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The applications of a higher-dimensional Lie algebra and its decomposed subalgebras

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Abstract

With the help of invertible linear transformations and the known Lie algebras, a higher-dimensional 6×6 matrix Lie algebra $su(6)$ is constructed. It follows a type of new loop algebra is presented. By using a $(2 + 1)$ -dimensional partial-differential equation hierarchy we obtain the integrable coupling of the $(2 + 1)$ -dimensional KN integrable hierarchy, then its corresponding Hamiltonian structure is worked out by employing the quadratic-form identity. Furthermore, a higher-dimensional Lie algebra denoted by E, is given by decomposing the Lie algebra $su(6)$, then a discrete lattice integrable coupling system is produced. A remarkable feature of the Lie algebras $su(6)$ and E is used to directly construct integrable couplings.

Keywords

Lie algebras; Hamiltonian; Integrable coupling

1 Introduction

Integrable couplings were proposed in view of the Virasoro algebra and the soliton theory, a few ways to construct integrable couplings were presented by use of perturbations[1-3]. Tu Guizhang once proposed a simple and efficient method for generating integrable couplings and Hamiltonian of soliton equations with infinite dimensions in Ref[4], Ma Wenxiu called the method as Tu scheme. By making use of Tu scheme, some well-known integrable hierarchies and corresponding Hamiltonian systems were worked out [5-18]. Its basis is the known algebra as follows:

$$\begin{cases} h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ [h, e] = 2e, [h, f] = -2f, [e, f] = h, \end{cases} \quad (1)$$

Li[19] presented the Lax pair from the self-dual Yang-Mills equation[15]

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$$\begin{cases} \varphi_x = M\varphi, \\ \varphi_t = P(\lambda)\varphi_y + N\varphi. \end{cases} \quad (2)$$

whose compatibility is the following $(2+1)$ -dimensional partial-differential equation hierarchy

$$M_t - N_x + [M, N] - P(\lambda)M_y = 0, \quad (3)$$

where $P(\lambda) = \alpha_0 + \alpha_1\lambda + \dots$, λ is a spectral parameter. If let $P(\lambda) = 0$, then (3) reduce to the standard zero curvature equation

$$M_t - N_x + [M, N] = 0 \quad (4)$$

In this paper, with the help of invertible linear transformations and the known Lie algebras, we construct a higher-dimensional 6×6 matrix Lie algebra $s\mu(6)$, the integrable coupling of the $(2+1)$ -dimensional KN integrable hierarchy is obtained under the framework of the $(2+1)$ -dimensional partial-differential equation hierarchy, then we obtain its Hamiltonian structure by using the quadratic-form identity [9]. Finally, In Ref[19,20], a decomposed Lie algebra E from $s\mu(6)$ is presented whose two subalgebras are denoted by E_1 and E_2 , which satisfy $E = E_1 \oplus E_2$, $[E_1, E_2] \subseteq E_2$. We also require the closure between E_1 and E_2 under the matrix multiplication $E_1 E_2, E_2 E_1 \subseteq E_2$. With the help of E , a discrete lattice Lax pair is given from which a discrete lattice integrable coupling system of an integrable hierarchy is obtained.

2 The application of a higher-dimensional Lie algebra

We construct a new higher-dimensional 6×6 matrix Lie algebra $s\mu(6)$ as follows[15]

$$\begin{cases} h_1 = \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix}, h_2 = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}, h_3 = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix}, \\ h_4 = \begin{pmatrix} 0 & h & 0 \\ 0 & 0 & h \\ 0 & 0 & 0 \end{pmatrix}, h_5 = \begin{pmatrix} 0 & e & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix}, h_6 = \begin{pmatrix} 0 & f & 0 \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}, \\ h_7 = \begin{pmatrix} 0 & 0 & h \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_8 = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_9 = \begin{pmatrix} 0 & 0 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ [h_1, h_2] = 2h_2, [h_1, h_3] = -2h_3, [h_2, h_3] = h_1, [h_1, h_4] = 0, [h_1, h_5] = 2h_5, [h_1, h_6] = -2h_6, [h_1, h_7] = 0, \\ [h_1, h_8] = 2h_8, [h_1, h_9] = -2h_9, [h_2, h_5] = 0, [h_2, h_6] = h_4, [h_2, h_7] = -2h_8, [h_2, h_8] = 0, \\ [h_2, h_9] = h_7, [h_3, h_4] = 2h_6, [h_3, h_5] = -h_4, [h_3, h_6] = 0, [h_3, h_7] = 2h_9, [h_3, h_8] = -h_7, [h_3, h_9] = 0, \\ [h_4, h_5] = 2h_8, [h_4, h_6] = -2h_9, [h_4, h_7] = [h_4, h_8] = [h_4, h_9] = 0, [h_5, h_6] = h_7, \\ [h_5, h_7] = [h_5, h_8] = [h_5, h_9] = [h_6, h_7] = [h_6, h_8] = [h_6, h_9] = [h_7, h_8] = [h_7, h_9] = [h_8, h_9] = 0. \end{cases}$$

It is easy to verify that $s\mu(6) = \text{span}\{h_i\}_{i=1}^9$ is a Lie algebra. Its resulting loop algebra $\tilde{s\mu}(6)$ with powers of λ being $\lambda^{2n}, \lambda^{2n+1}$ is given by

$$\begin{cases} h_1(n)=h_1\lambda^{2n}, h_2(n)=h_2\lambda^{2n+1}, h_3(n)=h_3\lambda^{2n+1}, h_4(n)=h_4\lambda^{2n}, h_5(n)=h_5\lambda^{2n+1} \\ h_6(n)=h_6\lambda^{2n+1}, h_7(n)=h_7\lambda^{2n}, h_8(n)=h_8\lambda^{2n+1}, h_9(n)=h_9\lambda^{2n+1}, \end{cases} \quad (5)$$

with a commutative operation defined as

$$\begin{cases} [h_1(m), h_2(n)]=2h_2(m+n), [h_1(m), h_3(n)]=-2h_3(m+n), [h_2(m), h_3(n)]=h_1(m+n+1), [h_1(m), h_4(n)]=0, \\ [h_1(m), h_5(n)]=2h_5(m+n), [h_1(m), h_6(n)]=-2h_6(m+n), [h_1(m), h_7(n)]=0, [h_1(m), h_8(n)]=2h_8(m+n), \\ [h_1(m), h_9(n)]=-2h_9(m+n), [h_2(m), h_5(n)]=0, [h_2(m), h_4(n)]=-2h_5(m+n), [h_2(m), h_6(n)]=h_4(m+n+1), \\ [h_2(m), h_7(n)]=-2h_8(m+n), [h_2(m), h_8(n)]=0, [h_2, h_9(n)]=h_7(m+n+1), [h_3(m), h_4(n)]=2h_6(m+n), \\ [h_3(m), h_5(n)]=-h_4(m+n+1), [h_3(m), h_6(n)]=0, [h_3(m), h_7(n)]=2h_9(m+n), [h_3(m), h_8(n)]=-h_7(m+n+1), \\ [h_3(m), h_9(n)]=0, [h_4(m), h_5(n)]=2h_8(m+n), [h_4(m), h_6(n)]=-2h_9(m+n), \\ [h_4(m), h_7(n)]=[h_4(m), h_8(n)]=[h_4(m), h_9(n)]=0, [h_5(m), h_6(n)]=h_7(m+n+1), \\ [h_5(m), h_7(n)]=[h_5(m), h_8(n)]=[h_5(m), h_9(n)]=[h_6(m), h_7(n)]=[h_6(m), h_8(n)]=[h_6(m), h_9(n)]=0, \\ [h_7(m), h_8(n)]=[h_7(m), h_9(n)]=[h_8(m), h_9(n)]=0, m, n \in \mathbb{Z}, n=0, \pm 1, \pm 2 \dots \end{cases} \quad (6)$$

Consider an isospectral problem

$$\varphi_x = M\varphi, M = h_1(1) + u_1h_2(0) + u_2h_3(0) + u_3h_5(0) + u_4h_6(0) + u_5h_8(0) + u_6h_9(0). \quad (7)$$

Denoting $V = \sum_{m \geq 0} (a_m h_1(-m) + b_m h_2(-m) + c_m h_3(-m) + d_m h_4(-m) + e_m h_5(-m) + f_m h_6(-m) + g_m h_7(-m) + k_m h_8(-m) + l_m h_9(-m))$, then the stationary linear equation

$$N_x = [M, N] \quad (8)$$

yields

$$\begin{cases} a_{mx} = u_1 c_{m+1} - u_2 b_{m+1}, b_{mx} = 2b_{m+1} - 2u_1 a_m \\ c_{mx} = -2c_{m+1} + 2u_2 a_m \\ d_{mx} = -u_2 e_{m+1} + u_1 f_{m+1} + u_3 c_{m+1} - u_4 b_{m+1}, \\ e_{mx} = 2e_{m+1} - 2u_1 d_m - 2u_3 a_m, \\ f_{mx} = -2f_{m+1} + 2u_2 d_m + 2u_4 a_m, \\ g_{mx} = u_1 l_{m+1} - u_2 k_{m+1} + u_3 f_{m+1} - u_4 e_{m+1} + u_5 c_{m+1} - u_6 b_{m+1}, \\ k_{mx} = 2k_{m+1} - 2u_1 g_m - 2u_3 d_m - 2u_5 a_m, \\ l_{mx} = -2l_{m+1} + 2u_2 g_m + 2u_4 d_m + 2u_6 a_m, \\ a_0 = \alpha = \text{const}, b_0 = c_0 = d_0 = e_0 = f_0 = g_0 = k_0 = l_0 = 0, \\ a_1 = -\frac{\alpha}{2} u_1 u_2, b_1 = \alpha u_1, c_1 = \alpha u_2, d_1 = 0, e_1 = \alpha u_3, f_1 = \alpha u_4, g_1 = 0, k_1 = \alpha u_5, l_1 = \alpha u_6. \end{cases} \quad (9)$$

Note

$$\begin{cases} N_+^{(n)} = \sum_{m=0}^n (a_m h_1(n-m) - b_m h_2(n-m) + c_m h_3(n-m) + d_m h_4(n-m) + e_m h_5(n-m) \\ + f_m h_6(n-m) + g_m h_7(n-m) + k_m h_8(n-m) + l_m h_9(n-m)), \\ N_-^{(n)} = \lambda^n N - N_+^{(n)}, \end{cases}$$

then Eq.(8) can be written as

$$-N_{+x}^{(n)} + [M, N_+^{(n)}] = N_{-x}^{(n)} - [M, N_-^{(n)}]. \quad (10)$$

a direct calculation reads

$$\begin{aligned} -N_{+x}^{(n)} + [M, N_+^{(n)}] = & -a_{nx}h_1(0) - 2b_{n+1}h_2(0) + 2c_{n+1}h_3(0) - d_{nx}h_4(0) - 2e_{n+1}h_5(0) \\ & + 2f_{n+1}h_6(0) - g_{nx}h_7(0) - 2k_{n+1}h_8(0) + 2l_{n+1}h_9(0). \end{aligned}$$

Taking $V^{(n)} = V_+^{(n)} + \Delta_n$, with $\Delta_n = -a_n h_1(0) - d_n h_4(0) - g_n h_7(0)$, then we obtain that

$$\begin{aligned} -N_{+x}^{(n)} + [M, N_+^{(n)}] = & (-2b_{n+1} + 2u_1 a_n)h_2(0) + (2c_{n+1} - 2u_2 a_n)h_3(0) - (2e_{m+1} - 2u_1 d_m - 2u_3 a_m)h_5(0) \\ & - (-2f_{m+1} + 2u_2 d_m + 2u_4 a_m)h_6(0) - k_{nx}h_8(0) - l_{nx}h_9(0) \\ = & -b_{nx}h_2(0) - c_{nx}h_3(0) - e_{nx}h_5(0) - f_{nx}h_6(0) - k_{nx}h_8(0) - l_{nx}h_9(0). \end{aligned}$$

Thus, the $(2+1)$ -dimensional partial-differential equation hierarchy

$$M_t - N_x^{(n)} + [M, N^{(n)}] - M_y = 0, \quad (11)$$

admits the following solution

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{t-y} = \begin{pmatrix} b_{nx} \\ c_{nx} \\ e_{nx} \\ f_{nx} \\ k_{nx} \\ l_{nx} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \partial \\ 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & \partial & 0 & -\partial \\ 0 & 0 & \partial & 0 & -\partial & 0 \\ 0 & \partial & 0 & -\partial & 0 & 0 \\ \partial & 0 & -\partial & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_n + f_n + l_n \\ b_n + e_n + k_n \\ c_n + f_n \\ b_n + e_n \\ c_n \\ b_n \end{pmatrix} = J W_n, \quad (12)$$

where J is a Hamiltonian operator. According to the definition of integrable couplings, we conclude that the system (12) is a type of the $(2+1)$ -dimensional integrable model of the KN hierarchy since taking $u_3 = u_4 = u_5 = u_6 = 0$, $u_1 = q$, $u_2 = r$, $\partial_y = 0$ the system (12) reduces to the KN hierarchy. In what follows, we look for the Hamiltonian structure of the system (12). We make use of the following quadratic-form identity. Let $a, b \in \mathfrak{su}(6)$, s -order matrix $R(b)$ is determined by [9]

$$[a, b]^T = (a_1, a_2, \dots, a_9) R(b) = a^T R(b),$$

and constant matrix $F = (f_{ij})_{s \times s}$, is determined by

$$F = F^T, R(b)F = -(R(b)F)^T. \quad (13)$$

We introduce quadratic-form identity functional

$$\{a, b\} = a^T F b, \quad a, b \in s \tilde{\mu} \quad (6),$$

then, we obtain

$$\frac{\delta}{\delta u_i} \{V, U_\lambda\} = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \left(\lambda^\gamma \left\{ V, \frac{\partial U}{\partial u_i} \right\} \right), \quad 1 \leq i \leq l, l \text{ is a constant} \quad (14)$$

where λ is a constant to be determined, we call (33) the quadratic-form identity[9]. we have

$$[a, b]^T = a^T \begin{pmatrix} 0 & 2b_2 & -b_3 & 0 & 2b_5 & -2b_6 & 0 & 2b_8 & -2b_9 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 & b_9 & -2b_7 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 & -b_8 & 0 & 2b_7 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ 0 & 0 & 0 & b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\ 0 & 0 & 0 & -b_1 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_3 & -2b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_2 & 0 & 2b_1 \end{pmatrix} = a^T R(b).$$

Solving the matrix equation $F^T = F$ and $R(b)F = -(R(b)F)^T$, give rise to

$$F = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In order to get the Hamiltonian structure, Let

$$\begin{cases} \varphi_x = \tilde{M} \varphi, \quad \tilde{M} = (\lambda^2, u_1 \lambda, u_2 \lambda, 0, u_3 \lambda, u_4 \lambda, 0, u_5 \lambda, u_6 \lambda)^T \\ \varphi_t = \tilde{N} \varphi + \varphi_y, \quad \tilde{N} = \sum_{m=0}^n (a_m, \lambda b_m, \lambda c_m, d_m, \lambda e_m, \lambda f_m, g_m, \lambda k_m, \lambda l_m)^T \lambda^{-2m} - (a_n, 0, 0, d_n, 0, 0, g_n, 0, 0)^T. \end{cases}$$

In terms of (13), it is easy to compute that

$$\begin{cases} \left\{ N, \frac{\partial M}{\partial u_1} \right\} = \lambda^2(c+f+l), & \left\{ N, \frac{\partial M}{\partial u_2} \right\} = \lambda^2(b+e+k), & \left\{ N, \frac{\partial M}{\partial u_3} \right\} = \lambda^2(c+f), \\ \left\{ N, \frac{\partial M}{\partial u_4} \right\} = \lambda^2(b+e), & \left\{ N, \frac{\partial M}{\partial u_5} \right\} = \lambda^2 c, & \left\{ N, \frac{\partial M}{\partial u_6} \right\} = \lambda^2 b, \end{cases}$$

$\left\{N, \frac{\partial M}{\partial \lambda}\right\} = 2\lambda(2a+2d+2g) + u_1\lambda(c+f+l) + u_2\lambda(b+e+k) + u_3\lambda(c+f) + u_4\lambda(b+e) + u_5\lambda c + u_6\lambda b$ where
 $a = \sum_{m \geq 0} a_m \lambda^{-2m}, b = \sum_{m \geq 0} b_m \lambda^{-2m}, \dots$ Substituting the above formulae into the quadratic identity, comparison of the coefficient of λ^{-2n+1} yields

$$\frac{\delta}{\delta u}(4a_n+4d_n+4g_n+u_1(c_n+f_n+l_n)+u_2(b_n+e_n+k_n)+u_3(c_n+f_n)+u_4(b_n+e_n)+u_5c_n+u_6b_n)=(\gamma-2n+2) \begin{pmatrix} c_n+f_n+l_n \\ b_n+e_n+k_n \\ c_n+f_n \\ b_n+e_n \\ c_n \\ b_n \end{pmatrix}.$$

Taking $n=1$ gives $\gamma = 0$. Therefore,

$$\begin{cases} \frac{\delta H_n}{\delta u} = W_n, \\ H_n = \frac{4a_n+4d_n+4g_n+u_1(c_n+f_n+l_n)+u_2(b_n+e_n+k_n)+u_3(c_n+f_n)+u_4(b_n+e_n)+u_5c_n+u_6b_n}{2-2n}. \end{cases} \quad (15)$$

Therefore, we obtain the Hamiltonian structure of the integrable coupling (12)

$$u_{t-y} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{t-y} = J \frac{\delta H_n}{\delta u} \quad (16)$$

Remark 1

In the paper, we construct a higher-dimensional 6×6 matrix Lie algebra $s\mu(6)$, which can obtain the integrable hierarchy with more-potential functions and its Hamiltonian structure. The method also can be used to generalize other integrable hierarchies, such as TC hierarchy, BPT hierarchy, Burgers hierarchy, etc.

3 A decomposed Lie algebra E from the Lie algebra $s\mu(6)$

Decompose h_1, h_4, h_7 in $s\mu(6)$ into the elements $e_1, e_2, e_5, e_6, e_9, e_{10}$, we have

$$\begin{aligned} e_1 &= \begin{pmatrix} h_1^* & 0 & 0 \\ 0 & h_1^* & 0 \\ 0 & 0 & h_1^* \end{pmatrix}, e_2 = \begin{pmatrix} h_2^* & 0 & 0 \\ 0 & h_2^* & 0 \\ 0 & 0 & h_2^* \end{pmatrix}, e_3 = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}, e_4 = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix}, \\ e_5 &= \begin{pmatrix} 0 & h_1^* & 0 \\ 0 & 0 & h_1^* \\ 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & h_2^* & 0 \\ 0 & 0 & h_2^* \\ 0 & 0 & 0 \end{pmatrix}, e_7 = \begin{pmatrix} 0 & e & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix}, e_8 = \begin{pmatrix} 0 & f & 0 \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}, \\ e_9 &= \begin{pmatrix} 0 & 0 & h_1^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_{10} = \begin{pmatrix} 0 & 0 & h_2^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_{11} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 0 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Where $h_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $h_2^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, it is easy to verify that E is a Lie algebra. Denote $E_1 = \text{span}\{e_1, e_2, e_3, e_4\}$, $E_2 = \text{span}\{e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\}$, we also see that

$$E = E_1 \oplus E_2, [E_1, E_2] \subset E_2.$$

Observe that E_1 and E_2 are closed under the following matrix multiplication

$$\begin{aligned} & e_1 e_1 = e_1, e_2 e_2 = e_2, e_5 e_5 = e_9, e_6 e_6 = e_{10}, \\ & e_1 e_2 = e_2 e_1 = e_3 e_3 = e_4 e_4 = e_7 e_7 = e_8 e_8 = e_9 e_9 = e_{10} e_{10} = 0, \\ & e_1 e_{10} = e_{10} e_1 = e_{11} e_1 = e_1 e_{11} = e_1 e_{12} = e_{12} e_1 = e_4 e_8 = e_8 e_4 = e_4 e_{12} = e_{12} e_4 = 0, \\ & e_2 e_5 = e_5 e_2 = 0, e_2 e_6 = e_6 e_2 = e_6, e_2 e_9 = e_9 e_2 = e_2 e_{11} = e_{11} e_2 = 0, \\ & \begin{cases} e_1 e_3 = e_3, \\ e_3 e_1 = 0, \end{cases} \begin{cases} e_1 e_4 = 0, \\ e_4 e_1 = e_4, \end{cases} \begin{cases} e_1 e_7 = e_7, \\ e_7 e_1 = 0, \end{cases} \begin{cases} e_1 e_8 = 0, \\ e_8 e_1 = e_8, \end{cases} \\ & e_1 e_5 = e_5 e_1 = e_5, e_1 e_6 = e_6 e_1 = 0, e_1 e_9 = e_9 e_1 = e_9, e_3 e_7 = e_7 e_3 = 0, \\ & \begin{cases} e_2 e_3 = 0, \\ e_3 e_2 = e_3, \end{cases} \begin{cases} e_2 e_4 = e_4, \\ e_4 e_2 = 0, \end{cases} \begin{cases} e_2 e_7 = 0, \\ e_7 e_2 = e_7, \end{cases} \begin{cases} e_2 e_8 = e_8, \\ e_8 e_2 = 0, \end{cases} \begin{cases} e_2 e_{10} = 0, \\ e_{10} e_2 = e_{10}, \end{cases} \begin{cases} e_2 e_{12} = e_{12}, \\ e_{12} e_2 = 0, \end{cases} \\ & \begin{cases} e_3 e_4 = e_1, \\ e_4 e_3 = e_2, \end{cases} \begin{cases} e_3 e_5 = 0, \\ e_5 e_3 = e_7, \end{cases} \begin{cases} e_3 e_6 = e_7, \\ e_6 e_3 = 0, \end{cases} \begin{cases} e_3 e_8 = e_5, \\ e_8 e_3 = e_6, \end{cases} \begin{cases} e_3 e_9 = 0, \\ e_9 e_3 = e_9, \end{cases} \begin{cases} e_3 e_{10} = e_{11}, \\ e_{10} e_3 = 0, \end{cases} \\ & \begin{cases} e_4 e_5 = e_8, \\ e_5 e_4 = 0, \end{cases} \begin{cases} e_4 e_6 = 0, \\ e_6 e_4 = e_8, \end{cases} \begin{cases} e_4 e_7 = e_6, \\ e_7 e_4 = e_5, \end{cases} \begin{cases} e_4 e_9 = e_{12}, \\ e_9 e_4 = 0, \end{cases} \begin{cases} e_4 e_{10} = 0, \\ e_{10} e_4 = e_{12}, \end{cases} \begin{cases} e_4 e_{11} = e_{10}, \\ e_{11} e_4 = e_9, \end{cases} \\ & \begin{cases} e_5 e_7 = e_{11}, \\ e_7 e_5 = 0, \end{cases} \begin{cases} e_5 e_8 = 0, \\ e_8 e_5 = e_{12}, \end{cases} \begin{cases} e_6 e_7 = 0, \\ e_7 e_6 = e_{11}, \end{cases} \begin{cases} e_6 e_8 = e_{12}, \\ e_8 e_6 = 0, \end{cases} \begin{cases} e_7 e_8 = e_9, \\ e_8 e_7 = e_{10}, \end{cases} \begin{cases} e_3 e_{12} = e_9, \\ e_{12} e_3 = e_{10}, \end{cases} \\ & e_5 e_6 = e_6 e_5 = e_5 e_9 = e_9 e_5 = e_6 e_9 = e_9 e_6 = e_6 e_{10} = e_{10} e_6 = e_7 e_8 = e_8 e_7 = 0. \end{aligned}$$

Remark 2

For computing convenience, the multiplication relations among e_{12} and the other elements in E are not presented here.

It is easy to see that $E_1 E_2, E_2 E_1 \subset E_2$, which is key idea to generate discrete integrable couplings. In addition, the spectral matrices) are of the form

$$\bar{N} = \begin{pmatrix} U & \bar{\bar{U}}_1 & \bar{\bar{U}}_2 \\ O & U & \bar{\bar{U}}_1 \\ O & O & U \end{pmatrix} \quad (17)$$

where $U, \bar{\bar{U}}_1, \bar{\bar{U}}_2$ are 2×2 matrices, respectively, O stands for a 2×2 zero matrix. According to various Jordan blocks, a general classification of the Lie algebras is followed. More detailed cases will be discussed in another paper.

Consider an isospectral problem

$$EU_n = U_n \psi, U_n = \begin{pmatrix} 0 & -\frac{1}{r_n} & 0 & u_{1n} & 0 & u_{4n} \\ r_n & \lambda + s_n & u_{2n} & v_n & u_{5n} & u_{3n} \\ 0 & 0 & 0 & -\frac{1}{r_n} & 0 & u_{1n} \\ 0 & 0 & r_n & \lambda + s_n & u_{2n} & v_n \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{r_n} \\ 0 & 0 & 0 & 0 & r_n & \lambda + s_n \end{pmatrix} \quad (18)$$

$$= \lambda e_2 + s_n e_2 - \frac{1}{r_n} e_3 + r_n e_4 + v_n e_6 + u_{1n} e_7 + u_{2n} e_8 + u_{3n} e_{10} + u_{4n} e_{11} + u_{5n} e_{12}.$$

Note $\Gamma = \sum_{i \geq 0} (a_i(e_1 - e_2) + b_i e_3 + c_i e_4 + f_i e_5 - f_i e_6 + g_i e_7 + h_i e_8 + k_i e_9 - k_i e_{10} + l_i e_{11} + w_i e_{12})$
 $\lambda^{-i} \equiv a(e_1 - e_2) + b e_3 + c e_4 + f e_5 - f e_6 + g e_7 + h e_8 + k e_9 - k e_{10} + l e_{11} + w e_{12},$

solving the discrete stationary zero curvature equation

$$(E\Gamma)U - U\Gamma = 0 \quad (19)$$

yields

$$\begin{cases} r_n b_m^{(1)} - \frac{1}{r_n} c_m = 0, \\ -\frac{1}{r_n} (a_m^{(1)} + a_m) + b_{m+1}^{(1)} + s_n b_m^{(1)} = 0, \\ r_n (a_m^{(1)} + a_m) + c_{m+1} + s_n c_m = 0, \\ \frac{1}{r_n} c_m^{(1)} + r_n b_m + s_n (a_m^{(1)} - a_m) + (a_{m+1}^{(1)} - a_{m+1}) = 0, \\ b_m^{(1)} u_{2n} + r_n g_m^{(1)} + \frac{1}{r_n} h_m - c_m u_{1n} = 0, \\ c_m^{(1)} u_{1n} - a_m^{(1)} v_n - \frac{1}{r_n} h_m^{(1)} - s_n f_m^{(1)} - f_{m+1}^{(1)} - r_n k_m^{(1)} - r_n g_m + f_{m+1} + s_n f_m - b_m u_{2n} + a_m v_n = 0, \\ a_m^{(1)} u_{1n} + v_n b_m^{(1)} + g_{m+1}^{(1)} + s_n g_m^{(1)} - \frac{1}{r_n} (f_m^{(1)} + f_m) + a_m u_{1n} = 0, \\ a_m^{(1)} u_{2n} + r_n f_m^{(1)} + f_m r_n + h_{m+1} + s_n h_m + v_n c_m + a_m u_{2n} = 0, \\ u_{5n} b_m^{(1)} + u_{2n} g_m^{(1)} + r_n l_m^{(1)} + \frac{1}{r_n} w_m - u_{1n} h_m - u_{4n} c_m = 0, \\ u_{4n} c_m^{(1)} - a_m^{(1)} u_{3n} + h_m^{(1)} u_{1n} - f_m^{(1)} v_n - \frac{1}{r_n} w_m^{(1)} - k_{m+1}^{(1)} - s_n k_m^{(1)} + s_n k_m + k_{m+1} - r_n l_m - \\ u_{2n} g_m + u_{3n} a_m - u_{5n} b_m + v_n f_m = 0, \\ u_{4n} a_m^{(1)} + b_m^{(1)} u_{3n} + k_m^{(1)} u_{1n} + g_m^{(1)} v_n - \frac{1}{r_n} k_m^{(1)} + u_{1n} f_m + l_{m+1}^{(1)} + s_n l_m^{(1)} - \frac{1}{r_n} k_m + u_{4n} a_m = 0, \\ u_{5n} a_m^{(1)} + u_{2n} f_m^{(1)} + k_m^{(1)} r_n + k_m r_n + w_{m+1} + s_n w_m + f_m u_{2n} + v_n h_m + a_m u_{5n} + c_m u_{3n} = 0. \end{cases} \quad (20)$$

Through some deduction, it is easy to see that the second and seventh and eleventh equation in (20) can be derived from the others. Hence, the three equations are superfluous.

Noting

$$\Gamma_+^{(m)} = \sum_{i=0}^m (a_i(e_1 - e_2) + b_i e_3 + c_i e_4 + f_i e_5 - f_i e_6 - g_i e_7 + h_i e_8 + k_i e_9 - k_i e_{10} + l_i e_{11} + w_i e_{12}) \lambda^{m-i} = \lambda^m \Gamma - \Gamma_-^{(m)}$$

, then Eq.(20) can be written as

$$(E\Gamma_+^{(m)})U_n - U_n \Gamma_+^{(m)} = - (E\Gamma_-^{(m)})U_n + U_n \Gamma_-^{(m)}. \quad (21)$$

A direct calculation reads

$$(E\Gamma_+^{(m)})U_n - U_n\Gamma_+^{(m)} = -b_{m+1}^{(1)}e_3 + c_{m+1}e_4 + (a_{m+1}^{(1)} - a_{m+1})e_2 + (f_{m+1}^{(1)} - f_{m+1})e_6 - g_{m+1}^{(1)}e_7 + h_{m+1}e_8 + (k_{m+1}^{(1)} - k_{m+1})e_{10} - l_{m+1}^{(1)}e_{11} + w_{m+1}e_{12}$$

Take $\Gamma^{(m)} = \Gamma_+^{(m)}$, the discrete zero curvature equation

$$(U_n)_{t_m} = (E\Gamma^{(m)})U_n - U_n\Gamma^{(m)} \quad (22)$$

generates the discrete integrable hierarchy

$$\begin{cases} (r_n)_{t_m} = c_{m+1}, \\ (s_n)_{t_m} = a_{m+1}^{(1)} - a_{m+1}, \\ (v_n)_{t_m} = -f_{m+1}^{(1)} - f_{m+1}, \\ (u_{1n})_{t_m} = -g_{m+1}^{(1)}, \\ (u_{2n})_{t_m} = h_{m+1}, \\ (u_{3n})_{t_m} = k_{m+1}^{(1)} - k_{m+1}, \\ (u_{4n})_{t_m} = -l_{m+1}^{(1)}, \\ (u_{5n})_{t_m} = w_{m+1}. \end{cases} \quad (23)$$

The first two equations in (23) were presented in Ref.[21]. Hence, the discrete lattice hierarchy (23) is the discrete integrable couplings of the first two equations.

Remark 3

The above example indicates a way to generate discrete integrable couplings by the Lie algebra, but there is an open problem: Could we generalize the continuous-type quadratic-form identity in [9] into the discrete-type one so that the Hamiltonian structures of the discrete integrable couplings could be obtained? which is worth studying in the future.

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