearly be seen that the coherent variance term only contributes at the main lobe. It is also seen that the amplitude saturates as the window width grows for the same reason as the expectation value, though the exact shape of this saturation varies depending on whether t_g or t_{on} is larger and is not the same as the expectation value. The amplitude also increases linearly with the decay constant $\tau_{\sigma}=1/\alpha$. The square root of the coherent variance (i.e., the coherent standard deviation) decays similarly to $E\{r(t)\}$ as t_{on} increases. Finally, it is important to note that the coherent variance scales with κ_1 , which increases as the pulse duration lengthens, independent of bandwidth.

2. Incoherent variance

The simplification of the incoherent variance based on the assumptions in Eq. (13) is³³

$$\text{VAR}_{\text{incoherent}} \approx \frac{\kappa_2(0)}{4\alpha M^2} e^{2\alpha(T-2t_{\text{on}}-t)} \begin{cases} (e^{-4\alpha(T-t_{\text{on}}-t)} - e^{-4\alpha \Delta t}) & \text{if } t < T - t_{\text{on}} \\ (1 - e^{-4\alpha \Delta t}) & \text{if } t \geq T - t_{\text{on}}, \end{cases} \quad (22)$$

where $\kappa_2(0)$ is approximated by the pulse-intensity integral (PII). Thus the incoherent variance scales with the PII. Noting that M will go as the maximum of $\sigma(t)$, which is approximately $e^{-\alpha t_{\text{on}}}$, it can be seen that the location of the window (i.e., t_{on}) does not change the peak magnitude of the incoherent variance term but rather shifts where it occurs. Specifically, as t_{on} increases, the time of the peak shifts in a linear manner. This contrasts with the expectation value and coherent variance, which do not shift in time as t_{on} changes but rather stay at the same location and decrease exponentially in amplitude. In general for $t < T - t_{\text{on}}$ the incoherent variance increases with t as a result of more energy being

transmitted into the system as described earlier. It then peaks at $t=T-t_{\text{on}}$ and decays exponentially for larger t since no additional energy is being transmitted into the system at this point. It is also clear that the incoherent variance grows as both the window width and decay constant increase in the same manner as the coherent variance.

3. Total variance

The total variance can now be obtained from

$$\text{VAR} = \text{VAR}_{\text{coherent}} + \text{VAR}_{\text{incoherent}}. \quad (23)$$

When $t_{\text{on}} < t_g$,

$$\text{VAR} \approx \frac{1}{4\alpha M^2} \begin{cases} \kappa_2(0) e^{2\alpha(T-2t_{\text{on}}-t)} (e^{-4\alpha(T-t_{\text{on}}-t)} - e^{-4\alpha \Delta t}) & \text{if } t \leq T - t_g \\ (\kappa_2(0) + \kappa_1 e^{2\alpha(T-t)}) e^{2\alpha(T-2t_{\text{on}}-t)} (e^{-4\alpha(T-t_{\text{on}}-t)} - e^{-4\alpha \Delta t}) & \text{if } T - t_g \leq t \leq T - t_{\text{on}} \\ (\kappa_2(0) + \kappa_1 e^{2\alpha(T-t)}) e^{2\alpha(T-2t_{\text{on}}-t)} (1 - e^{-4\alpha \Delta t}) & \text{if } T - t_{\text{on}} \leq t < T \\ \kappa_2(0) e^{2\alpha(T-2t_{\text{on}}-t)} (1 - e^{-4\alpha \Delta t}) & \text{if } T < t \end{cases} \quad (24)$$

and when $t_g \leq t_{\text{on}}$,

$$\text{VAR} \approx \frac{1}{4\alpha M^2} \begin{cases} \kappa_2(0)e^{2\alpha(T-2t_{\text{on}}-t)}(e^{-4\alpha(T-t_{\text{on}}-t)} - e^{-4\alpha\Delta t}) & \text{if } t < T - t_{\text{on}} \\ \kappa_2(0)e^{2\alpha(T-2t_{\text{on}}-t)}(1 - e^{-4\alpha\Delta t}) & \text{if } T - t_{\text{on}} \leq t < T - t_g \\ (\kappa_2(0) + \kappa_1 e^{2\alpha(T-t)})e^{2\alpha(T-2t_{\text{on}}-t)}(1 - e^{-4\alpha\Delta t}) & \text{if } T - t_g \leq t < T \\ \kappa_2(0)e^{2\alpha(T-2t_{\text{on}}-t)}(1 - e^{-4\alpha\Delta t}) & \text{if } T < t. \end{cases} \quad (25)$$

Figure 4 shows when each of the conditions in the piecewise function contributes. The relative magnitude of the coherent and incoherent terms is determined by $(\kappa_2(0) + \kappa_1 e^{4\alpha(T-t)})$. When $T - \tau_\sigma/4 \ln(\kappa_2(0)/\kappa_1) < t$ the incoherent term is larger. Recalling that there will only be a contribution from coherent term for $T - t_g < t$ and that $t_g \ll t_\sigma$, the condition $T - \tau_\sigma/4 \ln(\kappa_2(0)/\kappa_1) < t$ will always be satisfied. Thus the contribution from the incoherent term will always be larger by approximately $\kappa_2(0)/\max_{\text{time}}\{\kappa_1\}$. For tone-bursts greater than 5-cycles this ratio is approximately 4. The change in the total variance as a function of the decay constant, window width, window placement, and length of a tone-burst input function can be seen in Fig. 5. The broad portion of the plot being due to the incoherent variance and the sharp peak around $3000 \mu\text{s}$ being the coherent variance. The depen-

dence of both the coherent and incoherent variances on t_g is of particular note since it increases rapidly, independent of the input signal's bandwidth.

C. SNR

The SNR is defined as the ratio of the main lobe peak-to-peak amplitude to the side lobe standard deviation. Outside the main lobe only the incoherent term contributes.

$$\text{SNR} = \frac{\max\{E\{r(t)\}\} - \min\{E\{r(t)\}\}}{\sqrt{\text{VAR}_{\text{incoherent}}\{r(t)\}}}. \quad (26)$$

Based on the nature of the directivity pattern, the above equation describes both the SNR as a function of time at the focus and SNR associated with the directivity pattern. Applying Eq. (13) and simplifying,

$$\text{SNR}(t) = \frac{(\max\{g(T-t)\} - \min\{g(T-t)\})}{\sqrt{\alpha\kappa_2(0)}e^{\alpha(T-t)}} \begin{cases} \frac{1 - e^{-2\alpha\Delta t}}{\sqrt{(e^{-4\alpha(T-t_{\text{on}}-t)} - e^{-4\alpha\Delta t})}} & \text{if } t < T - t_{\text{on}} \\ \frac{1 - e^{-2\alpha\Delta t}}{\sqrt{(1 - e^{-4\alpha\Delta t})}} & \text{if } t \geq T - t_{\text{on}}. \end{cases} \quad (27)$$

As a function of time, it is seen that the SNR has a minimum at $t = T - t_{\text{on}}$ when the functional dependence on t switches from $e^{-4\alpha(T-t_{\text{on}}-t)}$ to 1. Thus, it should be noted that this minimum shifts with the window placement, while the center of the main lobe is always at $t = T$ (i.e., it does not change as the window placement changes). The window width also impacts the SNR. Both $(1 - e^{-2\alpha\Delta t})$ and $\sqrt{(1 - e^{-4\alpha\Delta t})}$ saturate to 1 as Δt grows, but the latter saturates more quickly. Since it is the denominator, the SNR will monotonically increase as the window width increases. Finally, it is seen that the SNR is inversely proportional to the $\sqrt{\kappa_2(0)} \approx \sqrt{\text{PII}}$. The PII increases as the length of a pulse grows. Thus the SNR will decrease as the pulse length increases. It is important to note that the above SNR equation was derived for an arbitrary input function $g(t)$ with the only constraint on $g(t)$ being that $t_g \ll \tau_\sigma$. Thus this result is entirely independent of bandwidth. It has previously been seen

that time-reversal requires a broadband signal; otherwise it becomes simple monochromatic phase conjugation. Good time-reversal focusing still requires broadband signals, but this is only a necessary and not a sufficient condition for good focusing. The result above shows that the PII (and thus pulse length) must also be small. The presence of $\kappa_2(0)$ comes from the incoherent variance term. It results from the fact that as the PII increases, $g(t) \otimes h(t)$ increases. Physically this can be associated with the long-range correlations created by the longer input function in the scattered signal. Previous work indicated that these long-range correlations come from the multiple-scattering medium; however, it is now evident that they may also arise due to the input signal.

Two time points are of particular interest, the minimum SNR ($t = T - t_{\text{on}}$) and the SNR near the main lobe ($t \approx T$). For $t = T - t_{\text{on}}$, the SNR becomes

$$\text{SNR}_{\text{min}} = \frac{1}{\sqrt{\alpha\kappa_2(0)}} \frac{1 - e^{-2\alpha\Delta t}}{\sqrt{(1 - e^{-4\alpha\Delta t})}} \frac{(\max\{g(T-t)\} - \min\{g(T-t)\})}{e^{\alpha t_{\text{on}}}}. \quad (28)$$

FIG. 4. The total variance plotted with each portion of the piecewise function shown with a different line style/color.

The directivity patterns can be defined by plotting the maximum signal over time at a particular location. $\text{SNR}_{\min}$ does this. Therefore Eq. (28) can also be used to describe how the SNR of the directivity pattern will change. Figure 6 shows the dependence of the minimum SNR on the decay constant, window width, window placement, and duration of $g(t)$ for a tone-burst input. Note that a negative SNR, on the decibel scale used, implies that the magnitude of the noise is greater than the signal itself. Previously it was mentioned that both the expectation value and the variance increase with τ_σ and Δt . Figure 6 shows, however, that the expecta-

tion value must increase faster since the SNR increases. Not surprisingly the SNR decreases with pulse length (independent of bandwidth) since the expectation value has no amplitude dependence on t_g but the variance does. Finally, it is seen that $\text{SNR}_{\min}$ decreases as t_{on} shifts. This is expected since the maximum of the incoherent variance does not change with t_{on} , but the expectation value decreases.

The SNR near the main lobe can be approximated from Eq. (27) evaluated at $t=T$ rather than $t=T \pm t_g/2$ because the incoherent variance is a slowly changing function on the time scale of t_g . In this case

$$\text{SNR}_{\text{near ML}} = \frac{1}{\sqrt{\alpha \kappa_2(0)}} \frac{1 - e^{-2\alpha \Delta t}}{\sqrt{(1 - e^{-4\alpha \Delta t})}} (\max\{g(T-t)\} - \min\{g(T-t)\}). \quad (29)$$

The trends are also shown in Fig. 6. It is seen that the SNR near the main lobe is very similar to $\text{SNR}_{\min}$, except that it has no dependence on the window placement.

Minimizing the ratio of the standard deviation to the expectation value [the coefficient of variance (CV)] at the main lobe is desirable so the total amplitude at the focus is predictable. Recalling the assumptions [Eq. (13)],

$$\text{CV} = \sqrt{\alpha} e^{\alpha(T-t)} \times \frac{\sqrt{\kappa_2(0)(1 + e^{-2\alpha \Delta t}) + \kappa_1 e^{2\alpha(T-t)}}}{\sqrt{(1 - e^{-2\alpha \Delta t})} \cdot g(T-t)} \bigg|_{t \in [T-t_g, T]}. \quad (30)$$

Recalling that

$$\kappa_1 = \int_{-t_g/2}^{t_g/2} g(T - (t + \tau)) g(T - (t - \tau)) d\tau,$$

$$\kappa_2(0) \approx \int_0^{t_g} g^2(\tau) d\tau,$$

$$\text{CV} \approx \sqrt{\alpha} e^{\alpha(T-t)} \frac{\sqrt{(1 + e^{-2\alpha \Delta t}) \int_0^{t_g} g^2(\tau) d\tau + e^{2\alpha(T-t)} \int_{-t_g/2}^{t_g/2} g(T - (t + \tau)) g(T - (t - \tau)) d\tau}}{\sqrt{(1 - e^{-2\alpha \Delta t})} \cdot g(T-t)} \bigg|_{t \in [T-t_g, T]}. \quad (31)$$

Both terms in the numerator increase as the total “on-time” and amplitude of the input $g(t)$ grow, whereas the denominator only increases with the amplitude. As the window shifts to later times, the second term in the denominator grows, increasing the CV. This originates from the fact that as the window shifts to later times, the expectation value decreases. Finally, for small window widths, the numerator remains finite, while the denominator goes to zero. For large windows, all the terms with Δt go to zero, and thus no longer contribute. Therefore the CV gets very large for small window widths and decreases to a saturation value as the win-

dow width grows. The rate of this saturation depends on α . Also note that the CV decreases as the decay time constant ($1/\alpha$) increases and that the window placement has no impact on the CV. These results are in Fig. 6 where the maximum CV (over time) is plotted.

IV. CONCLUSIONS

TRA has been a highly-successful method of focusing sound through high-order multiple-scattering media and has found application in many fields.^{20-24,26-31} An initial shot

FIG. 5. The total variance is shown as a function of time and how this dependence changes with (a) the decay time constant τ_σ , (b) the window width Δt , (c) the window placement t_{on} , and (d) the duration of tone-burst input t_g (using $\tau_\sigma=1500 \mu s$, $t_{on}=1000 \mu s$, $\Delta t=1000 \mu s$, and $t_g=20$ cycles as the respective constants).

noise model has been proposed by Derode *et al.*⁴ to describe the expected signal and noise of this process. This model has been extended so that it applies to arbitrary input signals and windowing. The equations resulting from the extended model are novel and they confirmed previous predictions and also provided new predictions and explanations. This includes an explanation of the origin of the noise observed in multiple-scattering time-reversal (coherent versus incoherent contributions). Additionally it predicted that windowing can cause the peak of the noise to occur far from the main lobe and that the SNR depends on the PII of the original input signal.

The relatively complex results of Sec. II were then simplified for a set of parameters commonly found in experimental work. This in turn has allowed many trends to be identified. In particular, increasing the length of the input function degrades the SNR, independent of bandwidth. This is the result of the expectation value not depending on t_g , but the variance increasing with t_g due to long-range correlations in the scattered signal $g(t) \otimes h(t)$ due to $g(t)$. Additionally, it is seen that time-reversing later windows of a recorded signal does not affect the SNR near the main lobe, but it does reduce the main lobe to side lobe ratio for side lobes far from the main lobe [Fig. 2(d)]. Many of these results are also

FIG. 6. The dependence of the SNR_{min} (dashed line), $SNR_{near ML}$ (dash-dot line), and CV_{max} (solid line) on (a) the decay time constant, (b) the window width, (c) the window placement, and (d) the duration of a tone-burst input (using $\tau_\sigma=1500 \mu s$, $t_{on}=1000 \mu s$, $\Delta t=1000 \mu s$, and $t_g=20$ cycles as the respective constants).

qualitatively similar to those seen for 1-bit time-reversal and an extension of this model to 1-bit time-reversal would be interesting but beyond the scope of this article.

Further analysis, specific to a particular application, can be done with these equations. This could include trying to maximize the SNR and minimize the CV for short pulses used in imaging. Alternatively, one might be interested in determining specific configurations that maximize the SNR and amplitude for long pulses that are used in thermal high-intensity focused ultrasound or acoustic radiation force experiments among other applications. Determining signals that maximize the amplitude for short pulses¹⁹ may be useful for histotripsy.³⁸ One approach for imaging may be to use coded pulses, which would effectively doubly encode the signal, first with coded pulse and second with the reverberations in the multiple-scattering media.

In addition to looking at how the input function impacts the time-reversal focused signal, it is possible to do time-reversal using multiple channels (i.e., multiple transmitter-receiver pairs). It has been shown that if multiple pairs are used to transmit the same information, the SNR increases.³⁹ Recognizing that the main lobe will add coherently, its amplitude will increase proportional to the number of channels, the side lobes, however, will add incoherently (based on the physical interpretation provided) and thus will only go as the square root of the number of channels. This has been verified by Derode *et al.*⁵

In addition, it has been proposed that if multiple transmitters and receivers are used simultaneously, one can increase the bit-rate of sending information.⁴⁰ This is because the multiple-scattering medium makes each set of paths from receiver to transducer unique and independent (to first approximation, ignoring weak localization effects, recurrent scattering, correlated scatters, etc.). While it may be possible to increase the bit-rate, it is important to verify that the amount of noise (i.e., variance in the signal) does not dominate the signal (i.e., expectation value) under the conditions being used. The work of this paper provides a method for estimating these parameters based on its derivation of signal and noise of transmitting arbitrary pulses through a multiple-scattering medium. In particular it is found that the physical explanation of the incoherent (uncorrelated) noise term will be the same for each transmitter-receiver pair and will add as uncorrelated noise does, thus increasing the noise-floor for all channels as the square of the rms of the signal.

ACKNOWLEDGMENTS

The authors would like to thank Dr. Matthew Schipper and Dr. Albert Levine for discussions on statistics. The authors would also like to thank the reviewers for their helpful and thought provoking comments. This work was supported in part by NIH Grant Nos. 5R01EB000281 and 1R21CA116043.

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