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A Note on the Existence of the Posteriors for One-way Random Effect Probit Models

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Abstract

The existence of the posterior distribution for one-way random effect probit models has been investigated when the uniform prior is applied to the overall mean and a class of noninformative priors are applied to the variance parameter. The sufficient conditions to ensure the propriety of the posterior are given for the cases with replicates at some factor levels. It is shown that the posterior distribution is never proper if there is only one observation at each factor level. For this case, however, a class of proper priors for the variance parameter can provide the necessary and sufficient conditions for the propriety of the posterior.

Keywords

Uniform Prior; Noninformative Prior; Probit Models; Propriety of Posteriors

1 Introduction

In Bayesian hierarchical models, the choice of priors for the variance components is crucial. The commonly used prior for a variance component follows the inverse-Gamma distribution, which is conjugate for the variance component of a normal distribution. However, the specification of hyperparameters in the inverse-Gamma prior is often subjective. In many applications, little or no prior information about the variance components is available. Then a “vague” or “noninformative” prior is used to reflect the uncertainty. One of the vague priors is so-called Inverse-Gamma(ε, ε) with ε as a small number (see Spiegelhalter et al., 1994, 2003). The improper uniform prior for the standard deviation was also recommended by Gelman et al. (2006). However, the uniform prior for the standard deviation and the Inverse-Gamma(ε, ε) as $\varepsilon \rightarrow 0$ belong to the class of improper priors $1/\delta^{a+1}$ for the variance component δ (Natarajan and McCulloch, 1995), where a is a real number.

Natarajan and McCulloch (1995) investigated the existence of the posteriors for a class of mixed models for binomial responses. Their model specification was

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$$\begin{aligned}(y_i|u) &\sim \text{Bernoulli}(h(x_i\beta + z_iu)), \\ (u|\delta) &\sim N_q(0, \delta I_q), \\ [\delta|a] &\propto \frac{1}{\delta^{a+1}},\end{aligned}$$

where h is a link function, β and u are fixed and random effects, respectively. Although their model is based on a general h , the fixed effect parameter β is assumed to be known or has a proper prior. They gave sufficient conditions to ensure the propriety of the joint posterior distribution of (β, u, δ) . One condition is about the likelihood (data outcomes). Another condition is to assume the range of the hyper parameter a to be in the interval $(-1/2, 0)$, which does not include commonly used priors $1/\delta$, $1/\sqrt{\delta}$, or the constant prior corresponding to $a = 0, -1/2$, and -1 , respectively.

In this note, we work with the one-way random effect probit models of data y_{ik} with factor-level effects α_i :

$$\begin{aligned}(y_{ik}|\mu, \alpha_i) &\stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(\Phi(\mu + \alpha_i)), \quad i=1, \dots, I, \quad k=1, \dots, k_i, \\ (\alpha_i|\delta) &\stackrel{\text{i.i.d.}}{\sim} N(0, \delta), \quad i=1, \dots, I,\end{aligned}\tag{1}$$

where $k_i \geq 1$. Model (1) belongs to the set of mixed models in Natarajan and McCulloch (1995), and has two hyperparameters μ and δ . The case here is different from Natarajan and McCulloch (1995) in the sense that an improper uniform prior is used to the overall mean μ instead of assuming μ known or having a proper prior. Although we consider only the probit link due to the ease of computation (Albert and Chib, 1993) and the practical similarity between the probit and logit link, it is not hard to generalize the results for a general link. For the variance component parameter δ , the same class of improper priors $1/\delta^{a+1}$ is considered first. This class of improper priors have been widely investigated for the propriety of posterior in hierarchical linear mixed models (Hobert and Casella, 1996). In Section 2, we prove that if there are replicate observations at some factor levels, the condition about the likelihood can be relaxed and the range of valid a can be wider. We also show that if there is no replicate at each factor level, the posterior would never be proper. For this case, in Section 3, we apply another class of priors incorporating a shrinkage effect, and gave the necessary and sufficient conditions to ensure the propriety of the posterior.

2 Propriety of Posterior for Prior $[\mu, \delta] \propto 1/\delta^{a+1}$

We consider the class of priors of (μ, δ) ,

$$[\mu, \delta] \propto \frac{1}{\delta^{a+1}},\tag{2}$$

where a is a real number. Let $\tau = 1/\delta$. Then the corresponding prior for (μ, τ) is

$$[\mu, \tau] \propto \tau^{a-1}.\tag{3}$$

Clearly this prior is improper. To ensure correct Bayesian inferences, the posterior distribution must be proper. In Section 2.1, we study the propriety of the posterior distribution for the case that there are replicates at some factor levels. In Section 2.2, we investigate the case without replication.

2.1 Case A: Some $k_i > 1$

Theorem 2.1 For model (1) with prior (2), the joint posterior of (μ, δ) is proper if the two conditions hold:

- i. for each i , $i = 1, 2, \dots, I_1$, $(2 \leq I_1 \leq I)$, there is at least one success and one failure;
- ii. $-(I_1 - 1)/2 < a < 0$.

Proof. To ensure a proper posterior, we need to show

$$G \equiv \int L(\mu, \alpha) [\alpha|\tau] [\mu, \tau] d\mu d\alpha d\tau < \infty, \quad (4)$$

where

$$L(\mu, \alpha) = \prod_{i=1}^I \prod_{k=1}^{k_i} \Phi(\mu + \alpha_i)^{y_{ik}} \bar{\Phi}(\mu + \alpha_i)^{1-y_{ik}}.$$

From Condition (i),

$$L(\mu, \alpha) \leq \prod_{i=1}^{I_1} \Phi(\mu + \alpha_i) \bar{\Phi}(\mu + \alpha_i) \leq \exp \left\{ -\frac{1}{2} \sum_{i=1}^{I_1} (\mu + \alpha_i)^2 \right\}.$$

Clearly,

$$\begin{aligned} G &\leq \int \exp \left\{ -\frac{1}{2} \sum_{i=1}^{I_1} (\mu + \alpha_i)^2 \right\} \tau^{\frac{I}{2}} \exp \left\{ -\frac{\tau}{2} \sum_{i=1}^I \alpha_i^2 \right\} \tau^{a-1} d\mu d\alpha d\tau \\ &\propto \int \exp \left\{ -\frac{1}{2} (\mu, \alpha_1, \dots, \alpha_{I_1})' H (\mu, \alpha_1, \dots, \alpha_{I_1}) \right\} \tau^{a+\frac{I_1}{2}-1} d\mu d\alpha_1 \dots d\alpha_{I_1} d\tau, \end{aligned}$$

where

$$H = \begin{pmatrix} I_1 & 1 & 1 & \dots & 1 \\ 1 & 1+\tau & 0 & \dots & 0 \\ 1 & 0 & 1+\tau & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1+\tau \end{pmatrix}.$$

It can be shown that $|H| = I_1 \tau (1 + \tau)^{I_1-1}$. Therefore, there exists a constant C such that

$$G \leq C \int \frac{\tau^{a+\frac{I_1}{2}-1}}{|H|^{\frac{1}{2}}} d\tau \propto \int_0^\infty \frac{\tau^{a+\frac{I_1-3}{2}}}{(1+\tau)^{\frac{I_1-1}{2}}} d\tau,$$

which is integrable when Condition (ii) holds.

Theorem 2.1 gives two properties. First there is no need for every level to have at least one success and one failure to guarantee a proper posterior. Second, if there are quite a few levels with both success and failure, then the range of a may be quite large. For example, if both success and failure occur at 4 or more levels, the posterior is proper under the constant prior

of (μ, δ) . More importantly, all valid a 's are less than zero, indicating that the prior (2) is not the limit of the vague inverse-Gamma $(\varepsilon, \varepsilon)$ prior when $\varepsilon \rightarrow 0$.

2.2 Case B: All $k_i = 1$

Theorem 2.2 Consider model (1) with prior (2). If all $k_i = 1$, then the joint posterior distribution of (μ, δ) is never proper.

Proof. For G defined in (4), by letting $\theta_i = \mu + \alpha_i$, for $i = 1, \dots, I$, and $\theta = (\theta_1, \dots, \theta_I)'$, we have

$$\begin{aligned} G &= \int \left[\prod_{i=1}^I \Phi(\theta_i)^{y_i} \bar{\Phi}(\theta_i)^{1-y_i} \right] \frac{1}{\delta^{I/2+a+1}} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^I (\theta_i - \mu)^2 \right\} d\mu d\theta d\delta \\ &\propto \int \left[\prod_{i=1}^I \Phi(\theta_i)^{y_i} \bar{\Phi}(\theta_i)^{1-y_i} \right] \frac{1}{\delta^{(I-1)/2+a+1}} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^I (\theta_i - \bar{\theta})^2 \right\} d\theta d\delta. \end{aligned} \quad (5)$$

It is easy to show that the subspace of $\mathbf{R}^I$, defined as $\Theta_{sub} = \{(\theta_1, \dots, \theta_I): \theta_i > 0, \text{ if } y_i = 1; \theta_i < 0, \text{ if } y_i = 0, \text{ for } i = 1, \dots, I\}$, satisfies that

$$(1/2)^I \leq \int \prod_{i=1}^I \Phi(\theta_i)^{y_i} \bar{\Phi}(\theta_i)^{1-y_i} \leq 1, \quad \text{for } \theta \in \Theta_{sub}. \quad (6)$$

Let $w_i = \theta_i / \sqrt{\delta}$, for $i = 1, \dots, I$. Define $\mathbf{W}_{sub} = \{w = (w_1, \dots, w_I)': \sqrt{\delta}w \in \Theta_{sub}\}$. Then

$$\begin{aligned} &\int_0^\infty \int_{\Theta_{sub}} \frac{1}{\delta^{(I-1)/2+a+1}} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^I (\theta_i - \bar{\theta})^2 \right\} d\theta d\delta \\ &= \int_0^\infty \frac{1}{\delta^{1/2+a}} d\delta \int_{\mathbf{W}_{sub}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^I (w_i - \bar{w})^2 \right\} dw, \end{aligned} \quad (7)$$

which is not integrable. Clearly, the integral G does not exist by (5)-(7). The result follows.

3 Propriety of Posterior for Prior $[\mu, \delta] \propto 1/(1+\delta)^a$

Theorem 2.2 shows that we never have a proper posterior distribution of (μ, δ) for model (1) without replication under prior (2). Indeed, in presence of a overall mean with the constant prior, prior (2) for δ will promote too low variance values obstructing the likelihood to contribute useful factors. We now consider another class of priors for (μ, δ) ,

$$[\mu, \delta] \propto \frac{1}{(1+\delta)^a}. \quad (8)$$

This prior incorporates a shrinkage effect. It is interesting that when $a = 2$, this prior corresponds to the uniform shrinkage prior on the shrinkage parameter $s = 1/(1+\delta)$ for the normal-normal hierarchical model, see Daniels (1999). The uniform shrinkage prior has several attractive properties as discussed by various authors, e.g., Strawderman (1971) and Daniels (1999). When $a > 1$, prior (8) is actually marginally proper for δ .

Theorem 3.1 Consider model (1) with prior (8) when all $k_i = 1$. The necessary and sufficient conditions to ensure the posterior propriety of (μ, δ) are

- i. there is at least one success and one failure among the I levels ($I \geq 2$);
- ii. $a > 3/2$.

Proof. For G defined in (4), by letting $\theta_i = \mu + \alpha_i$, for $i = 1, \dots, I$, and $\theta = (\theta_1, \dots, \theta_I)'$, we have

$$\begin{aligned} G &= \int \left\{ \prod_{i=1}^I \Phi(\theta_i)^{y_i} \bar{\Phi}(\theta_i)^{1-y_i} \right\} \frac{1}{\delta^{I/2}} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^I (\theta_i - \mu)^2 \right\} \frac{1}{(1+\delta)^a} d\mu d\theta d\delta \\ &\propto \int \left\{ \prod_{i=1}^I \Phi(\theta_i)^{y_i} \bar{\Phi}(\theta_i)^{1-y_i} \right\} \frac{1}{\delta^{(I-1)/2}} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^I (\theta_i - \bar{\theta})^2 \right\} \frac{1}{(1+\delta)^a} d\theta d\delta. \end{aligned} \quad (9)$$

Necessity of (i) and (ii). By the inequality (6) and by (9), one necessary condition that G is finite is

$$G_1 \equiv \int_0^\infty \int_{\Theta_{sub}} \frac{1}{\delta^{(I-1)/2}} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^I (\theta_i - \bar{\theta})^2 \right\} \frac{1}{(1+\delta)^a} d\theta d\delta < \infty.$$

Let $w_i = \theta_i / \sqrt{\delta}$, for $i = 1, \dots, I$. Define $\mathbf{W}_{sub} = \{w = (w_1, \dots, w_I)': \sqrt{\delta}w \in \Theta_{sub}\}$. Then

$$G_1 = G_2 \cdot G_3,$$

where

$$G_2 = \int_0^\infty \frac{\sqrt{\delta}}{(\delta+1)^a} d\delta, \quad (10)$$

$$G_3 = \int_{\mathbf{W}_{sub}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^I (w_i - \bar{w})^2 \right\} dw. \quad (11)$$

Obviously, Condition (ii) is required to make G_2 finite.

To prove the necessity of Condition (i), WLOG (Without Loss of Generality) we consider only the case that all $y_i = 1$, that is, all outcomes are successes.

Suppose $I = 2$. Let $\Theta_{sub} = \{(\theta_1, \theta_2) : \theta_1 > 0, \theta_2 > 0\}$. Then $1/2 \leq \Phi(\theta_i) \leq 1$, $i = 1, 2$ for $(\theta_1, \theta_2) \in \Theta_{sub}$, and $\mathbf{W}_{sub} = \{(w_1, w_2) : w_1 > 0, w_2 > 0\}$. Clearly G_3 in (11) is just

$$\begin{aligned} G_3 &= \int_0^\infty \int_0^\infty \exp \left\{ -\frac{1}{2} \sum_{i=2}^I (w_i - \bar{w})^2 \right\} dw_1 dw_2 \\ &= \int_0^\infty \int_0^\infty \exp \left\{ -\frac{1}{4} (w_1 - w_2)^2 \right\} dw_1 dw_2 \\ &= \int_0^\infty \int_{-w_2}^\infty \exp \left\{ -\frac{1}{4} z_1^2 \right\} dz_1 dw_2, \\ &\geq \int_0^\infty \int_0^\infty \exp \left\{ -\frac{1}{4} z_1^2 \right\} dz_1 dw_2 = \infty. \end{aligned} \quad (12)$$

When $I > 2$, $\Theta_{sub} = \{(\theta_1, \dots, \theta_I) : \theta_i > 0, \text{ for } i = 1, \dots, I\}$ and $\mathbf{W}_{sub} = \{(w_1, \dots, w_I) : w_i > 0, \text{ for } i = 1, \dots, I\}$. Using the equality,

$$\sum_{i=1}^I (w_i - \bar{w})^2 = \sum_{i=2}^I (w_i - \bar{w}_{(-1)})^2 + \frac{I-1}{I} \left(w_1 - \frac{1}{I-1} \sum_{i=2}^I w_i \right)^2,$$

where $\bar{w}$ is the mean value of all w_i 's, and $\bar{w}_{(-1)}$ is the mean value of all w_i 's but excluding w_1 , the integral G_3 can be written as

$$\int_0^\infty \cdots \int_0^\infty G_4 \cdot \exp \left\{ -\frac{1}{2} \sum_{i=2}^I (w_i - \bar{w}_{(-1)})^2 \right\} dw_2 \cdots dw_I, \quad (13)$$

where

$$\begin{aligned} G_4 &\equiv \int_0^\infty \exp \left\{ -\frac{I-1}{2I} \left(w_1 - \frac{1}{I-1} \sum_{i=2}^I w_i \right)^2 \right\} dw_1 \\ &\geq \int_0^\infty \exp \left\{ -\frac{I-1}{2I} z_1^2 \right\} dz_1. \end{aligned}$$

This inequality holds because all $w_i > 0$, for $i = 2, \dots, I$. Combining (12) with (13), it is clear that the integral G_3 is not finite if $I \geq 2$ and all observations are successes. Thus, Condition (i) is required to make G_3 finite.

Sufficiency of (i) and (ii). When $I = 2$, WLOG, we assume $y_1 = 1$ and $y_2 = 0$. From (9),

$$G \propto \int_0^\infty \int_{\mathbb{R}^2} \Phi(\theta_1) \bar{\Phi}(\theta_2) \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{1}{4\delta} (\theta_1 - \theta_2)^2 \right\} \frac{1}{(\delta+1)^a} d\theta d\delta.$$

We partition $\mathbb{R}^2$ for (θ_1, θ_2) as $\Theta_1 \cup \Theta_2 \cup \Theta_3$ with $\Theta_1 = \{(\theta_1, \theta_2) : \theta_1 < 0, -\infty < \theta_2 < \infty\}$, $\Theta_2 = \{(\theta_1, \theta_2) : \theta_1 > 0, \theta_2 > 0\}$, and $\Theta_3 = \{(\theta_1, \theta_2) : \theta_1 > 0, \theta_2 < 0\}$. Correspondingly,

$$G \propto \sum_{i=1}^3 Q_i,$$

where

$$Q_i \equiv \int_{\Theta_i} \Phi(\theta_1) \bar{\Phi}(\theta_2) \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{1}{4\delta} (\theta_1 - \theta_2)^2 \right\} \frac{1}{(\delta+1)^a} d\theta d\delta.$$

Note that

$$\begin{aligned} Q_1 &\leq \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^0 \exp \left\{ -\frac{\theta_1^2}{2} \right\} \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{1}{4\delta} (\theta_1 - \theta_2)^2 \right\} \frac{1}{(\delta+1)^a} d\theta_1 d\theta_2 d\delta \\ &\leq \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \exp \left\{ -\frac{\theta_1^2}{2} \right\} \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{1}{4\delta} (\theta_1 - \theta_2)^2 \right\} \frac{1}{(\delta+1)^a} d\theta_1 d\theta_2 d\delta \\ &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \exp \left\{ -\frac{1}{2} (\theta_1, \theta_2) \mathbf{H}(\theta_1, \theta_2)' \right\} \frac{1}{\sqrt{\delta}} \frac{1}{(\delta+1)^a} d\theta_1 d\theta_2 d\delta, \end{aligned}$$

where

$$\mathbf{H} = \begin{pmatrix} 1 + \frac{1}{2\delta} & -\frac{1}{2\delta} \\ -\frac{1}{2\delta} & \frac{1}{2\delta} \end{pmatrix}.$$

It is easy to show that $|\mathbf{H}| = 1/(2\delta)$. Then there is a constant C such that

$$\begin{aligned} Q_1 &\leq C \int_0^\infty \frac{1}{|\mathbf{H}|^{1/2}} \frac{1}{\sqrt{\delta}} \frac{1}{(\delta+1)^a} d\delta \\ &\propto \int_0^\infty \frac{1}{(\delta+1)^a} d\delta, \end{aligned}$$

which is integrable when $a > 1$. Similarly, $Q_2 < \infty$ if $a > 1$. Finally,

$$Q_3 \leq \int_0^\infty \int_{-\infty}^0 \int_0^\infty \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{1}{4\delta} (\theta_1 - \theta_2)^2 \right\} \frac{1}{(\delta+1)^a} d\theta_1 d\theta_2 d\delta.$$

Let $\theta_1 = r \sin \beta$ and $\theta_2 = r \cos \beta$. On Θ_3 , we have $0 < r < \infty$ and $\pi/2 < \beta < \pi$. Then

$$\begin{aligned} Q_3 &\leq \int_0^\infty \int_{\pi/2}^\pi \int_0^\infty \frac{r}{\sqrt{\delta}} \exp \left\{ -\frac{r^2}{4\delta} (\sin \beta - \cos \beta)^2 \right\} \frac{1}{(\delta+1)^a} dr d\beta d\delta \\ &\propto \int_0^\infty \frac{\sqrt{\delta}}{(\delta+1)^a} d\delta \int_{\pi/2}^\pi \frac{1}{(\sin \beta - \cos \beta)^2} d\beta, \end{aligned}$$

which is integrable when Condition (ii) holds.

Suppose $I > 2$. WLOG, we again assume $y_1 = 1$ and $y_2 = 0$. Now,

$$L(\mu, \alpha) \leq \Phi(\mu + \alpha_1) \bar{\Phi}(\mu + \alpha_2).$$

Then, there exists a constant C such that

$$\begin{aligned} G &\leq C \int \Phi(\mu + \alpha_1) \bar{\Phi}(\mu + \alpha_2) \frac{1}{\delta^{I/2}} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^I \alpha_i^2 \right\} \frac{1}{(\delta+1)^a} d\mu d\alpha d\delta \\ &= C \int \Phi(\mu + \alpha_1) \bar{\Phi}(\mu + \alpha_2) \frac{1}{\delta} \exp \left\{ -\frac{1}{2\delta} \sum_{i=1}^2 \alpha_i^2 \right\} \frac{1}{(\delta+1)^a} d\mu d\alpha_1 d\alpha_2 d\delta, \end{aligned}$$

which is exactly the case for $I = 2$ with $y_1 = 1$ and $y_2 = 0$. Hence, Conditions (i) and (ii) are sufficient for $I \geq 2$.

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References

Albert JH, Chib S. Bayesian analysis of binary and polychotomous response data. *Journal of the American Statistical Association* 1993;88:669–679.

- Daniels MJ. A prior for the variance in hierarchical models. *The Canadian Journal of Statistics* 1999;27:567–578.
- Gelman A. Prior distributions for variance parameters in hierarchical models. *Bayesian Analysis* 2006;1:515–533.
- Hobert PJ, Casella G. The effect of improper priors on Gibbs sampling in hierarchical linear mixed models. *Journal of the American Statistical Association* 1996;91:1461–1473.
- Natarajan R, McCulloch C. A note on the existence of the posterior distribution for a class of mixed models for binomial responses. *Biometrika* 1995;82:639–643.
- Spiegelhalter, DJ.; Abrams, KR.; Myles, JP. *Bayesian Approaches to Clinical Trials and Health-Care Evaluation*, section 5.7.3. Wiley; Chichester: 2004.
- Spiegelhalter, DJ.; Thomas, A.; Best, NG.; Gilks, WR.; Lunn, D. *BUGS: Bayesian inference using Gibbs sampling*. MRC Biostatistics Unit; Cambridge, England: 1994. p. 2003www.mrc-bsu.cam.ac.uk/bugs/
- Strawderman W. Proper Bayes minimax estimators of the multivariate normal mean. *Annals of Mathematical Statistics* 1971;42:385–388.