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Three semi-direct sum Lie algebras and three discrete integrable couplings associated with the modified KdV lattice equation

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Abstract

Three semi-direct sum Lie algebras are constructed, which is an efficient and new way to obtain discrete integrable couplings. As its applications, three discrete integrable couplings associated with the modified KdV lattice equation are worked out. The approach can be used to produce other discrete integrable couplings of the discrete hierarchies of soliton equations.

Keywords

lattice equation; modified KdV lattice equation; Lie algebra

1 Introduction

In recent years, the study of the nonlinear integrable lattice equations has become the focus of the common concern in the theory of integrable systems[1-3]. Many nonlinear integrable lattice equations have been proposed, such as The Bargmann lattice, the Toda-type lattice, the Volterra lattice, and so on. Integrable couplings is a new and significant direction in soliton theory, which were proposed in view of the Virasoro algebra and the soliton theory. A few ways to construct integrable couplings were presented by use of perturbations[4-6]. A simple and efficient method for generating integrable couplings was given in Refs.[7,8]. It followed that some integrable couplings of the known integrable solitary hierarchies of equations were obtained[9-19]. The study of integrable couplings not only generalizes the symmetry problem, but also provides clues towards complete classification of integrable systems.

But recently, Zhang constructed two semi-direct sum lie algebras to generate discrete integrable couplings of the Toda-type lattice equations[20]. Zhang gave an efficient and new way to obtain discrete integrable couplings with the help of the semi-direct sum Lie algebras. In this paper, we want to construct three semi-direct sum Lie algebras to generate discrete integrable couplings. As their applications, three discrete integrable couplings associated with the modified KdV lattice equation are worked out. The approach can be devoted to establishing other discrete integrable couplings of the discrete hierarchies of evolution equations.

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2 Three semi-direct sum Lie algebras

In this paper, we denote by E the shift operator as follows[12]

$$(E^m x)(n) = x(m+n), \text{ where } x: \mathbf{Z} \rightarrow \mathbf{R}, m, n \geq 0, D = E - 1. \quad (1)$$

Let G_1 be a matrix Lie algebra which is closed under matrix multiplication. Assume an integrable lattice equation of evolution type is given by

$$u_t = K(u) = K(u, Eu, E^{-1}u, \dots) \quad (2)$$

which is related to the G_1 , where $u = u(n, t)$. Usually, the system (2) is generated by the compatibility condition of the following isospectral problem

$$\begin{cases} E\varphi = U\varphi = U(u, \lambda)\varphi, \\ \varphi_t = V\varphi = V(u, Eu, E^{-1}u, \dots; \lambda), \end{cases} \quad (3)$$

where λ is a spectral parameter. To construct an integrable couplings of the lattice equation (2), we use semi-direct sums of Lie algebras to enlarge the original Lie algebra G_1 . Again take another matrix Lie algebra G_2 closed under matrix multiplication and then form a semi-direct sum G of G_1 and G_2 :

$$G = G_1 \dot{+} G_2. \quad (4)$$

We also require the closure between G_1 and G_2 under matrix multiplication

$$G_1 G_2, G_2 G_1 \subseteq G_2, \quad (5)$$

where $G_1 G_2 = \{U_1 U_2 \mid U_1 \in G_1, U_2 \in G_2\}$, to ensure a Lax pair from the semi-direct sum G_1 generating coupling system. In Ref.[12], Ma Wenxiu gave the general scheme for producing discrete integrable couplings by semi-direct sums of Lie algebras. Two types of general spectral matrix representations were proposed by him. The first one is that

$$\bar{U} = \begin{pmatrix} U & U_{12} & \dots & U_{1,\mu+1} \\ & \ddots & \ddots & \vdots \\ & & U & U_{\mu,\mu+1} \\ 0 & & & 0 \end{pmatrix}, \bar{V} = \begin{pmatrix} V & V_{12} & \dots & V_{1,\mu+1} \\ & \ddots & \ddots & \vdots \\ & & V & V_{\mu,\mu+1} \\ 0 & & & 0 \end{pmatrix},$$

The second one presents that

$$\bar{U} = \begin{pmatrix} U & U_{a_1} & \cdots & U_{a_v} \\ & U & \ddots & \vdots \\ & & \ddots & U_{a_1} \\ 0 & & & U \end{pmatrix}, \quad \bar{V} = \begin{pmatrix} V & V_{a_1} & \cdots & V_{a_v} \\ & V & \ddots & \vdots \\ & & \ddots & V_{a_1} \\ 0 & & & V \end{pmatrix}.$$

In the paper, we take the first case proposed by Ma and employ the following special Lie algebra

$$\tilde{G}_1 = \text{span}\{\tilde{h}_1, \tilde{h}_2, \tilde{e}, \tilde{f}\}, \quad (6)$$

where $\tilde{h}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\tilde{h}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\tilde{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tilde{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ along with the commutative operations $[\tilde{h}_1, \tilde{h}_2] = 0$, $[\tilde{h}_1, \tilde{e}] = \tilde{e}$, $[\tilde{h}_1, \tilde{f}] = -\tilde{f}$, $[\tilde{h}_2, \tilde{e}] = -\tilde{e}$, $[\tilde{h}_2, \tilde{f}] = \tilde{h}_1$, $[\tilde{e}, \tilde{f}] = \tilde{h}_1 - \tilde{h}_2$, enlarge G_1 as another Lie algebra

$$G = \text{span}\{h_1, h_2, e, f, e_+, e_-, f_+\}, \quad (7)$$

where $h_1, h_2, e, f, e_+, e_-, f_+$ is the same in Ref[20] then we find $\tilde{G}_1$ is isomorphic to $G_1 = \text{span}\{h_1, h_2, e, f\}$. Denote by $G_2 = \text{span}\{e_+, e_-, f_+\}$, we see that

$$G = G_1 \dot{+} G_2, [G_1, G_2] \subseteq G_2, \quad (8)$$

G_2 is an Abelian ideal of the Lie algebra G . Furthermore, we also see that G is closed under the following matrix multiplication in Ref[20]

It is easy to find that the spectral matrices U and V given in (3) present in the form in terms of $\{h_1, h_2, e, f, e_+, e_-\}$ except f_+ :

$$M = \begin{pmatrix} U & \tilde{U} \\ 0 & 0 \end{pmatrix}, \quad (9)$$

where U is a 2×2 matrix, $\tilde{U} = (e, f)^T$.

Actually, the Lie algebras satisfying the relation (8) are not unique, maybe there are a lot of ones. For example, take

$$\tilde{G} = \text{span}\{g_i\}_{i=1}^9, \quad (10)$$

where $g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9$ is the same in Ref[20].

It is easy to verify that $\tilde{G}$ is a Lie algebra. Denote $\tilde{G}_1 = \text{span}\{g_1, g_2, g_3, g_4\}$, $\tilde{G}_2 = \text{span}\{g_5, g_6, g_7, g_8, g_9\}$, we also see that

$$\tilde{G} = \tilde{G}_1 + \tilde{G}_2, [\tilde{G}_1, \tilde{G}_2] \subseteq \tilde{G}_2. \quad (11)$$

$\tilde{G}_2$ is also an Abelian ideal of the Lie algebra $\tilde{G}$. Observe that $\tilde{G}_1$ and $\tilde{G}_2$ are closed under the following matrix multiplication in Ref[20].

It is easy to see that $\tilde{G}_1\tilde{G}_2, \tilde{G}_2\tilde{G}_1 \subseteq \tilde{G}_2$, which is key idea to generate discrete integrable couplings. In addition, the spectral matrices appeared in (3) are of the form

$$N = \begin{pmatrix} U & \bar{U}_1 \\ O & \bar{U}_2 \end{pmatrix}, \quad (12)$$

where $U, \bar{U}_1, \bar{U}_2$ are 2×2 matrices, respectively, O stands for a 2×2 zero matrix. We enlarge $\tilde{G}$ as another Lie algebra $\bar{G}$ again. Take

$$\bar{G} = \text{span}\{e_i\}_{i=1}^{12}, \quad (13)$$

where

$$\begin{aligned} e_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_7 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to verify that $\bar{G}$ is a Lie algebra, $\bar{G}$ is isomorphic to $\bar{G}_1 = \text{span}\{e_1, e_2, e_3, e_4\}$, Denote by $\bar{G}_2 = \text{span}\{e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\}$, we also see that

$$\overline{\overline{G}} = \overline{\overline{G}}_1 + \overline{\overline{G}}_2, \left[\overline{\overline{G}}_1, \overline{\overline{G}}_2 \right] \subseteq \overline{\overline{G}}_2. \quad (14)$$

$\overline{\overline{G}}_2$ is also an Abelian ideal of the Lie algebra $\overline{\overline{G}}$. Observe that $\overline{\overline{G}}_1$ and $\overline{\overline{G}}_2$ are closed under the following matrix multiplication

$$\begin{aligned} & e_1 e_1 = e_1, e_2 e_2 = e_2, e_5 e_5 = e_9, e_6 e_6 = e_{10}, \\ & e_1 e_2 = e_2 e_1 = e_3 e_3 = e_4 e_4 = e_7 e_7 = e_8 e_8 = e_9 e_9 = e_{10} e_{10} = 0, \\ & e_1 e_{10} = e_{10} e_1 = e_{11} e_1 = e_1 e_{11} = e_1 e_{12} = e_{12} e_1 = 0, \\ & e_2 e_5 = e_5 e_2 = 0, e_2 e_6 = e_6 e_2 = e_6, e_2 e_9 = e_9 e_2 = e_2 e_{11} = e_{11} e_2 = 0, \\ & \begin{cases} e_1 e_3 = e_3, \\ e_3 e_1 = 0, \end{cases} \begin{cases} e_1 e_4 = 0, \\ e_4 e_1 = e_4, \end{cases} \begin{cases} e_1 e_7 = e_7, \\ e_7 e_1 = 0, \end{cases} \begin{cases} e_1 e_8 = 0, \\ e_8 e_1 = e_8, \end{cases} \\ & e_1 e_5 = e_5 e_1 = e_5, e_1 e_6 = e_6 e_1 = 0, e_1 e_9 = e_9 e_1 = e_9, e_3 e_7 = e_7 e_3 = 0, \\ & \begin{cases} e_2 e_3 = 0, \\ e_3 e_2 = e_3, \end{cases} \begin{cases} e_2 e_4 = e_4, \\ e_4 e_2 = 0, \end{cases} \begin{cases} e_2 e_7 = 0, \\ e_7 e_2 = e_7, \end{cases} \begin{cases} e_2 e_8 = e_8, \\ e_8 e_2 = 0, \end{cases} \begin{cases} e_2 e_{10} = 0, \\ e_{10} e_2 = e_{10}, \end{cases} \begin{cases} e_2 e_{12} = e_{12}, \\ e_{12} e_2 = 0, \end{cases} \\ & \begin{cases} e_3 e_4 = e_1, \\ e_4 e_3 = e_2, \end{cases} \begin{cases} e_3 e_5 = 0, \\ e_5 e_3 = e_7, \end{cases} \begin{cases} e_3 e_6 = e_7, \\ e_6 e_3 = 0, \end{cases} \begin{cases} e_3 e_8 = e_5, \\ e_8 e_3 = e_6, \end{cases} \begin{cases} e_3 e_9 = 0, \\ e_9 e_3 = e_9, \end{cases} \begin{cases} e_3 e_{10} = e_{11}, \\ e_{10} e_3 = 0, \end{cases} \\ & \begin{cases} e_3 e_{12} = e_9, \\ e_{12} e_3 = e_{10}, \end{cases} e_4 e_8 = e_8 e_4 = e_4 e_{12} = e_{12} e_4 = 0, \\ & \begin{cases} e_4 e_5 = e_8, \\ e_5 e_4 = 0, \end{cases} \begin{cases} e_4 e_6 = 0, \\ e_6 e_4 = e_8, \end{cases} \begin{cases} e_4 e_7 = e_6, \\ e_7 e_4 = e_5, \end{cases} \begin{cases} e_4 e_9 = e_{12}, \\ e_9 e_4 = 0, \end{cases} \begin{cases} e_4 e_{10} = 0, \\ e_{10} e_4 = e_{12}, \end{cases} \begin{cases} e_4 e_{11} = e_{10}, \\ e_{11} e_4 = e_9, \end{cases} \\ & \begin{cases} e_5 e_7 = e_{11}, \\ e_7 e_5 = 0, \end{cases} \begin{cases} e_5 e_8 = 0, \\ e_8 e_5 = e_{12}, \end{cases} \begin{cases} e_6 e_7 = 0, \\ e_7 e_6 = e_{11}, \end{cases} \begin{cases} e_6 e_8 = e_{12}, \\ e_8 e_6 = 0, \end{cases} \begin{cases} e_7 e_8 = e_9, \\ e_8 e_7 = e_{10}, \end{cases} \\ & e_5 e_6 = e_6 e_5 = e_5 e_9 = e_9 e_5 = e_6 e_9 = e_9 e_6 = e_6 e_{10} = e_{10} e_6 = e_7 e_8 = e_8 e_7 = 0, \end{aligned}$$

Remark 1: For computing convenience, the multiplication relations among e_{12} and the other elements in $\overline{\overline{G}}_2$ are not presented here.

It is easy to see that $\overline{\overline{G}}_1 \overline{\overline{G}}_2, \overline{\overline{G}}_2 \overline{\overline{G}}_1 \subseteq \overline{\overline{G}}_2$, which is key idea to generate discrete integrable couplings. In addition, the spectral matrices appeared in (3) are of the form

$$\overline{N} = \begin{pmatrix} U & \overline{\overline{U}}_1 & \overline{\overline{U}}_2 \\ O & U & \overline{\overline{U}}_1 \\ O & O & U \end{pmatrix} \quad (15)$$

where $U, \overline{\overline{U}}_1, \overline{\overline{U}}_2$ are 2×2 matrices, respectively, O stands for a 2×2 zero matrix. According to various Jordan blocks expressed such as (13), a general classification of the Lie algebras is followed. More detailed cases will be discussed in another paper.

In this paper, we want to use the Lie algebra $G, \tilde{G}$ and $\overline{\overline{G}}$ to generate three discrete integrable couplings of the modified KdV lattice type equation respectively.

3 Three discrete integrable couplings

The modified KdV lattice equation refers to

$$r_{n_t} = r_n^2 (r_{n-1} - r_{n+1}) \quad (16)$$

which is a simplified form of one result presented in [2]. We want to enlarge (16) into a larger discrete integrable system which can be expressed as the form

$$\begin{cases} u_l = K(u), \\ v_l = S(u, v) \end{cases} \quad (17)$$

where $S(u, v) = S(u, v, Eu, Ev, E^{-1}u, E^{-1}v, \dots; \lambda)$ should satisfy the non-triviality condition

$\frac{\partial S}{\partial [u]} \neq 0$, $[u]$ denotes a vector consisting of all derivatives of u with respect to the space variable. For the above sake, take an isospectral problem resorting to the Lie algebra G

$$E\psi = U_n\psi, U_n = \begin{pmatrix} 0 & r_n & u_n \\ \lambda s_n & \lambda & \lambda v_n \\ 0 & 0 & 0 \end{pmatrix} = \lambda h_2 + r_n e + \lambda s_n f + u_n e_+ + \lambda v_n e_-. \quad (18)$$

Note $\Gamma = \sum_{i \geq 0} (a_i(h_1 - h_2) + b_i e + \lambda c_i f + d_i e_+ + g_i e_-) \lambda^{-i} \equiv a(h_1 - h_2) + be + \lambda cf + de_+ + ge_-$, solving the discrete stationary zero curvature equation for Γ

$$(E\Gamma)U_n - U_n\Gamma = 0 \quad (19)$$

yields

$$\begin{cases} s_n b_m^{(1)} - r_n c_m = 0, \\ r_n (a_m^{(1)} + a_m) + b_{m+1}^{(1)} = 0, \\ s_n (a_m^{(1)} + a_m) + c_{m+1} = 0, \\ r_n c_m^{(1)} - (a_m^{(1)} - a_m) - s_n b_m = 0, \\ a_m^{(1)} u_n + b_{m+1}^{(1)} v_n - g_m r_n = 0, \\ c_m^{(1)} u_n + a_m^{(1)} v_n - s_n d_m - g_m = 0, \\ a_0 = -\frac{1}{2}, b_0 = 0, c_0 = d_0 = g_0 = 0, \\ a_1 = r_n^{(-1)} s_n, b_1 = r_n^{(-1)}, c_1 = s_n, \\ d_1 = \frac{r_n s_n^{(-1)} v_n + (r_n^{(-1)} s_n + r_n s_n^{(1)}) v_n}{s_n}, \\ g_1 = s_n^{(1)} u_n - (r_n^{(-1)} s_n + r_n s_n^{(1)}) v_n. \end{cases} \quad (20)$$

It is easy to see that

$$\begin{aligned} s_n(r_n(a_m^{(1)} + a_m) + b_{m+1}^{(1)}) &= s_n r_n a_m^{(1)} + s_n b_{m+1}^{(1)} + s_n r_n a_m = s_n r_n a_m^{(1)} + r_n c_{m+1} + s_n r_n a_m = \\ s_n r_n a_m^{(1)} + r_n(-s_n a_m^{(1)} - s_n a_m) + s_n r_n a_m &= 0. \end{aligned}$$

That is to say, the second equation in (20) can be derived from the others. Thus, it is superfluous, where $Df_n = (E - 1)f_n = f_{n+1} - f_n$.

Denoting $\Gamma_+^{(m)} = \sum_{i=0}^m (a_i(h_1 - h_2) + b_i e + \lambda c_i f + d_i e_+ + g_i e_-) \lambda^{m-i} = \lambda^m \Gamma - \Gamma_-^{(m)}$, then Eq.(19) can be written as

$$(E\Gamma_+^{(m)})U_n - U_n\Gamma_+^{(m)} = -(E\Gamma_-^{(m)})U_n + U_n\Gamma_-^{(m)}. \quad (21)$$

A direct calculation reads

$$\begin{aligned} (E\Gamma_+^{(m)})U_n - U_n\Gamma_+^{(m)} &= -b_{m+1}^{(1)}e + \lambda c_{m+1}f - b_{m+1}^{(1)}v_n e_+ \\ &= \begin{pmatrix} 0 & -b_{m+1}^{(1)} & -b_{m+1}^{(1)}v_n \\ \lambda c_{m+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Take $\Delta_1 = \begin{pmatrix} -2a_m & 0 & 0 \\ 0 & 0 & -g_m \\ 0 & 0 & 0 \end{pmatrix}$, $\Gamma^{(m)} = \Gamma_+^{(m)} + \Delta_1$ then we have

$$(E\Gamma^{(m)})U_n - U_n\Gamma^{(m)} = \begin{pmatrix} 0 & -b_{m+1}^{(1)} - 2a_m^{(1)}r_n & -b_{m+1}^{(1)}v_n - 2a_m^{(1)}u_n + r_n g_n \\ \lambda c_{m+1} + 2\lambda a_m s_n & 0 & \lambda g_m \\ 0 & 0 & 0 \end{pmatrix},$$

Therefore, the discrete zero curvature equation

$$(U_n)_{t_m} = (E\Gamma^{(m)})U_n - U_n\Gamma^{(m)} \quad (22)$$

admits the discrete integrable hierarchy

$$\begin{cases} (r_n)_{t_m} = -b_{m+1}^{(1)} - 2a_m^{(1)}r_n = r_n(a_m - a_m^{(1)}), \\ (s_n)_{t_m} = c_{m+1} + 2a_m s_n = s_n(a_m - a_m^{(1)}), \\ (u_n)_{t_m} = -b_{m+1}^{(1)}v_n - 2a_m^{(1)}u_n + r_n g_m = -a_m^{(1)}u_n, \\ (v_n)_{t_m} = g_m. \end{cases} \quad (23)$$

Taking $m = 1$, $r = s$ from (23) we can obtain

$$\begin{cases} r_{n1} = r_n^2(r_{n-1} - r_{n+1}), \\ u_{n1} = -r_{n+1}r_n u_n, \\ v_{n1} = r_{n+1}u_n - (r_{n-1}r_n + r_{n+1}r_n)v_n. \end{cases} \quad (24)$$

then the first equations in (24) reduces to Eq.(16). Therefore, we call them the modified KdV lattice equations and call (23) the discrete integrable couplings of the modified KdV lattice equation (16) resorting to the definition of integrable couplings[4-6].

In what follows, we make use of the Lie algebra $\tilde{G}$ to generate another discrete integrable couplings of the modified KdV lattice equations appeared in (16).

Consider an isospectral problem $E\psi = U_n\psi$,

$$U_n = \begin{pmatrix} 0 & r_n & u_{3n} & u_{1n} \\ \lambda s_n & \lambda & \lambda u_{2n} & \lambda u_{4n} \\ 0 & 0 & 0 & r_n \\ 0 & 0 & \lambda s_n & \lambda \end{pmatrix} = \lambda g_2 + r_n g_3 + \lambda s_n g_4 + u_{3n} g_5 + \lambda u_{4n} g_6 + u_{1n} g_7 + \lambda u_{2n} g_8$$

$$\Gamma = a(g_1 - g_2) + b g_3 + \lambda c g_4 + f(g_5 - g_6) + e g_7 + \lambda h g_8 = \begin{pmatrix} a & b & f & g \\ \lambda c & -a & \lambda h & -f \\ 0 & 0 & a & b \\ 0 & 0 & \lambda c & -a \end{pmatrix}$$

Denote

where $a = \sum_{i \geq 0} a_i \lambda^{-i}$, $b = \sum_{i \geq 0} b_i \lambda^{-i}$, A solution for Γ from the discrete stationary zero curvature equation $(E\Gamma)U_n - U_n\Gamma = 0$, gives

$$\begin{cases} s_n b_m^{(1)} - r_n c_m = 0, \\ r_n (a_m^{(1)} + a_m) + b_{m+1}^{(1)} = 0, \\ s_n (a_m^{(1)} + a_m) + c_{m+1} = 0, \\ r_n c_m^{(1)} - (a_m^{(1)} - a_m) - s_n b_m = 0, \\ a_m^{(1)} u_{3n} + b_{m+1}^{(1)} u_{2n} + s_n e_{m+1}^{(1)} - r_n h_{m+1} - a_m u_{3n} - c_{m+1} u_{1n} = 0, \\ c_m^{(1)} u_{3n} - a_m^{(1)} u_{2n} - s_n f_m - s_n f_m^{(1)} - h_{m+1} - a_m u_{3n} - c_{m+1} u_{4n} = 0, \\ a_m^{(1)} u_{1n} + b_{m+1}^{(1)} u_{4n} + r_n f_m^{(1)} + e_{m+1}^{(1)} + r_n f_m - b_m u_{3n} + a_m u_{1n} = 0, \\ c_m^{(1)} u_{1n} - a_m^{(1)} u_{4n} + r_n h_m^{(1)} - f_m^{(1)} - e_m s_n + f_m - b_m u_{2n} + a_m u_{4n} = 0, \\ a_0 = -\frac{1}{2}, b_0 = 0, c_0 = e_0 = f_0 = h_0 = 0, \end{cases} \quad (25)$$

Similar to Eqs.(20), we see that $s_n (a_m^{(1)} u_{1n} + b_{m+1}^{(1)} u_{4n} + r_n f_m^{(1)} + e_{m+1}^{(1)} + r_n f_m - b_m u_{3n} + a_m u_{1n}) = 0$. Thus, the second and seventh equation in (25) can be derived from the others.

Note $\Gamma_+^{(m)} = \sum_{i=0}^m (a_i(g_1 - g_2) + b_i g_3 + \lambda c_i g_4 + f_i g_5 - f_i g_6 + e_i g_7 + \lambda h_i g_8) \lambda^{m-i} = \lambda^m \Gamma - \Gamma_-^{(m)}$,

$$\Delta_2 = \begin{pmatrix} -2a_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -f_m \\ 0 & 0 & -2a_m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Gamma^{(m)} = \Gamma_+^{(m)} + \Delta_2, \text{ then a direct computation reads}$$

$$\begin{aligned} (E\Gamma^{(m)}U_n - U_n\Gamma^{(m)}) &= (-b_{m+1}^{(1)} - 2a_m^{(1)}r_n)g_3 + (\lambda c_{m+1}^{(1)} + 2\lambda a_m s_n)g_4 + (r_n h_{m+1} - b_{m+1}^{(1)}u_{2n} + c_{m+1}u_{1n} \\ &\quad - g_{m+1}^{(1)}s_n - 2a_m^{(1)}u_{3n})g_5 + (\lambda f_m^{(1)} - \lambda f_m)g_6 + (-e_{m+1}^{(1)} - r_n f_m - 2a_m^{(1)}u_{1n})g_7 + (\lambda h_{m+1} + \lambda c_{m+1}u_{4n})g_8 \\ &= \begin{pmatrix} 0 & -b_{m+1}^{(1)} - 2a_m^{(1)}r_n & -u_{3n}(a_m + a_m^{(1)}) & -e_{m+1}^{(1)} - r_n f_m - 2a_m^{(1)}u_{1n} \\ \lambda c_{m+1} + 2\lambda a_m s_n & 0 & \lambda h_{m+1} + \lambda c_{m+1}u_{4n} & \lambda f_m^{(1)} - \lambda f_m \\ 0 & 0 & 0 & -b_{m+1}^{(1)} - 2a_m^{(1)}r_n \\ 0 & 0 & \lambda c_{m+1} + 2\lambda a_m s_n & 0 \end{pmatrix} \end{aligned}$$

such that the discrete zero curvature equation

$$(U_n)_{t_m} = (E\Gamma^{(m)})U_n - U_n\Gamma^{(m)}$$

generates the discrete integrable hierarchy

$$\begin{cases} (r_n)_{t_m} = -b_{m+1}^{(1)} - 2a_m^{(1)} r_n = r_n(a_m - a_m^{(1)}), \\ (s_n)_{t_m} = c_{m+1} + 2a_m s_n = s_n(a_m - a_m^{(1)}), \\ (u_{1n})_{t_m} = -e_{m+1}^{(1)} - 2a_m^{(1)} u_{1n} - r_n f_m, \\ (u_{2n})_{t_m} = h_{m+1} + c_{m+1} u_{4n}, \\ (u_{3n})_{t_m} = -u_{3n}(a_m + a_m^{(1)}), \\ (u_{4n})_{t_m} = f_m^{(1)} - f_m. \end{cases} \quad (26)$$

The first two equations in (26) are the same with those in (23). Hence, the hierarchy (26) is also the discrete integrable couplings of the KdV lattice equations.

Consider an isospectral problem resorting to the Lie algebra $\bar{G}$ again to obtain the third discrete integrable couplings of the modified KdV lattice equations appeared in (16).

$$E\psi = U_n\psi, U = \begin{pmatrix} 0r_n & 0 & u_{1n}u_{5n} & u_{3n} \\ \lambda s_n\lambda & \lambda u_{2n}0 & \lambda u_{4n}\lambda u_{6n} & \\ 00 & 0r_n & 0u_{1n} & \\ 00 & \lambda s_n\lambda & \lambda u_{2n}0 & \\ 00 & 00 & 0r_n & \\ 00 & 00 & \lambda s_n\lambda & \end{pmatrix} \quad (27)$$

$$= \lambda e_2 + r_n e_3 + \lambda s_n e_4 + u_{1n} e_7 + \lambda u_{2n} e_8 + u_{5n} e_9 + \lambda u_{6n} e_{10} + u_{3n} e_{11} + \lambda u_{4n} e_{12}.$$

Note $\Gamma = \sum_{i \geq 0} (a_i(e_1 - e_2) + b_i e_3 + \lambda c_i e_4 + f_i e_5 + f_i e_6 + g_i e_7 + \lambda h_i e_8 + k_i e_9 - k_i e_{10} + l_i e_{11} + \lambda w_i e_{12}) \lambda^{-i} \equiv a(e_1 - e_2) + b e_3 + \lambda c e_4 + f e_5 - f e_6 + g e_7 + \lambda h e_8 + k e_9 - k e_{10} + l e_{11} + \lambda w e_{12}$, solving the discrete stationary zero curvature equation for Γ

$$(E\Gamma)U - U\Gamma = 0 \quad (28)$$

yields

$$\begin{cases} s_n b_m^{(1)} - r_n c_m = 0, \\ r_n(a_m^{(1)} + a_m) + b_{m+1}^{(1)} = 0, \\ s_n(a_m^{(1)} + a_m) + c_{m+1} = 0, \\ r_n c_m^{(1)} - (a_m^{(1)} - a_m) - s_n b_m = 0, \\ b_{m+1}^{(1)} u_{2n} + s_n g_{m+1}^{(1)} - r_n h_{m+1} - c_{m+1} u_{1n} = 0, \\ a_m^{(1)} u_{2n} + s_n f_m^{(1)} + h_{m+1} = 0, \\ a_m^{(1)} u_{1n} + r_n f_m^{(1)} + g_m^{(1)} + r_n f_m + a_m u_{1n} = 0, \\ c_m^{(1)} u_{1n} + r_n h_m^{(1)} - f_m^{(1)} - g_m s_n + f_m - b_m u_{2n} = 0, \\ u_{5n}(a_m^{(1)} - a_m) + u_{4n} b_{m+1}^{(1)} - u_{3n} c_{m+1} + u_{2n} g_{m+1}^{(1)} + s_n l_{m+1}^{(1)} - r_n w_{m+1} - u_{1n} h_{m+1} = 0, \\ u_{3n} c_m^{(1)} - a_m^{(1)} u_{6n} + h_m^{(1)} u_{1n} + r_n w_m^{(1)} - k_m^{(1)} - s_n l_m + k_m - u_{2n} g_m - u_{4n} b_m - u_{6n} a_m = 0, \\ u_{3n} a_m^{(1)} + b_{m+1}^{(1)} u_{6n} + k_m^{(1)} r_n + k_m r_n + u_{1n} f_m^{(1)} + l_{m+1} + r_n k_m + u_{1n} f_m + u_{3n} a_m - u_{5n} b_m = 0, \\ -u_{4n} a_m^{(1)} - u_{2n} f_m^{(1)} - k_m^{(1)} s_n + c_m^{(1)} u_{5n} - k_m s_n - w_{m+1} - f_m u_{2n} - u_{4n} a_m + a_m u_{6n} = 0. \end{cases} \quad (29)$$

Similar to Eqs.(20), we see that $s_n = (a_m^{(1)} u_{1n} + r_n f_m^{(1)} + e_m^{(1)} + r_n f_m + a_m u_{1n}) = 0$.

$s_n(u_{3n} a_m^{(1)} + b_{m+1}^{(1)} u_{6n} + k_m^{(1)} r_n + k_m r_n + u_{1n} f_m^{(1)} + l_{m+1} + r_n k_m + u_{1n} f_m + u_{3n} a_m - u_{5n} b_m) = 0$ Thus, the second and seventh and eleventh equation in (29) can be derived from the others.

Noting

$$\Gamma_+^{(m)} = \sum_{i=0}^m (a_i(e_1 - e_2) + b_i e_3 + \lambda c_i e_4 + f_i e_5 - f_i e_6 + g_i e_7 + \lambda h_i e_8 + k_i e_9 - k_i e_{10} + l_i e_{11} + \lambda w_i e_{12}) \lambda^{m-i} \equiv (a(e_1 - e_2) + b e_3 + \lambda c e_4 + f e_5 - f e_6 + g e_7 + \lambda h e_8 + k e_9 - k e_{10} + l e_{11} + \lambda w e_{12}) \lambda^{m-i} = \lambda^m \Gamma - \Gamma_-^{(m)}$$

$$\Delta_3 = \begin{pmatrix} -2a_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_m \\ 0 & 0 & -2a_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2a_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

, Taking $\Gamma^{(m)} = \Gamma_+^{(m)} + \Delta_3$ then a direct computation reads

$$(E\Gamma^{(m)})U_n - U_n\Gamma^{(m)} = \begin{pmatrix} 0 & \Delta_{12} & 0 & \Delta_{14} & \Delta_{15} & \Delta_{16} \\ \Delta_{21} & 0 & \Delta_{23} & 0 & \Delta_{25} & \Delta_{26} \\ 0 & 0 & 0 & \Delta_{12} & 0 & \Delta_{14} \\ 0 & 0 & \Delta_{21} & 0 & \Delta_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_{12} \\ 0 & 0 & 0 & 0 & \Delta_{21} & 0 \end{pmatrix}.$$

$$\begin{aligned} \Delta_{12} &= -b_{m+1}^{(1)} - 2a^{(m)}r_n, & \Delta_{21} &= \lambda c_{m+1} + 2\lambda a_m s_n, \\ \Delta_{14} &= -g_{m+1}^{(1)}s_n - 2a_m^{(1)}u_{1n}, & \Delta_{23} &= \lambda h_{m+1} + \lambda 2a_m u_{2n}, \\ \Delta_{15} &= u_{1n}h_{m+1} - u_{4n}b_{m+1}^{(1)} + u_{3n}c_{m+1} - 2u_{5n}a_m^{(1)} + 2u_{5n}a_m, \\ \Delta_{16} &= -b_{m+1}^{(1)}u_{6n} + k_m r_n - l_{m+1} + 2u_{3n}a_m, \\ \Delta_{25} &= \lambda k_m^{(1)}s_n + \lambda w_{m+1} + 2\lambda u_{4n}a_m, & \Delta_{26} &= \lambda(k_m^{(1)} - k_m), \end{aligned}$$

such that the discrete zero curvature equation

$$(U_n)_{t_m} = (E\Gamma^{(m)})U_n - U_n\Gamma^{(m)} \quad (30)$$

generates the discrete integrable hierarchy

$$\begin{cases} (r_n)_{t_m} = -b_{m+1}^{(1)} - 2r_n a_m^{(1)} = r_n(a_m - a_m^{(1)}), \\ (s_n)_{t_m} = c_{m+1}^{(1)} + 2a_m s_n = s_n(a_m - a_m^{(1)}), \\ (u_{1n})_{t_m} = -g_{m+1}^{(1)} - 2a_m^{(1)}u_{1n}, \\ (u_{2n})_{t_m} = h_{m+1} + 2a_m u_{2n}, \\ (u_{3n})_{t_m} = -b_{m+1}^{(1)}u_{6n} + k_m r_n - l_{m+1} + 2u_{3n}a_m, \\ (u_{4n})_{t_m} = k_m^{(1)}s_n + w_{m+1} + 2u_{4n}a_m, \\ (u_{5n})_{t_m} = u_{1n}h_{m+1} - u_{4n}b_{m+1}^{(1)} + u_{3n}c_{m+1} - 2u_{5n}a_m^{(1)} + 2u_{5n}a_m, \\ (u_{6n})_{t_m} = k_m^{(1)} - k_m. \end{cases} \quad (31)$$

The first two equations in (30) are the same with those in (23). Hence, the hierarchy (30) still is the discrete integrable couplings of the KdV lattice equations.

Remark 2: The problem on Hamiltonian structures of the continuous integrable couplings has been overcome by using the quadratic-form identity [11], however, it is necessary to construct proper Lie algebras so that the aim could be come true. There is an open problem: Could we generalize the continuous-type quadratic-form identity in [11] into the discrete-type one so that

the Hamiltonian structures of the discrete integrable couplings could be obtained? which is worth studying in the future.

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