Published in final edited form as:

Opt Lett. 2006 June 15; 31(12): 1845–1847.

Wavefront sensing and reconstruction from gradient and Laplacian data measured with a Hartmann-Shack wavefront sensor

Sergio Barbero,

School of Optometry, Indiana University, Bloomington IN 47405, USA

Jacob Rubinstein, and

Department of Mathematics, Indiana University, Bloomington IN 47405, USA

Larry N. Thibos

School of Optometry, Indiana University, Bloomington IN 47405, USA

Abstract

A new wavefront sensing and reconstruction technique is presented. It is possible to measure Laplacian and gradient information of a wavefront with a Hartmann-shack set-up. Using simultaneously the Laplacian and gradient data, the wavefront is reconstructed by solving sequentially two partial differential equations.

Wavefront sensing and reconstruction is a wide area in optics with many applications, including adaptive optics technology,¹ phase imaging² and visual optics.³ Wavefront sensing is the physical technique used to obtain information of the wavefront, typically in the form of gradient data (Hartmann-Shack sensor (H-S), shear interferometry, etc.) or curvature (Laplacian) data (e.g. curvature sensors). Wavefront reconstruction is the mathematical technique to reconstruct the wavefront surface from the data collected in wavefront sensing. In this letter we present two novel ideas, one in the field of wavefront sensing and the other one in wavefront reconstruction. We propose a wavefront sensor that measures gradient *and* Laplacian data simultaneously with a single experimental set-up (H-S). In addition, we derive the necessary mathematical formulation to reconstruct the wavefront surface from both data simultaneously, rather than using gradient or Laplacian data independently.

There is a common basis which relates gradient and curvature wavefront sensors. The phase and the intensity of a wave propagating in a homogeneous medium are related by a partial differential equation derived by I. Runge and A Sommerfeld in 1911.³ Gradient and curvature sensors assume that the wavefront obeys the paraxial approximation of this equation, the so-called Transport-of-Intensity-Equation (TIE). This way, TIE is the basic tool in curvature sensing following the suggestion of Teague⁴ and others. The link between TIE and gradient sensors (specifically H-S sensors) was shown by Bara.⁵ This common theoretical framework for Laplacian and gradient measurements suggests the possibility of using a H-S set up to measure not only gradient data but also Laplacian data.

Figure 1 shows the procedure to measure combined gradient and Laplacian data. Two H-S images of an incoming wavefront are captured at two axial locations. This can be done by moving the H-S sensor axially or using a beam splitter. The intensity difference between corresponding spots in the two images is used to approximate the partial derivative of the intensity with respect to the z-axis (optical axis). In most practical applications it is reasonable to assume that the spatial intensity distribution is nearly constant over the surface of a microlens in the H-S array. This assumption is used to simplify the paraxial TIE to:

$$\Delta u = - \frac{\partial I / \partial z}{I} \quad (1)$$

Here u and I are the phase and intensity respectively, and the Laplacian operator (Δ) is considered over a plane perpendicular to the axis of propagation. The left hand side of equation (1) quantifies the curvature of the wavefront and the right hand side is the ratio between the derivative of the intensity with respect to z (evaluated as spot energy difference) and the total intensity —computed as the energy of each spot. Such a procedure is expected to be robust with respect to the important problem of photo-detection noise in curvature sensing⁶ because the intensity is integrated over the pixels covering each spot. In a recent paper Paterson et al⁷ suggested to use a H-S sensor with cylindrical lenses to measure Laplacian data, although they did not propose an algorithm for reconstructing the wavefront from such data.

In practice, the experimentally measured gradient and Laplacian data will be affected by noise. Therefore, one should seek an optimal estimate of the wavefront. For this purpose we estimate the phase function u by minimizing the least squares functional

$$\min K(u) \equiv \min \iint_D \left(w |\nabla u - \vec{f}|^2 + |\Delta u - g|^2 \right) dx dy. \quad (2)$$

Here $\vec{f}$ and g are the experimentally measured gradient and Laplacian of u , respectively, and D is the domain where u is defined. Finally, w is a weight that balances the relative importance of the gradient data and the Laplacian data. For simplicity it is assumed here that w is constant. The variational problem expressed in equation (2) can be solved by computing the associated Euler-Lagrange equation:

$$w(\nabla \cdot \vec{f} - \Delta u) + (\Delta^2 u - \Delta g) = 0. \quad (3)$$

Equation (3) is supplemented by two natural boundary conditions:

$$w \partial_n u - \partial_n (\Delta u) = w \vec{f} \cdot \hat{n} - \partial_n g \quad \Delta u = g \quad (4)$$

where $\hat{n}$ is the normal to the boundary of D , and ∂_n denotes partial differentiation in the $\hat{n}$ direction. Solving the PDE (3) together with the boundary conditions (4) provides the optimal phase estimate u . Differential equations of fourth order such as equation (3) are hard to solve numerically. Fortunately u appears in the equation only under the Laplace operator Δ and under the biharmonic operator Δ^2 . This feature, together with the special form of the boundary conditions enables us to split the biharmonic problem into two second order PDEs that are much easier to solve. Thus, we define $v = \Delta u$ and write:

$$\Delta v - wv = \Delta g - w \nabla \cdot \vec{f} \quad (5)$$

$$\begin{aligned}
 &\text{Boundary condition: } v=g \\
 &\Delta u=v \\
 &\text{Boundary condition: } w\partial_n u = w\vec{f} \cdot \vec{n} - \partial_n g + \partial_n v
 \end{aligned} \tag{6}$$

Notice that equation (5) does not depend on u . Thus, the single biharmonic equation was split into two second order PDEs that can be solved sequentially.

In practice, the final PDEs in (5) and (6) are solved using numerical techniques, such as the Finite Difference Method (FDM) or the Finite Element Method (FEM). We adopt a FEM solver mainly because for typical circular samplings of the wavefront the mesh generated in FEM is more adequate for points close to the edge than in the FDM.

Briefly, the FEM technique consists of discretizing the PDE, i.e. approximating the continuous solution by a discrete solution in a set of points defined by a mesh. The continuous problem is replaced by a system of linear equations defined with respect to the mesh. In the simulations done for this paper we used the mesh generator and linear algebra code available in Matlab Partial Differential Equation toolbox (version 7.0).

We simulated an asymmetric wavefront $u=0.2Z_4^0+0.1Z_3^{-1}$. Here Z_m^n are Zernike polynomials and the units are microns. The wavefront is defined over a circular aperture of radius 3 mm, where we used 6000 points in the FEM mesh. Gaussian random noise was introduced in the gradient ($\sigma=0.06$) and Laplacian data of the wavefront. We simulated three different amounts of noise in the Laplacian: noise level A ($\sigma=0.07\mu m^{-1}$), noise level B ($\sigma=0.05\mu m^{-1}$) and noise level C ($\sigma=0.03\mu m^{-1}$). Finally we use 5 different values for the weight w (0.01, 0.1, 1, 10, 100) to study the relative contribution of gradient and Laplacian data in the accuracy of the reconstruction.

Figure 2 shows the results for this simulation. The optimal weight depends on the relative noise levels of the gradient and Laplacian data, which obviously depends on the specific setup and shape of the wavefront measured. It is observed that when the weight w is very small, the solver is using mainly the Laplacian data, and the RMS errors are ranked by the induced noise. As the weight w increases, the behavior in the three cases is not the same. For noise level A, the Laplacian data is too noisy, and in this case one clearly benefits from increasing w , at least up to some value. On the other hand, the RMS error for noise level C always increases with w , since in this case the noise in the Laplacian data is less important than the noise in the gradient data. When w gets too large, though, the RMS errors increase for all three noise levels. We suspect that this happens because the large w limit is a singular limit for the differential equation (3).

A detailed description of the numerical method, together with extensive simulations and experimental data will be give by us in a forthcoming publication.

Finally it is worth noting that in the special situation of periodic wavefronts, inside the domain, it is possible to use a fast Fourier solver. Applying the Discrete Fourier Transform directly to the PDE (3) and performing some analytical computations, the Fourier coefficients of the solution ($U(l, m)$) are found to satisfy:

$$U(l, m) = \frac{wk^2G(l^2+m^2) - k(lF_x+mF_y)}{wk^4(l^2+m^2)^2 - k^2(l^2+m^2)} \tag{7}$$

where G are the discrete Fourier coefficients of g (Laplacian data), F_x and F_y of f_x and f_y respectively, and k is $2\pi i$. The reconstructed wavefront is obtained from this Fourier coefficients using the inverse Fourier transform.

References

1. Tyson, RK. Principles of adaptive optics. Academic Press; 1998.
2. Nugent KA, Paganin D, Gureyev TE. A phase odyssey. *Physics Today* 2001;54(8):27–32.
3. Atchison, DA.; Smith, G. Optics of the human eye. Butterworth-Heinemann; 2000.
4. Sommerfeld A, Runge JI. Anwendung der Vektorrechnung auf die Grundlagen der geometrischen. *Optik Annalen der Physik* 1911;35:277–298.
5. Teague MR. Deterministic Phase Retrieval - A Green-Function Solution. *J Opt Soc Am A* 1983;73(11):1434–1441.
6. Bara S. Measuring eye aberrations with Hartmann-Shack wave-front sensors: Should the irradiance distribution across the eye pupil be taken into account? *J Opt Soc Am A* 2003;20(12):2237–2245.
7. Barbero S, Thibos LN. Error analysis and correction in wavefront reconstruction from Transport-of-Intensity-Equation Optical Engineering. *Optical Engineering* 45:94001–94006.
8. Paterson C, Dainty JC. Hybrid curvature and gradient wave-front sensor. *Optics Letters* 2000;25(23):1687–1689. [PubMed: 18066314]

Figure 1.

Two H-S images (a)–(b) are taken in two axial locations in the optical axis. The energy contained in each spot of the H-S images is computed to generate two spot energy maps (c)–(d). The difference between the energy maps (d) and (c) is the energy map (e) necessary to compute equation (1).

Figure 2.

Error in the wavefront reconstruction —Root mean square error (RMS)—for different level of simulated noise in the Laplacian and values of the weight factor w . The FEM solver used 6000 mesh points.