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Bayesian Analysis of Multivariate Latent Curve Models With Nonlinear Longitudinal Latent Effects

Xin-Yuan Song and **Sik-Yum Lee**

Department of Statistics, Chinese University of Hong Kong

Yih-Ing Hser

Integrated Substance Abuse Programs, University of California, Los Angeles

Abstract

In longitudinal studies, investigators often measure multiple variables at multiple time points and are interested in investigating individual differences in patterns of change on those variables. Furthermore, in behavioral, social, psychological, and medical research, investigators often deal with latent variables that cannot be observed directly and should be measured by 2 or more manifest variables. Longitudinal latent variables occur when the corresponding manifest variables are measured at multiple time points. Our primary interests are in studying the dynamic change of longitudinal latent variables and exploring the possible interactive effect among the latent variables.

Much of the existing research in longitudinal studies focuses on studying change in a single observed variable at different time points. In this article, we propose a novel latent curve model (LCM) for studying the dynamic change of multivariate manifest and latent variables and their linear and interaction relationships. The proposed LCM has the following useful features: First, it can handle multivariate variables for exploring the dynamic change of their relationships, whereas conventional LCMs usually consider change in a univariate variable. Second, it accommodates both first- and second-order latent variables and their interactions to explore how changes in latent attributes interact to produce a joint effect on the growth of an outcome variable. Third, it accommodates both continuous and ordered categorical data, and missing data.

Latent curve models (LCMs) are popular longitudinal techniques in analyzing individual differences in the patterns of change. The pattern of change usually involves random intercept and random slope (grouped as a latent growth factor), with each pair forming a different trajectory over time. Techniques of LCMs have been developed through the incorporation of many features of structural equation models (see Bollen & Curran, 2006; Meredith & Tisak, 1990). For instance, the random intercept and slope are considered as latent variables representing initial status and rate of change of the outcome variable, respectively. Indeed, structural equation modeling (SEM) techniques provide powerful tools for developing more useful LCMs for analyzing complex dynamic changes. For example, based on the product

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Correspondence should be addressed to Sik-Yum Lee, Department of Statistics, Chinese University of Hong Kong, Shatin, N.T., Hong Kong. sylee@sta.cuhk.edu.hk.

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indicator procedure of Jöreskog and Yang (1996), Li, Duncan, and Acock (2000) developed LCMs with interaction between latent growth factors for assessing how changes in latent attributes produce a joint effect in the growth of the outcome latent variables. See Blozis (2007), Liu (2007), Blozis and Cho (2007) for more recent developments of this model.

In this article, we use a comprehensive LCM and efficient statistical methods to better understand and predict the dynamic changes of manifest and latent variables. Here, we describe the important features of our proposed LCM via an empirical application to a longitudinal study about cocaine use. Cocaine use is a major social and health problem in many countries. The main interest is in studying the dynamic influences of various latent variables on the dynamic change in cocaine use. In the literature on cocaine use, many studies have shown that psychiatric problems have a substantial impact on cocaine use (Browne et al., 1998; Hser et al., 2006). Moreover, the treatment of a cocaine addict and family support are other important influences on cocaine use. To achieve a better understanding of the aforementioned issues, the UCLA Center for Advancing Longitudinal Drug Abuse Research collected various measures from patients admitted in 1989 to the West Los Angeles Veterans Affairs Medical Center who met the Diagnostic and Statistical Manual of Mental Disorders (*DSM-III-R*, American Psychiatric Association, 1989), criteria for cocaine dependence (Kasarabada, Anglin, Khalsa-Denison, & Paredes, 1999). These patients were assessed at baseline, 1 year after treatment, 2 years after treatment, and 12 years ($t = 1, 2, 3, 4$) after treatment in 2002. Collected measures at each time point include days of cocaine use per month (CC), times per month in formal treatment (TR), Beck Inventory (BI), depression (DEP), anxiety (AN), and family support (FS), where CC, TR, BI, DEP, and AN are continuous, and FS is an ordered categorical with a 3-point scale. As some patients were confirmed to be deceased, some declined to be interviewed, and some were out of the country, there is a considerable amount of missing data. To efficiently analyze the data, it is necessary to take into account the following features in developing LCMs and associated statistical methods.

1. It is natural to consider an LCM based on cocaine use measured at $t = 1, 2, 3, 4$ with a latent variable η_1 on initial status (intercept) and another latent variable η_2 on rate of change (slope) of cocaine use. Another LCM is developed on the basis of TR measured at $t = 1, 2, 3, 4$ with latent variables ξ_{11} and ξ_{21} on the initial status of TR and the rate of change of TR, respectively. The effects of $\{\xi_{11}, \xi_{21}\}$ on $\{\eta_1, \eta_2\}$ can be assessed by a simple structural equation. Moreover, we similarly consider one more LCM based on FS measured at $t = 1, 2, 3, 4$, and obtain the initial status of FS (ξ_{12}) and rate of change of FS (ξ_{22}) via growth curve modeling. The effects of $\{\xi_{12}, \xi_{22}\}$ together with $\{\xi_{11}, \xi_{21}\}$ on $\{\eta_1, \eta_2\}$ are assessed by incorporating them in a structural equation. Based on the same rationale, to address the effect of initial status and rate of change of DEP on cocaine use, we could first obtain these latent variables (intercept, slope) developed on the basis of the measures of DEP at each time point, and then assess the effects of these latent variables on $\{\eta_1, \eta_2\}$ by incorporating them in the structural equation (see Li et al., 2000). With the inclusion of BI and AN, however, our interest is not only on DEP, but also a latent variable, psychiatric problem (PP), that should be assessed by the three manifest variables BI, DEP, and AN. Hence, at each time point we first use a measurement equation to group BI, DEP, and AN into latent variable PP, then construct an LCM on the basis of PP at $t = 1, 2, 3, 4$ in obtaining the initial status of PP (ξ_{13}) and rate of change of PP (ξ_{23}). In this article, we call a latent variable obtained from several manifest variables a first-order latent variable (e.g., $\eta_1, \eta_2, \xi_{11}, \xi_{21}, \xi_{12}, \xi_{22}, PP$), and call a latent variable obtained from several latent variables (e.g., ξ_{13}, ξ_{23}) a second-order latent variable (see Jöreskog, 1970). Hence, a key feature of the proposed comprehensive LCM is the accommodation of the effects from both first-order and second-order latent variables on the growth factors of the outcome variables.

2. As pointed out by Li et al. (2000), it is important to include the interaction of latent variables in assessing the joint effect of the dynamic latent variables. Hence, the proposed LCM accommodates a nonlinear structural equation (see Lee & Song, 2003) that subsumes polynomials (including interaction and higher order terms) of the first-order and second-order latent variables.
3. The proposed LCM accommodates mixed continuous and ordered categorical data. This kind of data is very common in practice. As pointed out by Lee, Song, Skevington, and Hao (2005), most ordered categorical variables are skewed, and treating them as variables with a multivariate normal distribution is not appropriate (see also Lee, 2007; Shi & Lee, 2000).
4. The proposed LCM accommodates missing data, which are very common in longitudinal studies.
5. The statistical methods for analyzing the proposed LCM are developed with a Bayesian approach combined with appropriate Markov chain Monte Carlo (MCMC) techniques. It has been shown that this approach is better than the indicator approach in estimating the nonlinear terms (including interaction) of latent variables (see Lee, Song, & Poon, 2004). Moreover, the Bayesian approach has the following features: (a) It directly incorporates genuine prior knowledge in the analysis. In longitudinal studies, good prior information on the parameters in the whole models for all time points could be available from the preliminary analysis at the previous time points. (b) The sampling-based Bayesian methods do not rely on asymptotic theory and give reliable statistical inference even with small sample sizes (see Ansari & Jedidi, 2000; Dunson, 2000; Lee & Song, 2004). (c) The sampling-based Bayesian methods produce latent variable estimates with good statistical properties, reliable model comparison statistics, estimates of residuals, and so on.

The article is organized as follows. The comprehensive LCM is described in the following section, using the cocaine study as an illustrative example. A Bayesian approach coupled with the Gibbs sampler is discussed after that. Analysis of the cocaine study data by using the freely available WinBUGS software is presented next. A simulation study is then provided to demonstrate the methodology. We conclude with a discussion. Technical details are presented in the appendices.

MODEL AND DATA DESCRIPTIONS

Basic Latent Curve Models

The basic latent curve model can be viewed as the following common factor analysis model:

$$\mathbf{y} = \mathbf{\Lambda}\boldsymbol{\eta} + \boldsymbol{\epsilon}_y, \quad (1)$$

where $\mathbf{y}$ is a $T \times 1$ vector of repeated measures, $\mathbf{\Lambda}$ is a $T \times m$ parameter matrix of sequential known values of the growth curve records, $\boldsymbol{\eta}$ is an $m \times 1$ endogenous latent growth factor that contains scores on the m factors for a given individual, and $\boldsymbol{\epsilon}_y$ is a $T \times 1$ vector of residual errors. The pattern of $\mathbf{\Lambda}$ can be interpreted as representing a particular aspect of change in y across the T occasions. It is assumed that $\boldsymbol{\epsilon}_y$ follows the distribution $N[\mathbf{0}, \boldsymbol{\Psi}_y]$, where $\boldsymbol{\Psi}_y$ is a diagonal matrix with diagonal elements ψ_{yt} , for $t = 1, \dots, T$. When $m = 2$, Equation 1 is expressed in the following matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_T \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \epsilon_{y1} \\ \epsilon_{y2} \\ \vdots \\ \epsilon_{yT} \end{pmatrix} \quad (2)$$

The first column in $\mathbf{\Lambda}$ can be defined as an intercept factor, which represents an initial status of change in y . The second column in $\mathbf{\Lambda}$ represents known times of measurement (t_1 through t_T) and constraints (the values of t should reflect the spacing between measurement occasions), and the latent growth factor $\boldsymbol{\eta}$ contains the initial status η_1 (intercept) and rate of change η_2 (slope). The two-factor linear growth model is specified in Equation 2 so that the intercept factor (fixed at a constant value of 1) serves as the starting point (initial status) for any change (growth) across time and the slope factor captures the rate of change of the trajectory over time. The scaling of the slope can be specified by using either fixed value restrictions (e.g., 0, 1, . . . , $T - 1$) representing a straightline growth, or unspecified value restrictions (where $t_1 = 0$ and $t_2 = 1$ are fixed for model identification, and the remaining $t_3, \dots, t_T$ are freely estimated) allowing estimation of an optimal pattern of change over measurement occasions (Meredith & Tisak, 1990).

The equation for the latent growth factor $\boldsymbol{\eta}$ can be simply written as:

$$\boldsymbol{\eta} = \boldsymbol{\beta} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\delta}, \quad (3)$$

where $\boldsymbol{\beta}$ is an $m \times 1$ vector of the population average of the latent individual growth factors, $\mathbf{B}$ is an $m \times m$ matrix of coefficients expressing the structural relations between the $\boldsymbol{\eta}$ variables, and $\boldsymbol{\delta}$ is an $m \times 1$ vector of equation residuals.

To illustrate the model framework, we use a longitudinal study of cocaine use across four measurement occasions. Suppose that we wish to specify an LCM representing linear change on cocaine use. Let $\mathbf{y} = (y_1, y_2, y_3, y_4)'$, and $\mathbf{B} = \mathbf{0}$. Then Equations 2 and 3 can be expressed in the following form to represent the model in Figure 1:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 12 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \epsilon_{y1} \\ \epsilon_{y2} \\ \epsilon_{y3} \\ \epsilon_{y4} \end{pmatrix}, \quad (4)$$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \quad (5)$$

Note that the four measurement occasions are 1989 (baseline), 1990 (1 year after treatment), 1991 (2 years after treatment), and 2002 (12 years after treatment). Hence, the known times of measurement t_1, t_2, t_3 , and t_4 are set to be 0, 1, 2, and 12 to reflect the spacing between measurement occasions.

Latent Curve Models With Exogenous Latent Variables

The previous model can be extended to include the following exogenous latent growth factors:

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_r \end{pmatrix} = \begin{pmatrix} \Lambda & 0 & \dots & 0 \\ 0 & \Lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{pmatrix} + \begin{pmatrix} \epsilon_{x1} \\ \epsilon_{x2} \\ \vdots \\ \epsilon_{xr} \end{pmatrix}, \quad (6)$$

where 0 is a $T \times m$ matrix with elements 0. For $k = 1, \dots, r$, ξ_k is the k th latent growth factor that includes growth parameters (intercept, slope), $\mathbf{x}_k$ and ϵ_{xk} are similarly defined as $\mathbf{y}$ and ϵ_y , and Λ is similarly defined as in Equation 1. It is assumed that ϵ_{xk} follows the distribution $N[0, \Psi_{xk}]$, where Ψ_{xk} is a diagonal matrix with diagonal elements $\psi_{x1k}, \dots, \psi_{xTk}$. The latent growth factors ξ_k ($k = 1, \dots, r$) are the first-order exogenous latent variables. Note that generalization of Equation 6 to allow different Λ 's is straightforward.

In the cocaine study data, the outcome variable is cocaine use (CC), and η_1 and η_2 are the endogenous latent growth parameters where η_1 (intercept) specifies the initial status of individual growth and η_2 (slope) specifies the rate of change over the four measurement time points. Let $\mathbf{x}_1 = (x_{11}, x_{21}, x_{31}, x_{41})'$ be a vector of treatment measured at four time occasions, and $\xi_1 = (\xi_{11}, \xi_{21})'$ such that

$$\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 12 \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{21} \end{pmatrix} + \begin{pmatrix} \epsilon_{x1} \\ \epsilon_{x2} \\ \epsilon_{x3} \\ \epsilon_{x4} \end{pmatrix}. \quad (7)$$

The growth factor in relation to treatment, ξ_1 , can be regarded as an exogenous latent growth factor with intercept (ξ_{11}) and slope (ξ_{21}). The LCM with this exogenous latent growth factor can be expressed by Equations 4 and 7, and the following equation:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{21} \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \quad (8)$$

The path diagram of this LCM is depicted in Figure 2.

Latent Curve Models With Longitudinal Latent Variables

Latent variables are common in behavioral, social, psychological, and medical research. To incorporate longitudinal latent variables, we further consider the following measurement model:

$$\mathbf{u}_t = \Lambda_{ut} \omega_{ut} + \epsilon_{ut}, \quad t = 1, \dots, T \quad (9)$$

where $\mathbf{u}_t$ is the $p \times 1$ vector of measurements, Λ_{ut} is the $p \times q$ loading matrix, ω_{ut} is a $q \times 1$ vector of latent variables, and ϵ_{ut} is a $p \times 1$ vector of unique variances in $\mathbf{u}_t$. It is assumed that ϵ_{ut} follows the distribution $N[0, \Psi_{ut}]$, where Ψ_{ut} is a diagonal matrix with diagonal elements $\psi_{ut1}, \dots, \psi_{utp}$. Equation 9 can be expressed in the following form:

$$\begin{pmatrix} u_{t1} \\ u_{t2} \\ \vdots \\ u_{tp} \end{pmatrix} = \begin{pmatrix} \lambda_{ut,11} & \lambda_{ut,12} & \dots & \lambda_{ut,1q} \\ \lambda_{ut,21} & \lambda_{ut,22} & \dots & \lambda_{ut,2q} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{ut,p1} & \lambda_{ut,p2} & \dots & \lambda_{ut,pq} \end{pmatrix} \begin{pmatrix} \omega_{ut1} \\ \omega_{ut2} \\ \vdots \\ \omega_{utq} \end{pmatrix} + \begin{pmatrix} \epsilon_{ut1} \\ \epsilon_{ut2} \\ \vdots \\ \epsilon_{utp} \end{pmatrix}, \quad (10)$$

where some $\lambda_{ut,ij}$ are fixed at 0 or 1 for model identification. As the latent variables ω_{utk} , $k = 1, \dots, q$ defined in Equation 9 or 10 are related to manifest variables measured at multiple time points $t_1, \dots, t_T$, they are longitudinal. Let $\omega_k = (\omega_{u1k}, \dots, \omega_{uTk})'$, and $\epsilon_{wk} = (\epsilon_{w1k}, \dots, \epsilon_{wTk})'$, we can further use the following LCM to investigate the patterns of change in these longitudinal latent variables:

$$\omega_k = \Lambda \xi_{r+k} + \epsilon_{wk}, \quad k=1, \dots, q, \quad (11)$$

where $\xi_{r+k} = (\xi_{1,r+k}, \xi_{2,r+k})'$, in which $\xi_{1,r+k}$ and $\xi_{2,r+k}$ indicate the intercept and slope of latent factor ξ_{r+k} , r is specified in Equation 6, and Λ is similarly defined as in Equation 1. It is assumed that ϵ_{wk} follows the distribution $N[0, \Psi_{wk}]$, where Ψ_{wk} is a diagonal matrix with diagonal elements ψ_{wtk} , $t = 1, \dots, T$, and $k = 1, \dots, q$. Equation 11 has the form:

$$\begin{pmatrix} \omega_{u1k} \\ \omega_{u2k} \\ \vdots \\ \omega_{uTk} \end{pmatrix} = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_T \end{pmatrix} \begin{pmatrix} \xi_{1,r+k} \\ \xi_{2,r+k} \end{pmatrix} + \begin{pmatrix} \epsilon_{w1k} \\ \epsilon_{w2k} \\ \vdots \\ \epsilon_{wTk} \end{pmatrix}, \quad k=1, \dots, q. \quad (12)$$

In the cocaine study example, PP is a longitudinal latent variable, which is formed by three manifest variables at multiple time points: BI, DEP, and AN. The related measurement equation, which is a special case of Equation 10, is given by the following equation:

$$\begin{pmatrix} u_{t1} \\ u_{t2} \\ u_{t3} \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_{ut,21} \\ \lambda_{ut,31} \end{pmatrix} \omega_{ut1} + \begin{pmatrix} \epsilon_{ut1} \\ \epsilon_{ut2} \\ \epsilon_{ut3} \end{pmatrix}, \quad t=1, \dots, 4, \quad (13)$$

where 1s in Λ_{ut} are fixed to identify the factor analysis model. The LCM (see Equation 12) that describes the pattern of change in this longitudinal latent variable is defined as:

$$\begin{pmatrix} \omega_{u11} \\ \omega_{u21} \\ \omega_{u31} \\ \omega_{u41} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 12 \end{pmatrix} \begin{pmatrix} \xi_{13} \\ \xi_{23} \end{pmatrix} + \begin{pmatrix} \epsilon_{w11} \\ \epsilon_{w21} \\ \epsilon_{w31} \\ \epsilon_{w41} \end{pmatrix}, \quad (14)$$

where ξ_{13} and ξ_{23} can be regarded as the second-order latent variables. The path diagram in relation to Equations 13 and 14 is depicted in Figure 3.

In general, let $\xi = (\xi'_1, \dots, \xi'_r, \xi'_{r+1}, \dots, \xi'_{r+y})'$ be the vector of exogenous latent growth factors that are the first- and second-order latent variables. To assess the linear and nonlinear effects of the exogenous latent growth factors in ξ on the endogenous latent growth factor η , we further propose the following nonlinear structural equation:

$$\eta = \beta + \mathbf{B}\eta + \mathbf{\Gamma}\mathbf{P}(\xi) + \delta, \quad (15)$$

where $\mathbf{\Gamma}$ is a matrix of unknown regression coefficients, $\mathbf{F}(\xi) = (f_1(\xi), \dots, f_h(\xi))'$ is a vector-valued function containing nonzero differentiable functions $f_1, \dots, f_h$, and $h \geq q$, ξ is distributed as $N[0, \Phi]$, δ is a vector of residuals with distribution $N[0, \Psi_\eta]$, in which Ψ_η is a diagonal matrix with diagonal elements $\psi_{\eta,k}$, $k = 1, 2$, and ξ and δ are independent. Note that Φ contains variances and covariances of exogenous latent growth factors that describe the relationships

between the aspects of change represented by the various growth curve parameters for different response variables. These relationships represent the valuable information obtained by extending the univariate model to the multivariate case.

In analyzing cocaine study data, we are interested in the longitudinal effects of treatment (TR), family support (FS), and psychiatric problem (PP) on cocaine use (CC), where PP is a latent variable. Specifically, we are interested in studying how the dynamic changes of three exogenous latent growth factors, $\xi_1(\text{TR})$, $\xi_2(\text{FS})$, and $\xi_3(\text{PP})$, influence the endogenous latent factor $\eta(\text{CC})$. Here, in Equation 12, $r = 2$, $q = 1$, and each $\xi_k = (\xi_{k1}, \xi_{k2})'$ ($k = 1, 2, 3$) includes the intercept and slope that represent the initial status and rate of changes in TR, FS, and PP. An interesting model is described as follows and depicted in Figure 4:

$$\eta_1 = \beta_1 + \gamma_{11}\xi_{11} + \gamma_{12}\xi_{12} + \gamma_{13}\xi_{13} + \gamma_{14}\xi_{11}\xi_{12} + \gamma_{15}\xi_{11}\xi_{13} + \gamma_{16}\xi_{12}\xi_{13} + \delta_1, \quad (16)$$

$$\eta_2 = \beta_2 + \gamma_{21}\xi_{21} + \gamma_{22}\xi_{22} + \gamma_{23}\xi_{23} + \gamma_{24}\xi_{21}\xi_{22} + \gamma_{25}\xi_{21}\xi_{23} + \gamma_{26}\xi_{22}\xi_{23} + \delta_2, \quad (17)$$

where the regression coefficients γ_{11} , γ_{12} , and γ_{13} represent the effect of initial status in treatment, family support, and psychiatric problem, respectively, on the initial status in cocaine use, and γ_{21} , γ_{22} , and γ_{23} represent the effect of rate of change in treatment, family support, and psychiatric problem, respectively, on the rate of change in cocaine use. The other coefficients represent various interactive effects of exogenous growth factors on the endogenous growth factor. For example, Equation 17 can be used to examine the impact of joint changes in TR, FS, and PP and the influences of changes in outcome variable CC. Hence, the significance of parameters γ_{24} , γ_{25} , and γ_{26} essentially answers this question: Does the simultaneous change (interaction effect) in exogenous latent growth factors in treatment, psychiatric problem, or family support influence the rate of change in the endogenous growth factor in cocaine use?

Latent Curve Models With Ordered Categorical Repeated Measures

So far we have focused attention on continuous repeated measures. In practice, dichotomous or ordered categorical variables are common. In the cocaine use study, there are three kinds of manifest variables y , x , and u defined in Equations 1, 6, and 9. Some of these are ordered categorical variables. For example, we assume that x is ordered categorical. Let z be the underlying continuous unobservable variable. An auxiliary threshold model is a natural way to show the relation between x and z . For $k = 1, \dots, r$, $t = 1, \dots, T$:

$$x_{kt} = j, \quad \text{if} \quad \alpha_{kt,j} \leq z_{kt} < \alpha_{kt,j+1}, \quad j = 0, \dots, h_k - 1, \quad (18)$$

where $\{-\infty = \alpha_{kt,0} < \alpha_{kt,1} < \dots < \alpha_{kt,h} = \infty\}$ is the set of threshold parameters that defines the h_k categories. Equation 18 connects the ordered categorical variable to its underlying continuous counterpart such that as the continuous variable lies between two thresholds, the ordered categorical variable will register in the category that falls between those thresholds. To identify the auxiliary threshold model and enable the estimation of change in the mean over time, we follow Bollen and Curran (2006) and set the first threshold $\alpha_{kt,1}$ to 0 and ψ_{xtk} to 1 for $k = 1, \dots, r$, $t = 1, 2, \dots, T$. The extension of more complex situations that all y , x , and u include ordered categorical components is straightforward.

In this article, we proposed a comprehensive LCM defined by Equations 1, 6, 9, 11, and 15 applied to data from the cocaine study. Equation 18 will be incorporated if we encounter

ordered categorical variables. Moreover, data that are missing at random are allowed. Hence, the proposed model can be used to conduct the following analyses: (a) studying multivariate dynamic change of the manifest and latent variables, (b) exploring interaction effects between the first-order and second-order latent variables, and (c) handling mixed continuous and ordered categorical data with missing entries.

BAYESIAN ANALYSIS

The Bayesian approach is well recognized in the literature as an attractive approach for analyzing a wide variety of statistical models (Zellner, 1971), as well as structural equation models (e.g., Lee & Song, 2003, Lee, Song, & Tang, 2007, and the references therein). To introduce the Bayesian approach for the proposed LCM, let θ be the vector that contains the unknown free parameters of the model, and α be the vector that contains the unknown thresholds. In a Bayesian approach, (θ, α) is considered to be random with a prior distribution $p(\theta, \alpha)$. Let $\mathbf{D} = \{\mathbf{Y}, \mathbf{X}, \mathbf{U}\}$, where $\mathbf{Y}$, $\mathbf{X}$, and $\mathbf{U}$ include all three kinds of manifest variables defined in Equations 2, 6, and 10, and $\mathbf{Z}$ includes all underlying continuous variables that correspond to ordered categorical variables in $\mathbf{D}$. The Bayesian inference is based on the observed data set $\mathbf{D}$ and $p(\theta, \alpha)$. Let $p(\theta, \alpha, \mathbf{D})$ be the probability density function of the joint distribution of θ , α , and $\mathbf{D}$. It follows that $p(\theta, \alpha, \mathbf{D}) = p(\theta, \alpha | \mathbf{D})p(\mathbf{D})$, and $p(\theta, \alpha | \mathbf{D}) \propto p(\mathbf{D} | \theta, \alpha)p(\theta, \alpha)$. The function $p(\theta, \alpha | \mathbf{D})$ is called the posterior density function. It is common to use the posterior mean as the Bayesian estimate of (θ, α) .

In the context of the proposed LCM, it is rather difficult to compute the posterior mean mainly because of the existence of first- and second-order latent variables, unobservable continuous variables, and the nonlinearity of the structural equation. To solve this difficulty, we adopt a commonly used strategy in statistics that involves two key steps. The first step is to augment the observed data with all the latent variables to form a complete data set. Let Ξ include all latent growth factors (η and ξ), and Ω include all longitudinal latent variables (ω). The complete data set is $\{\mathbf{Z}, \Xi, \Omega, \mathbf{D}\}$. Treating $\mathbf{Z}$, Ξ , and Ω as random unknown quantities, the joint posterior density is $p(\theta, \alpha, \mathbf{Z}, \Xi, \Omega | \mathbf{D})$. The second step is to draw a sufficiently large sample of observations from the joint posterior distribution $[\theta, \alpha, \mathbf{Z}, \Xi, \Omega | \mathbf{D}]$, say $\{(\theta^{(j)}, \alpha^{(j)}, \mathbf{Z}^{(j)}, \Xi^{(j)}, \Omega^{(j)}), j = 1, \dots, J\}$. The Bayesian estimate of (θ, α) can be obtained through the sample mean, and the standard error estimates can be obtained through the sample covariance matrix. Moreover, estimates of the various latent variables can be obtained from $\Xi^{(j)}$ and $\Omega^{(j)}$ for $j = 1, \dots, J$, respectively. Hence, the main problem is to develop feasible and efficient methods to simulate observations from the joint posterior distributions. The task can be carried out with the well-known Gibbs sampler (Geman & Geman, 1984), which iteratively simulates observations from $p(\theta | \alpha, \mathbf{Z}, \Xi, \Omega, \mathbf{D})$, $p(\alpha | \mathbf{Z}, \Xi, \Omega, \mathbf{D})$, $p(\Xi | \theta, \alpha, \mathbf{Z}, \Omega, \mathbf{D})$, and $p(\Omega | \theta, \alpha, \mathbf{Z}, \Xi, \mathbf{D})$. Note that $p(\theta, \alpha | \mathbf{Z}, \Xi, \Omega, \mathbf{D})$ is much simpler than $p(\theta, \alpha | \mathbf{D})$. This is because in the corresponding conditional distribution $(\theta, \alpha | \mathbf{Z}, \Xi, \Omega, \mathbf{D})$, $\mathbf{Z}$, Ξ , and Ω are given rather than random, hence the corresponding equations in the LCM are essentially the regression equations with continuous scale.

Because $p(\theta, \alpha | \mathbf{Z}, \Xi, \Omega, \mathbf{D})$ depends on the prior distribution of (θ, α) , the selection of the appropriate prior distribution for (θ, α) is an important issue in the Bayesian approach. In selecting an informative prior distribution for (θ, α) , it is desirable from a methodological point of view to select a prior distribution such that the conditional distribution $p(\theta, \alpha | \mathbf{Z}, \Xi, \Omega, \mathbf{D})$ is not too complicated and has a convenient form from which to simulate observations. There are many choices of prior distributions. It is natural to assume that $p(\theta, \alpha) = p(\theta)p(\alpha)$. To deal with the situation with little or no information about threshold parameters, the following noninformative prior distribution is used for the prior distribution $p(\alpha)$:

$$p(\alpha) = p(\alpha_{kt,2}, \dots, \alpha_{kt,h_k-1}) \propto C, \quad k=1, \dots, r, \quad t=1, \dots, T. \quad (19)$$

where C is a constant. Guided by the recommendation of Lindley and Smith (1972) for the analysis of linear models and most Bayesian analyses of latent variable models (Ansari, Jedidi, & Jagpal, 2000, Lee & Song, 2003, among others), we use conjugate prior distributions for $p(\theta)$. A key advantage of conjugate prior distributions is that the resulting posterior distribution will follow the same parametric form of the prior distribution (see Gelman, Carlin, Stern, & Rubin, 2004), hence conjugacy ensures proper posterior distribution and easy derivation of the conditional distributions.

Based on the context of the proposed LCM, the following conjugate prior distributions are considered:

$$\begin{aligned} \psi_{yt}^{-1} &\stackrel{D}{=} \text{Gamma}(a_{0yt}, b_{0yt}), \quad \psi_{xtk}^{-1} \stackrel{D}{=} \text{Gamma}(a_{0xk}, b_{0xk}), \quad k=1, \dots, r, \quad t=1, \dots, T, \\ \psi_{\omega tk}^{-1} &\stackrel{D}{=} \text{Gamma}(a_{0\omega k}, b_{0\omega k}), \quad k=1, \dots, q, \quad t=1, \dots, T, \\ \psi_{utk}^{-1} &\stackrel{D}{=} \text{Gamma}(a_{0utk}, b_{0utk}), \quad \Lambda_{utk} \stackrel{D}{=} N[\Lambda_{0utk}, \psi_{utk} \Sigma_{0utk}], \quad k=1, \dots, p, \quad t=1, \dots, T, \\ \psi_{\eta k}^{-1} &\stackrel{D}{=} \text{Gamma}(a_{0\eta k}, b_{0\eta k}), \quad \Pi_k \stackrel{D}{=} N[\Pi_{0k}, \psi_{\eta k} \Sigma_{0\eta k}], \quad k=1, 2, \\ \beta &\stackrel{D}{=} N[\beta_0, \Sigma_0], \quad \text{and} \quad \Phi^{-1} \stackrel{D}{=} W_{r+q}(\mathbf{R}_0, \rho_0). \end{aligned} \quad (20)$$

where ψ_{yt} , ψ_{xtk} , and $\psi_{\omega tk}$ are the t th diagonal element of Ψ_y , Ψ_{xk} , and $\Psi_{\omega k}$, and ψ_{utk} and $\psi_{\eta k}$ are the k th diagonal element of Ψ_{ut} and Ψ_{η} , respectively; $\Pi = (\mathbf{B}', \mathbf{\Gamma}')'$, Λ'_{utk} and Π'_k are the k th row of Λ_{ut} and Π , respectively; a_{0yt} , b_{0yt} , a_{0xk} , b_{0xk} , $a_{0\omega k}$, $b_{0\omega k}$, a_{0utk} , b_{0utk} , β_0 , Σ_0 , Λ_{0utk} , Σ_{0utk} , Π_{0k} , $\Sigma_{0\eta k}$, $\mathbf{R}_0$, and ρ_0 are the hyperparameters in the prior distributions; and W_{r+q} is an $r+q$ dimensional Wishart distribution that is associated with a positive definite matrix $\mathbf{R}_0$ and degree of freedom ρ_0 . Under the preceding conjugate prior distributions, the full conditional distributions required for implementing the Gibbs sampler are presented in Appendix A.

Under mild regularity conditions, it has been shown that an observation simulated after a sufficiently large number of iterations can be regarded as an observation from the joint posterior distribution of $(\theta, \alpha, \mathbf{Z}, \Xi, \Omega)$ given $\mathbf{D}$ (Geman & Geman, 1984). The convergence of the algorithm can be determined by examining the plots of a number (usually three) of different sequences of simulated observations with different initial values. At convergence, these sequences of observations should be well mixed together (WinBUGS provides the diagnostic figure of convergence). Suppose that the algorithm reaches convergence after the K th iteration, then observations in the early iterations (called the burn-in iterations) are discarded because they do not belong to the joint posterior distribution. A sufficient number of simulated observations, say J , after the K th iteration are then collected to conduct statistical inference.

AN ILLUSTRATIVE APPLICATION USING COCAINE USE DATA

The data set for this example was from a longitudinal study conducted by the UCLA Center for Advancing Longitudinal Drug Abuse Research. Various measures were collected from 321 patients who were admitted in 1989 to the West Los Angeles Veterans Affairs Medical Center and who met the *DSM-III-R* criteria for cocaine dependence (Kasarabada et al., 1999). The patients were assessed at baseline, 1 year after treatment, 2 years after treatment, and 12 years after treatment in 2002 ($t_1 = 0$, $t_2 = 1$, $t_3 = 2$, $t_4 = 12$). Among the patients located at the 12-year follow-up (96.9%), some were confirmed to be deceased (8.7%), some declined to be interviewed, and some were either out of the country or too ill to be interviewed after 1 year, 2 years, or at the 12-year follow-up. Consequently, there is a large amount of missing data in this data set, treated as missing at random (MAR; see Little & Rubin, 1987). Interview

questionnaires covered information on participants' background characteristics, history of drug use, mental health, treatment, and so on. In this analysis, seven variables are involved. Days of cocaine use per month (CC), times per month in formal treatment (TR), family support (FS), which is an ordered categorical variable with a 3-point scale. The other three variables, Beck Inventory (BI), depression (DEP), and anxiety (AN), associate with the mental health or psychiatric problems of the patients; the latter two are measured based on the Hopkins Symptom Check List (HSCL-58; see Hser et al., 2006). Figure 5 displays the records in the days of cocaine use per month for 5 patients selected at random. The days are represented by an x and are connected by a dotted line summarizing the change trajectories. These individual trajectories show that patients usually have a significant decrease in the days of cocaine use after 2 years of treatment, although a few patients have different behaviors. The overall mean of days in cocaine use for all fully interviewed (i.e., at all time points) patients is also displayed in Figure 5. The mean of days are represented by an o and are connected by a solid line summarizing the change trajectory for most of the patients. From Figure 5, we can see on average that the patients dramatically reduced their cocaine use after 1 or 2 years of treatment, and then steadily over 12 years.

In this analysis, the endogenous latent growth factor (η) represents the pattern of change in CC. The first exogenous latent growth factor ξ_1 represents the pattern of change in TR. The second exogenous latent growth factor ξ_2 represents the pattern of change in FS. The variables BI, DEP, and AN with repeated measurements (at $t = 1, 2, 3, 4$) formed a longitudinal latent variable ω_{ut} , which can be interpreted as PP. Hence, the third exogenous latent growth factor ξ_3 represents the patterns of change in PP. To study the possible effects of treatment, family support, and psychiatric problems of the patients on cocaine use, we proposed the multivariate LCM described in Figure 6. The interaction effect between the initial status or between the rate of changes is assessed through the structural equations given in Equations 16 and 17. Because FS is an ordered categorical variable with a 3-point scale, we fixed $\alpha_{t,2} = 0.0$, and $\psi_{xt2} = 1.0$ to identify the model.

The Bayesian estimates of the unknown parameters were obtained by WinBUGS. In specifying the hyperparameters in the prior distributions, we use the following hierarchical prior inputs: $\beta_0 \sim N[0, 4\mathbf{I}]$, $\Lambda_{outk} \sim N[0, 4\mathbf{I}]$, and $\Pi_{0k} \sim N[0, 4\mathbf{I}]$, where $\mathbf{I}$ is an identity matrix with appropriate dimension. The values of other hyperparameters are given by $\alpha_{0yt} = \alpha_{0ok} = \alpha_{outk} = \alpha_{\eta k} = 10$, $\beta_{0yt} = \beta_{0ok} = \beta_{outk} = \beta_{0dk} = 4$, $\rho_0 = 6$, and $\mathbf{R}_0 = \mathbf{I}$. With three different starting values, the WinBUGS outputs showed that the sequences produced by the Gibbs sampler algorithm mixed well in less than 4,000 iterations. We took a burn-in phase of 4,000 iterations, and further collected $J = 30,000$ observations to produce the Bayesian estimates. The Bayesian estimates from WinBUGS are reported in Table 1, and the WinBUGS codes are presented in Appendix B. The estimated structural equations of Equations 16 and 17 are given as follows. The values in parentheses provide the standard error estimates of the associated parameters in the following equations and interpretations.

$$\eta_1 = 0.566 - 0.064\xi_{11} + 0.383\xi_{12} + 0.159\xi_{13} - 0.123\xi_{11}\xi_{12} - 0.016\xi_{11}\xi_{13} - 0.014\xi_{12}\xi_{13} \quad (0.173) \quad (21)$$

(0.254) (0.291) (0.362) (0.353) (0.293) (0.254)

$$\eta_2 = -0.060 + 0.006\xi_{21} - 0.034\xi_{22} + 0.517\xi_{23} + 0.619\xi_{21}\xi_{22} + 0.951\xi_{21}\xi_{23} + 0.704\xi_{22}\xi_{23} \quad (0.465) \quad (22)$$

(0.014) (0.135) (0.065) (0.219) (0.410) (0.494)

From Table 1 and the preceding equations, we have the following conclusions. First, all estimates of $\lambda_{ut,i1}$, $t = 1, \dots, 4$, $i = 2, 3$ are large, which indicate the strong associations between

manifest variables and corresponding latent variables. Second, in the preceding estimated structural equations, $\beta_1 = 0.566$ (0.254), $\beta_2 = -0.060$ (0.014), and $\lambda_{23} = 0.517$ (0.219) are statistically significant based on a p value of .05. The value of λ_{23} represents the positive effect of rate of change in PP on the rate of change in CC. This positive effect shows that as patients increase the rate of change in PP over time, the rate of change in CC also increases. The following interesting phenomenon is also found in predicting the longitudinal change in cocaine use (η_2 , rate of change). Although the coefficients $\gamma_{21} = 0.006$ (0.135), $\gamma_{22} = -0.034$ (0.065) are small and insignificant, the coefficients of interaction terms, $\gamma_{24} = 0.619$ (0.410), $\gamma_{25} = 0.951$ (0.494), and $\gamma_{26} = 0.704$ (0.465), are relatively large. These marginally significant coefficients indicate interactions between the dynamic latent shape factors (ξ_{21} and ξ_{22} , ξ_{21} and ξ_{23} , or ξ_{22} and ξ_{23}). These interactions suggest joint effects of longitudinal changes in cocaine use behavior. The interpretation of the positive value of γ_{24} is that the relation between longitudinal change in treatment (decreasing times of formal treatments per month) and longitudinal change in cocaine use (increasing days of cocaine use per month) becomes more positive over time as patients increase their family support. Thus, to reduce the rate of growth in cocaine use, patients must simultaneously increase their time in formal treatments and increase their family support. Similarly, γ_{25} indicates that to reduce the rate of growth in cocaine use, patients must simultaneously increase their time in formal treatments and improve (decrease) their psychiatric problems, and γ_{26} indicates that to reduce the rate of growth in cocaine use, patients must simultaneously increase their family support and improve (decrease) their psychiatric problems. These findings show that the integrated approach (combining formal medical treatment, increasing family support, and reducing psychiatric problems) is more efficient than simple medical treatment in reducing cocaine use among patients.

Third, the significant variances in both the intercept and the slope, $\psi_{\eta 1} = 0.340$ (0.060) and $\psi_{\eta 2} = 0.038$ (0.003), indicate that there are significant individual differences in both initial level and growth in cocaine use over time. Similarly, the significant variances $\phi_{11} = 0.669$ (0.077), $\phi_{33} = 1.430$ (0.293), and $\phi_{55} = 2.064$ (0.190) indicate that there are significant individual differences in initial level in treatment, family support, and psychiatric problems. Although variances $\phi_{22} = 0.015$ (0.001), $\phi_{44} = 0.053$ (0.014), and $\phi_{66} = 0.007$ (0.000) are small, they are also significant due to their small standard errors, which indicate that there are significant individual differences in the rate of change in treatment, family support, and psychiatric problems.

Finally, most of the covariances in Φ are small except for ϕ_{13} , ϕ_{15} , and ϕ_{35} . The large positive covariance $\phi_{13} = 0.528$ (0.109) indicates that there is a relatively strong positive correlation between the initial levels of treatment and family support. Similarly, $\phi_{15} = 0.708$ (0.091) indicates there is a relatively strong positive correlation between the initial levels of treatment and psychiatric problems. $\phi_{35} = 1.596$ (0.209) indicates there is a very strong positive correlation between the initial levels of family support and psychiatric problems. Interpretations of other less important parameters are not presented here. To cross-validate the results, we used different prior inputs to conduct the Bayesian analysis. We found that the obtained results were very close to those given in Table 1. This finding is consistent with the fact that the Bayesian estimates are not sensitive to the prior inputs.

A SIMULATION STUDY

To demonstrate the LCM and statistical methodology, we conducted a simulation study with similar design as the example in the preceding section. The model is defined by Equations 4, 7, 13, 14, 16, and 17 and is depicted in Figure 6, except that the linear time trajectories are designated as $t_1 = 0$, $t_2 = 1$, $t_3 = 2$, $t_4 = 3$. The sample size is 500 and the population values of the parameters are set as follows. $\lambda_{ut,21} = \lambda_{ut,31} = 0.8$, $t = 1, \dots, 4$; $\gamma_{11} = \gamma_{13} = \gamma_{22} = 0.6$, $\gamma_{12} = \gamma_{21} = \gamma_{23} = -0.6$, $\gamma_{14} = \gamma_{16} = \gamma_{25} = 0.4$, $\gamma_{15} = \gamma_{24} = \gamma_{26} = -0.4$, $\beta_1 = 0.5$, $\beta_2 = -0.5$, $\psi_{\eta t} = \psi_{\eta t1} =$

$\psi_{\omega t} = \psi_{utj} = 0.5, t = 1, \dots, 4, j = 1, 2, 3; \psi_{\eta 1} = \psi_{\eta 2} = 0.36; \Phi = (\phi_{kj})_{6 \times 6}$ with $\phi_{kk} = 1.0$ and $\phi_{kj} = 0.3, k \neq j$. We assume that $\mathbf{x}_2$ in Equation 6 includes ordered categorical measurements at each time point with a 5-point scale. The population values of thresholds are $\alpha_{t,1} = 0.0, \alpha_{t,2} = 1.0, \alpha_{t,3} = 2.0$, and $\alpha_{t,4} = 3.0$, for $t = 1, \dots, 4$. Note that $\alpha_{t,1}$ and ψ_{xt2} ($t = 1, 2, 3, 4$) are fixed at 0 and 1, respectively, to identify the auxiliary threshold model. To accelerate the convergence of the MCMC algorithm in the simulation, $\alpha_{t,4}$ ($t = 1, 2, 3, 4$) are also fixed at their true values. About 30% are created to be missing, and they are designated as MAR in this simulation.

The Bayesian estimates of the unknown parameters based on 100 replications were conducted by R2WinBUGS (Sturtz, Ligges, & Gelman, 2005). First, we performed a few tests using WinBUGS, and found that the Gibbs sampler algorithm converged in less than 4,000 iterations. We took a burn-in phase of 4,000 iterations, and further collected $J = 6,000$ observations to produce the Bayesian estimates in each replication. The mean and the root mean squares of the Bayesian estimates are reported in Table 2. From Table 2, we see that the mean of Bayesian estimates of the unknown parameters are close to the true values, and the root mean squares of the Bayesian estimates based on 100 replications are small, which indicates the Bayesian estimates are accurate.

DISCUSSION

Most methodological and empirical articles involving the analysis of change focus on the case of univariate change. In this article, we use a new comprehensive LCM that accommodates dynamic changes and interactions of multivariate manifest and latent variables. We introduce second-order latent variables in LCMs to explore dynamic changes of longitudinal latent variables. Although there has been an increasing application of LCMs in various fields, modeling the interaction among the first-order and second-order latent variables (growth factors) has not been widely applied in practice. A simulation study and an illustrative example involving cocaine study data have been used to demonstrate the proposed model and Bayesian methodology. The simulation results show that the new multivariate LCM is plausible and useful in exploring interactive dynamic changes of multivariate longitudinal variables in real applications. The Bayesian analysis and freely available software WinBUGS were used to obtain the Bayesian estimates in the real data about cocaine use, which include some ordered categorical variables and a lot of missing data. Due to the complexity of the proposed model and the moderate sample size ($n = 321$), we treated the missing data as MAR for simplicity. According to our understanding, commercial software such as LISREL (Jöreskog & Sörbom, 1996) cannot perform Bayesian analysis of the current LCM. Moreover, this software cannot be used to analyze interactive effects of the first-order and second-order latent variables (see Figure 6). Treating the missing data as nonignorable and exploring (comparing) different missing mechanisms, incorporating fixed covariates in the model, dealing with more complex types of data (e.g., unordered categorical data, nonnormal data, etc.), and selecting the best model (model comparison or selection) represent further research topics.

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APPENDIX A CONDITIONAL DISTRIBUTIONS

Conditional Distribution $p(\alpha, \mathbf{Z}|\cdot)$

Here, we only discuss the situation that $\mathbf{z}_k$ includes the underlying unobservable continuous variables corresponding to $\mathbf{x}_k$, and $\mathbf{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_r)$. Other situations that $\mathbf{Z}$ corresponds to $\mathbf{Y}$ and $\mathbf{U}$ are similar. Let $\alpha_{kt} = (\alpha_{kt,2}, \dots, \alpha_{kt,h-1})$, Λ'_t be the t th row of Λ , z_{kt} is the t th element of $\mathbf{z}_k$, then

$$p(\alpha, \mathbf{Z}) = \prod_{k=1}^r \prod_{t=1}^T p(\alpha_{kt}, z_{kt}|\cdot) \propto \prod_{k=1}^r \prod_{t=1}^T \Phi^* \left\{ \psi_{xtk}^{-1/2} (z_{kt} - \Lambda'_t \xi_k) \right\} I_{(\alpha_{kt,x_{kt}}, \omega_{kt,x_{kt}+1})}(z_t), \quad (\text{A.1})$$

where Φ^* is the standard normal density function.

Conditional Distribution $p(\Xi|\cdot)$

Note that once $\mathbf{Z}$ is given, Ξ is independent of α and $\mathbf{Y}$. Let $\zeta = (\eta', \xi')'$, we have

$$p(\zeta|\theta, \omega, \mathbf{z}, \mathbf{x}, \mathbf{u}) \propto p(\omega, \mathbf{z}, \mathbf{x}, \mathbf{u}|\theta, \zeta) p(\zeta|\theta) \propto p(\mathbf{z}|\theta, \eta) p(\eta|\xi) p(\xi|\theta) \prod_{k=1}^r p(\mathbf{x}_k|\theta, \xi_k) \prod_{k=1}^q p(\omega_k|\theta, \xi_{r+k}).$$

It follows that $p(\zeta|\cdot)$ is proportional to

$$\exp \left[-\frac{1}{2} \left\{ (\mathbf{z} - \Lambda \eta)' \Psi_y^{-1} (\mathbf{z} - \Lambda \eta) + [\eta - \mathbf{B} \eta - \Gamma \mathbf{F}(\xi)]' \Psi_\delta^{-1} [\eta - \mathbf{B} \eta - \Gamma \mathbf{F}(\xi)] + \xi' \Phi^{-1} \xi \right. \right. \\ \left. \left. + \sum_{k=1}^r (\mathbf{x}_k - \Lambda \xi_k)' \Psi_{xk}^{-1} (\mathbf{x}_k - \Lambda \xi_k) + \sum_{k=1}^q (\omega_k - \Lambda \xi_{r+k})' \Psi_{\omega k}^{-1} (\omega_k - \Lambda \xi_{r+k}) \right\} \right]. \quad (\text{A.2})$$

Conditional Distribution $p(\Omega|\cdot)$

Note that given Ξ , Ω is not depending on α , $\mathbf{Z}$, $\mathbf{Y}$, and $\mathbf{X}$, and $\{\omega_k, k = 1, \dots, q\}$ is the appropriate permutation of $\{\omega_{ut}, t = 1, \dots, T\}$, thus we have

$$p(\omega|\cdot) = p(\omega|\theta, \xi, \mathbf{u}) \propto p(\mathbf{u}|\theta, \omega) p(\omega|\theta, \xi) = \prod_{k=1}^T p(\mathbf{u}_t|\theta, \omega_{ut}) \prod_{k=1}^q p(\omega_k|\theta, \xi_{r+k}).$$

It follows that $p(\omega|\cdot)$ is proportional to

$$\exp \left[-\frac{1}{2} \left\{ \sum_{t=1}^T (\mathbf{u}_t - \Lambda_{ut} \omega_{ut})' \Psi_{ut}^{-1} (\mathbf{u}_t - \Lambda_{ut} \omega_{ut}) + \sum_{k=1}^q (\omega_k - \Lambda \xi_{r+k})' \Psi_{\omega k}^{-1} (\omega_k - \Lambda \xi_{r+k}) \right\} \right], \quad (\text{A.3})$$

where $\omega_{ut} = (\omega_{ut1}, \omega_{ut2})'$ for $t = 1, \dots, T$, and $\omega_k = (\omega_{u1k}, \dots, \omega_{uTk})'$ for $k = 1, \dots, q$.

Conditional Distribution of $p(\theta|\cdot)$

Let $\mathbf{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_n)$, $\mathbf{X}_k = (\mathbf{x}_{k1}, \dots, \mathbf{x}_{kn})$, $\mathbf{U}_t = (\mathbf{u}_{t1}, \dots, \mathbf{u}_{tm})$, $\Omega_1 = (\eta_1, \dots, \eta_n)$, $\Omega_{2k} = (\omega_{k1}, \dots, \omega_{kn})$, $\Omega_{3t} = (\omega_{t1}, \dots, \omega_{tm})$, $\Xi_k = (\xi_{k1}, \dots, \xi_{kn})$, and $F = (\mathbf{F}(\xi_1), \dots, \mathbf{F}(\xi_n))$, where n is sample size. Let $\mathbf{Z}_t$, $\mathbf{X}_{tk}$, Ω_{2tk} , Λ_t be the t th row of $\mathbf{Z}$, $\mathbf{X}_k$, Ω_{2k} , Λ , respectively, and $\mathbf{U}_{tk}$,

Ω_{1k} , Λ_{utk} be the k th row of $\mathbf{U}_t$, Ω_1 , and Λ_{ut} , respectively. It can be shown (see Lee & Song, 2003) that

$$\begin{aligned}
 p(\psi_{yt}^{-1}|\mathbf{z}_t) &\stackrel{D}{=} \text{Gamma}[n/2+a_{0yt}, b_{yt}], \quad t=1, \dots, T, \\
 p(\psi_{xtk}^{-1}|\mathbf{x}_k, \xi_k) &\stackrel{D}{=} \text{Gamma}[n/2+a_{0xk}, b_{xtk}], \quad k=1, \dots, r, \quad t=1, \dots, T, \\
 p(\psi_{otk}^{-1}|\omega_k, \xi_{r+k}) &\stackrel{D}{=} \text{Gamma}[n/2+a_{0ok}, b_{otk}], \quad k=1, \dots, q, \quad t=1, \dots, T, \\
 p(\Lambda_{utk}|\mathbf{u}_t, \omega_{ut}, \psi_{utk}) &\stackrel{D}{=} N[\mu_{utk}, \psi_{utk}\Sigma_{utk}], \quad p(\psi_{utk}^{-1}|\mathbf{u}_t, \omega_{ut}) \stackrel{D}{=} \text{Gamma}[n/2+a_{0utk}, b_{utk}], \\
 p(\Pi_k|\eta, \psi_{\eta k}) &\stackrel{D}{=} N[\mu_{\eta k}, \psi_{\eta k}\Sigma_{\eta k}], \quad p(\psi_{\eta k}^{-1}|\eta) \stackrel{D}{=} \text{Gamma}[n/2+a_{0\eta k}, b_{\eta k}], \\
 p(\beta|\eta, \Gamma, \Psi_\eta) &\stackrel{D}{=} N[\mu_\beta, \Sigma_\beta], \quad p(\Phi^{-1}|\Omega_2) \stackrel{D}{=} W_{r+q}[(\Xi\Xi' + \mathbf{R}_0^{-1})^{-1}, n+\rho_0],
 \end{aligned} \tag{A.4}$$

$$\begin{aligned}
 \text{where } b_{yt} &= b_{0yt} + 2^{-1} [\mathbf{Z}_t \mathbf{Z}_t' - 2\Lambda_t' \Omega_1 \mathbf{Z}_t' + \Lambda_t' (\Omega_1 \Omega_1') \Lambda_t], \\
 b_{xtk} &= b_{0xk} + 2^{-1} (\mathbf{X}_{tk} \mathbf{X}_{tk}' - 2\Lambda_t' \Xi_k \mathbf{X}_{tk}' + \Lambda_t' (\Xi_k \Xi_k') \Lambda_t), \\
 b_{otk} &= b_{0ok} + 2^{-1} (\Omega_{2tk} \Omega_{2tk}' - 2\Lambda_t' \Xi_{r+k} \Omega_{2tk}' + \Lambda_t' (\Xi_{r+k} \Xi_{r+k}') \Lambda_t), \quad \mu_{utk} = \Sigma_{utk} (\Sigma_{0utk}^{-1} \Lambda_{0utk} + \Omega_{3t} \mathbf{U}_{tk}'), \\
 \Sigma_{utk} &= (\Sigma_{0utk}^{-1} + \Omega_{3t} \Omega_{3t}')^{-1}, \quad b_{utk} = b_{0utk} + 2^{-1} (\mathbf{U}_{tk} \mathbf{U}_{tk}' - 2\Lambda_{utk}' \Omega_{3t} \mathbf{U}_{tk}' + \Lambda_{utk}' (\Omega_{3t} \Omega_{3t}') \Lambda_{utk}), \\
 \mu_{\eta k} &= \Sigma_{\eta k} (\Sigma_{0\eta k}^{-1} \Pi_{0k} + \mathbf{F} \Omega_{1k}'), \quad \Sigma_{\eta k} = (\Sigma_{0\eta k}^{-1} + \mathbf{F} \mathbf{F}')^{-1}, \\
 b_{\eta k} &= b_{0\eta k} + 2^{-1} [\Omega_{1k} \Omega_{1k}' - \mu_{\eta k}' \Sigma_{\eta k}^{-1} \mu_{\eta k} + \Pi_{0k}' \Sigma_{0\eta k}^{-1} \Pi_{0k}], \\
 \mu_\beta &= \Sigma_\beta [\Psi_\eta^{-1} \Sigma_{i=1}^n \{\eta_i - \mathbf{B} \eta_i - \Gamma \mathbf{F} (\xi_i)\} + \Sigma_0^{-1} \beta_0], \quad \Sigma_\beta = (\Sigma_0^{-1} + n \Psi_\eta^{-1})^{-1}.
 \end{aligned}$$

In this article, the missing data are treated as MAR. For handling these missing data, we should add an extra step in the Gibbs sampler, simulating the missing entries from their conditional distributions. For simplicity, we only discuss the situation when there are missing entries in $\mathbf{y}$. Let $\mathbf{y}_{mis}$ and $\mathbf{y}_{obs}$ be the vectors that include the missing and observed components in $\mathbf{y}$, and $\mathbf{z}_{mis}$ and $\mathbf{z}_{obs}$ be underlying continuous vectors that correspond to $\mathbf{y}_{mis}$ and $\mathbf{y}_{obs}$. Note that $\mathbf{z}_{obs}$ can be simulated from Equation A.1. To simulate $\mathbf{z}_{mis}$ from its conditional distribution, the following extra step should be added to the Gibbs sampler.

Conditional Distribution $p(\mathbf{z}_{mis}|\cdot)$

It follows from Equation 1 that

$$p(\mathbf{z}_{mis}|\cdot) = p(\mathbf{z}_{mis}|\theta, \eta) \stackrel{D}{=} N\Lambda_{y,mis}\eta, \Psi_{y,mis}, \tag{A.5}$$

where $\Lambda_{y,mis}$ is a $T_y \times m$ submatrix of Λ with rows that correspond to observed components deleted and $\Psi_{y,mis}$ is a $T_y \times T_y$ submatrix of Ψ_y with the appropriate rows and columns deleted, in which T_y is the dimension of $\mathbf{y}_{mis}$. The conditional distributions involved in Equations A.3, A.4, and A.5 are standard, so simulating observations from them is straightforward. As the conditional distributions that are involved in Equations A.1 and A.2 are not standard, the Metropolis–Hastings (MH, Metropolis, Rosenbluth, Rosenbluth, Teller, & Teller, 1953, see also Hastings, 1970) algorithm will be used for simulating observations from them (see Cowles, 1996, Lee & Song, 2003).

APPENDIX B WinBUGS CODES IN ANALYZING THE REAL EXAMPLE

```
odel{or(i in 1:N){# structural equationi[i,1:6]~dmnorm(u[1:6],phi[1:6,1:6])or
(j in 1:2){xxi[i,j]~dnorm(nu[i,j],psd[j,j])}u[i,1]<-vu[1]+gam[1]*xi[i,1]+gam
```



```

[2]*xi[i,3]+gam[3]*xi[i,5]gam[4]*xi[i,1]*xi[i,3]+gam[5]*xi[i,1]*xi[i,5]+gam
[6]*xi[i,3]*xi[i,5]u[i,2]<-vu[2]+gam[7]*xi[i,2]+gam[8]*xi[i,4]+gam[9]*xi[i,6]
gam[10]*xi[i,2]*xi[i,4]+gam[11]*xi[i,2]*xi[i,6]+gam[12]*xi[i,4]*xi[i,6]#
measurement model
for(j in 1:20){ y[i,j]~dnorm(mu[i,j],psi[j]) }u[i,1]<- xxi[i,
1] mu[i,2]<- xxi[i,1]+xxi[i,2] mu[i,3]<- xxi[i,1]+2*xxi[i,2]u[i,4]<- xxi[i,1]
+12*xxi[i,2] mu[i,5]<- xi[i,1] mu[i,6]<- xi[i,1]+xi[i,2]u[i,7]<- xi[i,1]+2*xi
[i,2] mu[i,8]<- xi[i,1]+12*xi[i,2] mu[i,9]<- xi[i,1]u[i,10]<- lam[1]*xi[i,
1] mu[i,11]<- lam[2]*xi[i,1] mu[i,12]<- xi[i,2]u[i,13]<- lam[3]*xi[i,2] mu
[i,14]<- lam[4]*xi[i,2] mu[i,15]<- xi[i,3]u[i,16]<- lam[5]*xi[i,3] mu[i,17]
<- lam[6]*xi[i,3] mu[i,18]<- xi[i,4]u[i,19]<- lam[7]*xi[i,4] mu[i,20]<- lam
[8]*xi[i,4]i1[i,1]<- xi[i,3]i1[i,2]<- xi[i,3]+xi[i,4]i1[i,3]<- xi[i,3]+2*xi
[i,4]i1[i,4]<- xi[i,3]+12*xi[i,4]or(j in 1:4){ z1[i,j]~dcat(p1[i,j,1:C1]) p1
[i,j,1]<-Q1[i,j,1]or(t in 2:C1-1){ p1[i,j,t]<- Q1[i,j,t]-Q1[i,j,t-1] } p1
[i,j,C1]<- 1-Q1[i,j,C1-1]or(t in 1:C1-1){ logit(Q1[i,j,t])<- alph1[j,t]-mu[i,
20+2*j-1] } } #end for(j in 1:4){ z2[i,j]~dcat(p2[i,j,1:C2])2[i,j,1]<-Q2[i,j,
1]or(t in 2:C2-1){ p2[i,j,t]<-Q2[i,j,t]-Q2[i,j,t-1] } p2[i,j,C2]<- 1-Q2
[i,j,C2-1]or(t in 1:C2-1){ logit(Q2[i,j,t])<- alph2[j,t]-mu[i,20+2*j] } }
#end ju[i,21]<- xi2[i,1]u[i,22]<- lam[9]*xi2[i,1]u[i,23]<- xi2[i,2]u[i,24]<-
lam[10]*xi2[i,2]u[i,25]<- xi2[i,3]u[i,26]<- lam[11]*xi2[i,3]u[i,27]<- xi2[i,
4]u[i,28]<- lam[12]*xi2[i,4]i2[i,1]<- xi[i,5]i2[i,2]<- xi[i,5]+xi[i,6]i2[i,3]
<- xi[i,5]+2*xi[i,6]i2[i,4]<- xi[i,5]+12*xi[i,6] }# thresholds for latent
variablesor(j in 1:4){ alph1[j,2]~ dnorm(0.0, 2.0)I(alph1[j,1]) }or(j in 1:4)
{ alph2[j,2]~ dnorm(0.0, 2.0)I(alph2[j,1],alph2[j,3])lph2[j,3]~ dnorm(0.0,
2.0)I(alph2[j,2],alph2[j,4])lph2[j,4]~ dnorm(0.0, 2.0)I(alph2[j,3]) }or(j in
1:4){ alph1[j, 1]<- 0 alph2[j, 1]<- 0 }# priors on loadings and coefficientsu
[1]~dnorm(0.0,4.0) vu[2]~dnorm(0.0,4.0)or(j in 1:12){ lam[j]~ dnorm(1.0,
4.0) }or(j in 1:12){ gam[j]~ dnorm(1.0, 4.0) }# priors on precisionsor(j in
1:20){ psi[j]~dgamma(10.0,4.0) v[j]<- 1/psi[j] }or(j in 1:2) { psd[j]~dgamma
(10.0,4.0) vd[j]<- 1/psd[j] }hi[1:6,1:6]~dwish(RR[1:6,1:6],6)hx[1:6,1:6]<-
inverse(phi[1:6,1:6])# put all the parameters' results into bbor(j in 1:2){ bb
[j]<- vu[j] }or(j in 1:12){ bb[2+j]<- lam[j] }for(j in 1:12){ bb[14+j]<- gam
[j] }for(j in 1:2){ bb[26+j]<- vd[j] }for(j in 1:20){ bb[28+j]<- v[j] }for(j
in 1:6){ bb[48+j]<- phx[1,j] }for(j in 2:6){ bb[54+j-1]<- phx[2,j] }for(j in
3:6){ bb[59+j-2]<- phx[3,j] }for(j in 4:6){ bb[63+j-3]<- phx[4,j] }for(j in
5:6){ bb[66+j-4]<- phx[5,j] }for(j in 6:6){ bb[68+j-5]<- phx[6,j] }for(j in
1:4){ bb[69+j]<- alph1[j,2] }for(j in 1:4){ bb[73+j]<- alph2[j,2] }for(j in
1:4){ bb[77+j]<- alph2[j,3] }for(j in 1:4){ bb[81+j]<- alph2[j,4] } }# init 1
list(paste initial value 1)# init 2 list(paste initial value 2)# init 3 list
(paste initial value 3)#Data list(N=321, u=c(0,0,0,0,0,0), C1=3,
C2=5,RR=structure(.Data= c(1.0, 0.0, 0.0, 0.0, 0.0, 0.0,0.0, 1.0, 0.0, 0.0,
0.0, 0.0,0.0, 0.0, 1.0, 0.0, 0.0, 0.0,0.0,0.0, 0.0, 0.0, 1.0, 0.0, 0.0,0.0, 0.0,
0.0, 0.0, 1.0, 0.0,0.0, 0.0, 0.0, 0.0, 0.0, 1.0), .Dim= c(6,6)),y=structure
(.Data=c(paste continuous variable), .Dim=c(321,20)),z1=structure(.Data=c
(paste the first ordinal variable), .Dim=c(321,4)),z2=structure(.Data=c(paste
the second ordinal variable), .Dim=c(321,4)))

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FIGURE 1.

Path diagram of the model defined in Equations 4 and 5. In this and the following figures, the error terms are not displayed for brevity.

FIGURE 2.
Path diagram of the model defined in Equations 4, 7, and 8.

FIGURE 3.
Path diagram of the model defined in Equations 10 and 11.

FIGURE 4.
Path diagram of the model defined in Equations 16 and 17.

FIGURE 5.
Trajectories of change in cocaine use for 5 patients.

FIGURE 6.
Path diagram of the proposed model in real example.

TABLE 1
Bayesian Estimates and Their Standard Error Estimates of the Parameters in the Cocaine Study Example

Par	Est	Std	Par	Est	Std	Par	Est	Std
$\lambda_{\eta 1,21}$	2.083	0.053	ψ_{y1}	1.361	0.136	ϕ_{11}	0.669	0.077
$\lambda_{\eta 1,31}$	1.828	0.051	ψ_{y2}	0.564	0.068	ϕ_{12}	-0.052	0.009
$\lambda_{\eta 2,21}$	2.169	0.052	ψ_{y3}	0.564	0.065	ϕ_{13}	0.528	0.109
$\lambda_{\eta 2,31}$	2.173	0.055	ψ_{y4}	0.293	0.070	ϕ_{14}	0.014	0.021
$\lambda_{\eta 3,21}$	2.028	0.046	ψ_{x11}	0.591	0.065	ϕ_{15}	0.708	0.091
$\lambda_{\eta 3,31}$	2.091	0.051	ψ_{x21}	0.746	0.073	ϕ_{16}	0.002	0.006
$\lambda_{\eta 4,21}$	1.959	0.065	ψ_{x31}	0.614	0.063	ϕ_{22}	0.015	0.001
$\lambda_{\eta 4,31}$	1.843	0.063	ψ_{x41}	0.249	0.052	ϕ_{23}	-0.031	0.012
γ_{11}	-0.064	0.291	$\psi_{\mu 11}$	0.554	0.054	ϕ_{24}	-0.007	0.002
γ_{12}	0.383	0.362	$\psi_{\mu 12}$	0.203	0.034	ϕ_{25}	-0.040	0.012
γ_{13}	0.159	0.353	$\psi_{\mu 13}$	0.384	0.048	ϕ_{26}	0.002	0.000
γ_{14}	-0.123	0.293	$\psi_{\mu 21}$	0.608	0.056	ϕ_{33}	1.430	0.293
γ_{15}	0.016	0.254	$\psi_{\mu 22}$	0.220	0.037	ϕ_{34}	0.023	0.034
γ_{16}	-0.014	0.173	$\psi_{\mu 23}$	0.437	0.056	ϕ_{35}	1.596	0.209
γ_{21}	0.006	0.135	$\psi_{\mu 31}$	0.547	0.051	ϕ_{36}	-0.009	0.009
γ_{22}	-0.034	0.065	$\psi_{\mu 32}$	0.159	0.024	ϕ_{44}	0.053	0.014
γ_{23}	0.517	0.219	$\psi_{\mu 33}$	0.344	0.043	ϕ_{45}	0.045	0.039
γ_{24}	0.619	0.410	$\psi_{\mu 41}$	0.604	0.060	ϕ_{46}	-0.000	0.001
γ_{25}	0.951	0.494	$\psi_{\mu 42}$	0.181	0.028	ϕ_{55}	2.064	0.190
γ_{26}	0.704	0.465	$\psi_{\mu 43}$	0.277	0.035	ϕ_{56}	-0.012	0.011
β_1	0.566	0.254	$\psi_{\theta 1}$	0.175	0.025	ϕ_{66}	0.007	0.000
β_2	-0.060	0.014	$\psi_{\theta 2}$	0.128	0.017	$\alpha_{1,2}$	2.703	0.196
$\psi_{\eta 1}$	0.340	0.060	$\psi_{\theta 3}$	0.126	0.016	$\alpha_{2,2}$	3.172	0.218
$\psi_{\eta 2}$	0.038	0.003	$\psi_{\theta 4}$	0.163	0.026	$\alpha_{3,2}$	3.425	0.236
						$\alpha_{4,2}$	2.502	0.269

TABLE 2

Bayesian Estimates, Standard Error Estimates, and Root Mean Squares of the Parameters in Simulation Study

True Value	M	RMS	True Value	M	RMS	True Value	M	RMS
$\lambda_{\eta 1,21} = 0.8$	0.810	0.038	$\psi_{\gamma 1} = 0.5$	0.479	0.056	$\phi_{11} = 1.0$	0.986	0.087
$\lambda_{\eta 1,31} = 0.8$	0.815	0.044	$\psi_{\gamma 2} = 0.5$	0.497	0.040	$\phi_{12} = 0.3$	0.310	0.058
$\lambda_{\eta 2,21} = 0.8$	0.805	0.023	$\psi_{\gamma 3} = 0.5$	0.497	0.049	$\phi_{13} = 0.3$	0.367	0.116
$\lambda_{\eta 2,31} = 0.8$	0.805	0.025	$\psi_{\gamma 4} = 0.5$	0.482	0.084	$\phi_{14} = 0.3$	0.332	0.078
$\lambda_{\eta 3,21} = 0.8$	0.800	0.013	$\psi_{\alpha 11} = 0.5$	0.509	0.056	$\phi_{15} = 0.3$	0.289	0.064
$\lambda_{\eta 3,31} = 0.8$	0.800	0.017	$\psi_{\alpha 21} = 0.5$	0.504	0.038	$\phi_{16} = 0.3$	0.303	0.063
$\lambda_{\eta 4,21} = 0.8$	0.799	0.012	$\psi_{\alpha 31} = 0.5$	0.499	0.046	$\phi_{22} = 1.0$	1.006	0.067
$\lambda_{\eta 4,31} = 0.8$	0.800	0.013	$\psi_{\alpha 41} = 0.5$	0.494	0.079	$\phi_{23} = 0.3$	0.378	0.117
$\gamma_{11} = 0.6$	0.621	0.070	$\psi_{\alpha 11} = 0.5$	0.506	0.049	$\phi_{24} = 0.3$	0.320	0.074
$\gamma_{12} = -0.6$	-0.567	0.085	$\psi_{\alpha 12} = 0.5$	0.502	0.045	$\phi_{25} = 0.3$	0.287	0.059
$\gamma_{13} = 0.6$	0.622	0.081	$\psi_{\alpha 13} = 0.5$	0.500	0.046	$\phi_{26} = 0.3$	0.302	0.050
$\gamma_{14} = -0.4$	-0.353	0.089	$\psi_{\alpha 21} = 0.5$	0.503	0.044	$\phi_{33} = 1.0$	1.182	0.277
$\gamma_{15} = 0.4$	0.454	0.108	$\psi_{\alpha 22} = 0.5$	0.499	0.039	$\phi_{34} = 0.3$	0.509	0.238
$\gamma_{16} = -0.4$	-0.375	0.088	$\psi_{\alpha 23} = 0.5$	0.501	0.040	$\phi_{35} = 0.3$	0.333	0.104
$\gamma_{21} = -0.6$	-0.573	0.054	$\psi_{\alpha 31} = 0.5$	0.506	0.054	$\phi_{36} = 0.3$	0.403	0.133
$\gamma_{22} = 0.6$	0.584	0.054	$\psi_{\alpha 32} = 0.5$	0.498	0.041	$\phi_{44} = 1.0$	1.132	0.210
$\gamma_{23} = -0.6$	-0.574	0.051	$\psi_{\alpha 33} = 0.5$	0.495	0.042	$\phi_{45} = 0.3$	0.345	0.087
$\gamma_{24} = 0.4$	0.390	0.057	$\psi_{\alpha 41} = 0.5$	0.491	0.048	$\phi_{46} = 0.3$	0.318	0.080
$\gamma_{25} = -0.4$	-0.392	0.055	$\psi_{\alpha 42} = 0.5$	0.500	0.042	$\phi_{55} = 1.0$	0.939	0.132
$\gamma_{26} = 0.4$	0.387	0.062	$\psi_{\alpha 43} = 0.5$	0.504	0.036	$\phi_{56} = 0.3$	0.311	0.056
$\beta_1 = 0.5$	0.464	0.070	$\psi_{\sigma 1} = 0.5$	0.513	0.073	$\phi_{66} = 1.0$	1.010	0.077
$\beta_2 = -0.5$	-0.466	0.053	$\psi_{\sigma 2} = 0.5$	0.503	0.062			
$\psi_{\eta 1} = 0.36$	0.388	0.062	$\psi_{\sigma 3} = 0.5$	0.491	0.058			
$\psi_{\eta 2} = 0.36$	0.359	0.043	$\psi_{\sigma 4} = 0.5$	0.501	0.089			
$\alpha_{1,2} = 1.0$	1.200	0.218	$\alpha_{1,3} = 2.0$	2.264	0.278			
$\alpha_{2,2} = 1.0$	1.104	0.132	$\alpha_{2,3} = 2.0$	2.211	0.173			
$\alpha_{3,2} = 1.0$	1.049	0.100	$\alpha_{3,3} = 2.0$	2.064	0.123			

True Value	M	RMS	True Value	M	RMS
$\alpha_{4,2} = 1.0$	0.967	0.096	$\alpha_{4,3} = 2.0$	1.962	0.109