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Influence of viscosity on the scattering of an air pressure wave by a rigid body: a regular boundary integral formulation

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Abstract

This paper gives a regular vector boundary integral equation for solving the problem of viscous scattering of a pressure wave by a rigid body. Firstly, single-layer viscous potentials and a generalized stress tensor are introduced. Correspondingly, generalized viscous double-layer potentials are defined. By representing the scattered field as a combination of a single-layer viscous potential and a generalized viscous double-layer potential, the problem is reduced to the solution of a vectorial Fredholm integral equation of the second kind. Generally, the vector integral equation is singular. However, there is a particular stress tensor, called pseudostress, which yields a regular integral equation. In this case, the Fredholm alternative applies and permits a direct proof of the existence and uniqueness of the solution. The results presented here provide the foundation for a numerical solution procedure.

Keywords

scattering; viscous fluid; regular integral equation

1. Introduction

Classical mathematical acoustics is concerned with the modelling of sound waves considered as small perturbations in an inviscid fluid (or gas). In the case of time-harmonic acoustic waves in inviscid (non-viscous) fluids, the basic equation is the scalar Helmholtz equation for the pressure. This modelling approach forms the basis for much of the theoretical and practical work in this area (see the classical book by Morse & Ingard 1968). Classical and modern theoretical results connected with direct and inverse scattering problems are presented by Colton & Kress (1998), which has been updated and condensed by Kress (2001). The linearity of the basic equation and applications on infinite domains has made the boundary integral method the most suitable mathematical instrument for solving classical acoustical problems. Correspondingly, the boundary-element method (BEM) has become a powerful approach in the computational acoustic analysis.

The modern technology of micro-electro-mechanical systems (MEMS) requires the study of the motion of gases in microscale geometries. When the dimensions of the body are of the order of the boundary-layer thickness of the surrounding air (as happens in the case of microphones built using MEMS technology), viscous effects cannot be neglected. The same is true for underwater acoustic waves that we expect to be strongly influenced by the viscosity.

A third important medium with properties close to those of water is the human body, i.e. biological tissue (ultrasound). These examples are sufficient to justify the development of a viscous acoustic scattering theory based on the linearization of the equations describing the motion of viscous compressible fluids.

The important difference in mathematically treating viscous acoustics versus the classical non-viscous case is related to boundary conditions. In the viscous case, the commonly used condition is the non-slip condition asserting that the fluid particles adhere to the surface of the body, while, in the non-viscous case, only the normal derivative of the pressure is specified on the surface. Consequently, instead of working with the pressure as the main dependent function, we have to include the velocity field as well. Instead of a scalar function, we have to determine a vector field.

The number of studies that includes viscous effects in acoustics is limited. Alblas (1957) solved the problem for the scattering of a pressure wave by a half-plane considering the viscosity of the air. Mechel (1989) discussed the problem of a slit array and the problem of a slit resonator considering both viscous and thermal losses. Homentcovschi *et al.* (2005) solved the problem of the diffraction of a plane sound wave by a grating for a viscous compressible fluid. Homentcovschi & Miles (2006) gave a solution to the problem of the viscous scattering of a pressure wave by an artificial hair-like biomimetic acoustic velocity sensor, allowing the calculation of the fluid tractions on the sensor.

Homentcovschi & Miles (2007) reduced the problem of viscous scattering of a pressure wave by a rigid body to the solution of a two-dimensional singular vector integral equation for the viscous traction (surface tension) on the surface of the body. By using some results from the theory of two-dimensional singular integral equations, it was shown that the equation for the viscous traction has a unique solution.

In the present paper, we succeed in obtaining a regular vector Fredholm integral equation of the second kind for solving the viscous scattering problem. First, a generalized stress operator, containing an arbitrary real parameter β , is defined. In the usual way, this generalized stress expression yields the Navier–Stokes system for all values of the parameter β . When $\beta = \mu$ (shear viscosity value), the generalized stress law reduces to the physical stress. The generalized stress gives a generalized traction on any portion of the surface. By the Fourier transform, a representation formula (of Green's type) for a general solution of the basic equation in the external domain in terms of a single- and a double-layer potential is given. By using a representation of velocities, as a sum of a simple-layer potential and a generalized double-layer potential, a Fredholm integral equation of the second kind on the boundary of the body is obtained. For a general value of the parameter β , the integral equation is singular. There is a single value β_p of the parameter that cancels the singular part, giving a regular integral equation on the surface of the body. The corresponding stress is called a pseudostress. In §4, by using the resulting regular integral equation, a theorem of existence for the exterior Dirichlet problem will be proved. Consequently, the viscous acoustic scattering problem is well posed. Its solution can be obtained by solving the regular integral equation, resulting in the case where we use the pseudostress. Once it is shown that the integral equation is uniquely solvable, its solutions can be obtained by using specific numerical methods.

2. Viscous acoustic waves

Let $V' = V'(x, t)$ be the velocity field in a fluid depending on the space variable x and the time variable t , and let $P' = P'(x, t)$ and $\rho' = \rho'(x, t)$ denote the pressure and density in a viscous fluid, respectively. The motion is then governed by the equation of continuity

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' V') = 0,$$

the equation of momentum

$$\rho' \left(\frac{\partial}{\partial t} + V' \cdot \nabla \right) V' - \nabla \cdot \Sigma' = 0$$

and the state equation, which, for isentropic flows, can be written as

$$P' = c'^2 \rho'.$$

The stress tensor Σ' has the components

$$\sigma'_{ij} [P', V'] = \left(-P' + \left(\mu_B - \frac{2}{3} \mu \right) \nabla \cdot V' \right) \delta_{ij} + \mu \left(\frac{\partial V'_i}{\partial x_j} + \frac{\partial V'_j}{\partial x_i} \right), \quad (2.1)$$

where c' is the isentropic speed of μ sound and μ and μ_B are the shear and bulk viscosities, respectively.

Assuming that v , p , ρ and c are small perturbations of the static state $V_0 = 0$, P_0 , ρ_0 and c_0 , the above equations can be linearized to obtain the Stokes system

$$\frac{1}{P_0} \frac{\partial p}{\partial t} + \nabla \cdot v = 0$$

and

$$\rho_0 \frac{\partial v}{\partial t} - \nabla \cdot \Sigma' = 0.$$

For time-harmonic acoustic waves of the form

$$[v(x, t), p(x, t)] = \text{Re} \left\{ [v(x), p(x)] e^{-i\omega t} \right\},$$

with angular frequency $\omega > 0$, it results that the complex-valued space-dependent parts $[v(x), p(x)]$ satisfy the reduced equations (repeated indices summation convention is used throughout the paper)

$$\frac{\partial v_j}{\partial x_j} = \frac{i\omega}{P_0} p \quad (2.2)$$

and

$$-i\omega\rho_0 v_i = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i=1, 2, 3. \quad (2.3)$$

The physical stress operator σ can be written as

$$\sigma_{ij} = \sigma_{ij}[v] = \lambda \left(\frac{\partial v_k}{\partial x_k} \right) \delta_{ij} + 2\mu e_{ij}, \quad e_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad (2.4)$$

where

$$\lambda \equiv \lambda_1 + i\lambda_2 = \left(\mu_B - \frac{2}{3}\mu \right) + i \frac{P_0}{\omega}.$$

By using the expression (2.4) in equation (2.3), the equation for velocity can be obtained as follows:

$$\mu \Delta v + (\lambda + \mu) \nabla \nabla \cdot v + i\omega\rho_0 v = 0. \quad (2.5)$$

By applying the operator ∇ to equation (2.5) and also accounting for equation (2.2), the equation for pressure is

$$[\Delta + k^2] p = 0, \quad (2.6)$$

where

$$k = \frac{\omega}{c_0 \sqrt{1 - i\omega\nu'/c_0^2}}, \quad \text{Im}(k) \leq 0.$$

Also, ν and ν' are the kinematic viscosities

$$\nu = \frac{\mu}{\rho_0} \quad \text{and} \quad \nu' = \frac{\mu_B}{\rho_0} + \frac{4\mu}{3\rho_0}.$$

By applying to equation (2.5) the operator $[\Delta + k^2]$, a factorized form of the equation for velocity is obtained as follows:

$$[\Delta + k^2][\Delta + k^{*2}] v = 0, \quad (2.7)$$

where

$$k^* = \sqrt{\frac{i\omega}{\nu}}, \quad \text{Im}(k^*) \leq 0.$$

Despite the compact form of equation (2.7), we prefer to use the system of equations (2.2)–(2.4) that contains all the physical variables v , p , σ .

(a) The direct scattering problem

(i) The incident field—The function

$$p^{\text{in}}(x) = P_0 \exp \{ik m \cdot x\}, \quad (2.8)$$

where m is a unit vector, gives a plane wave travelling in the direction m with velocity c_0 . It can be verified directly that the function $p^{\text{in}}(x)$ satisfies the pressure equation (2.6) for all $x \in \mathbb{R}^3$. Also, equations (2.2) and (2.3) provide the corresponding velocity field as

$$v^{\text{in}}(x) = i\delta k c_0^2 m \exp \{ik m \cdot x\}, \quad (2.9)$$

where

$$\delta = \frac{1 + (v - v')i\omega/c_0^2}{i\omega - vk^2}.$$

(ii) The direct scattering problem—The viscous scattering of time-harmonic acoustic waves by a rigid, bounded obstacle $D \subset \mathbb{R}^3$ yields the following problem. Given an incident field $(p^{\text{in}}, v^{\text{in}})$ by formulae (2.8) and (2.9), find the scattered field (p^s, v^s) as a radiating solution to the system (2.2) and (2.3) in $\mathbb{R}^3 \setminus D$, such that the total velocity field,

$$V = v^{\text{in}} + v^s,$$

satisfies the boundary condition $V = 0$ on ∂D . Here, ∂D denotes the boundary surface of the domain D , which is assumed to be of class C^2 .

Clearly, the direct scattering problem is in fact a special *exterior Dirichlet problem* for the bounded domain D . Find a vectorial field $v^s(x) \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ solution of the basic equation

$$\mu \Delta v + (\lambda + \mu) \nabla \nabla \cdot v + i\rho_0 \omega v = 0, \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \quad (2.10)$$

that satisfies the boundary condition (f is a continuous vectorial field defined on ∂D)

$$v = f \quad \text{on } \partial D.$$

The system of equations (2.10) is similar to the system describing the steady-state oscillation problems in elasticity (Kupradze 1979; Natroshvili 1997). The main difference is due to the term containing $i\omega v$ that combines the real and imaginary parts of v .

(iii) The uniqueness theorem

Theorem 2.1: In the case

$$v_j = O\left(\frac{1}{|x|}\right), \quad \frac{\partial v_j}{\partial x_k} = o\left(\frac{1}{|x|}\right), \quad j, k=1, 2, 3, \quad (2.11)$$

the exterior Dirichlet problem for equation (2.10) has at most one solution.

Proof: We must show that the homogeneous boundary condition $v = 0$ on ∂D implies that v vanishes identically. Denote $\Omega_r := \{x \in \mathbb{R}^3 : |x| < r\}$ and $D_r = (\mathbb{R}^3 \setminus D) \cap \Omega_r$. Then,

$$-i\omega\rho_0 \int_{D_r} v_i \overline{v_i} \, dx = \int_{D_r} \overline{v_i} \frac{\partial \sigma_{ij}}{\partial x_j} \, dx = \int_{D_r} \frac{\partial}{\partial x_j} (\overline{v_i} \sigma_{ij}) \, dx - \int_{D_r} \frac{\partial \overline{v_i}}{\partial x_j} \sigma_{ij} \, dx.$$

Hence,

$$-i\omega\rho_0 \int_{D_r} v_i \overline{v_i} \, dx = \int_{\partial D} \overline{v_i} t_i \, ds + \int_{\partial \Omega_r} \overline{v_i} t_i \, ds - \int_{D_r} \sigma_{ij} \overline{e_{ij}} \, dx,$$

where $t_i = \sigma_{ij} n_j$ is the traction on the surface and n denotes the outer normal unit vector. Substituting σ_{ij} by its expression given in equation (2.4), results in

$$\int_{D_r} \left\{ \lambda (\nabla \cdot v) (\overline{\nabla \cdot v}) + 2\mu e_{ij} \overline{e_{ij}} - i\omega\rho_0 v_i \overline{v_i} \right\} \, dx = \int_{\partial D} \overline{v_i} t_i \, ds + \int_{\partial \Omega_r} \overline{v_i} t_i \, ds.$$

By virtue of (2.11), the integral over $\partial \Omega_r$ tends to zero when $r \rightarrow \infty$. Therefore, the limit of the r.h.s. and hence the limit of the l.h.s. exist and are equal. By taking the real part in the resulting relationship, we obtain, for the homogeneous problem,

$$2\mu \int_{\mathbb{R}^3 \setminus D} \sum_{i,j} |e_{ij}|^2 \, dx + \left(\mu_B - \frac{2}{3}\mu \right) \int_{\mathbb{R}^3 \setminus D} |\nabla \cdot v|^2 \, dx = 0. \quad (2.12)$$

Now, by using the inequality

$$\frac{2}{3}\mu |\nabla \cdot v|^2 \leq 2\mu \left(\left| \frac{\partial v_1}{\partial x_1} \right|^2 + \left| \frac{\partial v_2}{\partial x_2} \right|^2 + \left| \frac{\partial v_3}{\partial x_3} \right|^2 \right),$$

formula (2.12) becomes

$$\int_{\mathbb{R}^3 \setminus D} \left\{ 2\mu \sum_{i \neq j} |e_{ij}|^2 \, dx + \mu_B \sum_{i=1}^3 \left| \frac{\partial v_i}{\partial x_i} \right|^2 \right\} \, dx \leq 0.$$

Hence, $|e_{ij}| = |\partial v_i / \partial x_j| = 0$. Considering the boundary conditions results in $v \equiv 0$ in $\mathbb{R}^3 \setminus D$.

Remark 2.2: The above proof supposes that the field v is continuously differentiable up to the boundary, while the formulation of the exterior Dirichlet problem requires v to be continuous only up to the boundary. The methods for overcoming this difficulty can be found in Colton & Kress (1983, theorem 3.27 or 1998, lemma 3.8).

3. The generalized Green's formula and viscous acoustic layer potentials

The main objective of this section is to prove the generalized Green's formula. To obtain this formula, we applied the three-dimensional Fourier transform to some discontinuous fields that coincide with the velocity and pressure fields outside the body and which are zero inside the domain D . The classical way to obtain this formula is based on Lorentz's reciprocal identity, fundamental solution and Green's formulae for bounded and unbounded domains (Homentcovschi & Miles 2007). The deduction given here is shorter.

(a) The generalized stress operator

Because the incident field (p^i, v^i) satisfies the basic system of equations (2.2) and (2.3), the scattered field (p^s, v^s) will be the solution of the system

$$\frac{\partial v_j^s}{\partial x_j} = \frac{i\omega}{P_0} p^s \quad (3.1)$$

and

$$-i\omega\rho_0 v_i^s = \frac{\partial \sigma_{ij}^g}{\partial x_j}, \quad i=1, 2, 3, \quad (3.2)$$

where

$$\sigma_{ij}^g = \sigma_{ij}^g[v^s] := (\lambda + \mu - \beta) \left(\frac{\partial v_k^s}{\partial x_k} \right) \delta_{ij} + \mu \frac{\partial v_i^s}{\partial x_j} + \beta \frac{\partial v_j^s}{\partial x_i} \quad (3.3)$$

is the generalized stress operator (Kupradze 1979).

By substituting the relationship (3.3) into equation (3.2), the same equation (2.3) for velocity results. The real parameter β is arbitrary. Particularly, for $\beta = \mu$, the generalized stress operator σ^g coincides with the physical stress operator σ , defined by formula (2.4).

(b) The generalized Green's formula

The total field (p, v) will be extended with the value $(0, 0)$ inside the domain D and we take the Fourier transform of the scattered field

$$(\tilde{p}^s(\alpha), \tilde{v}^s(\alpha)) = \mathcal{F}[p^s, v^s] := \int_{\mathbb{R}^3} [p^s, v^s] \exp(-i\alpha \cdot x) dx.$$

Since ∂D is a discontinuity surface (of the first kind) for the field (p^s, v^s) , we have

$$\mathcal{F} \left[\frac{\partial v_1^s}{\partial x_1} \right] = \int_{\mathbb{R}^3} \frac{\partial v_1^s}{\partial x_1} \exp(-i\alpha \cdot x) dx = \int_{\mathbb{R}^3} \frac{\partial}{\partial x_1} [v_1^s \exp(-i\alpha \cdot x)] dx + i\alpha_1 \mathcal{F}[v_1^s],$$

with the derivatives being considered in the sense of differential calculus (not distributional derivatives). But,

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial x_1} \left[v_1^s \exp(-i\alpha \cdot x) \right] dx = \int_D \frac{\partial}{\partial x_1} \left[v_1^s \exp(-i\alpha \cdot x) \right] dx, \\ + \int_{\mathbb{R}^3 \setminus D} \frac{\partial}{\partial x_1} \left[v_1^s \exp(-i\alpha \cdot x) \right] dx = \int_{\partial D} n_1 v_1^{s-} \exp(-i\alpha \cdot x) ds$$

and

$$- \int_{\partial D} n_1 v_1^{s+} \exp(-i\alpha \cdot x) ds = \int_{\partial D} [n_1 v_1^s] \exp(-i\alpha \cdot x) ds.$$

Suppose $D \in C^2$, n denotes the normal unit vector directed outside D and $[a]$ denotes the jump of the function a when crossing the surface ∂D in the direction given by n .

Hence,

$$\mathcal{F} \left[\frac{\partial v_1^s}{\partial x_1} \right] = i\alpha_1 \mathcal{F} [v_1^s] - \int_{\partial D} [n_1 v_1^s] \exp(-i\alpha \cdot x) ds.$$

Because the incident field is continuous on the surface of the body, the last relationship can also be written as

$$\mathcal{F} \left[\frac{\partial v_1^s}{\partial x_1} \right] = i\alpha_1 \mathcal{F} [v_1^s] - \int_{\partial D} n_1 v_1^s \exp(-i\alpha \cdot x) ds.$$

Similarly,

$$\mathcal{F} \left[\frac{\partial v_j^s}{\partial x_j} \right] = i\alpha_j \mathcal{F} [v_j^s] - \int_{\partial D} n_j v_j^s \exp(-i\alpha \cdot x) ds$$

and

$$\mathcal{F} \left[\frac{\partial \sigma_{ij}^g}{\partial x_j} \right] = i\alpha_j \mathcal{F} [\sigma_{ij}^g] - \int_{\partial D} t_i^g \exp(-i\alpha \cdot x) ds.$$

In the last integral,

$$t_i^g [V] := n_j \sigma_{ij}^g [V]$$

are the components of the generalized traction (generalized stress force) of the fluid on the body surface.

The system of equations (3.1)–(3.3) in the Fourier transform space becomes

$$i\alpha \cdot \tilde{v}^s - \int_{\partial D} V_n \exp(-i\alpha \cdot x) ds = \frac{i\omega}{c_0^2} \tilde{p}^s, \quad (3.4)$$

$$-i\omega \rho_0 \tilde{v}_i^s = i\alpha_j \cdot \tilde{\sigma}_{ij}^g - \int_{\partial D} t_i^g \exp(-i\alpha \cdot x) ds \quad (3.5)$$

and

$$\begin{aligned} \tilde{\sigma}_{ij}^g = & (\lambda + \mu - \beta) \left(i\alpha \cdot \tilde{v}^s - \int_{\partial D} V_n \exp(-i\alpha \cdot x) ds \right) \delta_{ij} + \mu i\alpha_j \tilde{v}_i^s \\ & + \beta i\alpha_i \tilde{v}_j^s - \int_{\partial D} [\mu n_j V_i + \beta n_i V_j] \exp(-i\alpha \cdot x) ds. \end{aligned} \quad (3.6)$$

By substituting equation (3.6) into equation (3.5), the equation for $\tilde{v}^s$ becomes

$$\begin{aligned} i\omega \rho_0 \tilde{v}_i^s + (\lambda + \mu) i\alpha_i (i\alpha \cdot \tilde{v}^s) - \mu |\alpha|^2 \tilde{v}_i^s \\ = \int_{\partial D} [t_i^g + (\lambda + \mu - \beta) i\alpha_i V_n + \beta (i\alpha_j V_j) n_i + \mu (i\alpha_j n_j) V_i] \exp(-i\alpha \cdot x) ds. \end{aligned} \quad (3.7)$$

Equations (3.4) and (3.7) yield the following expressions for pressure and velocity Fourier transforms:

$$\begin{aligned} \frac{i\omega}{c_0^2} \tilde{p}^s = & \int_{\partial D} \frac{-i\alpha \cdot t^g + (\lambda + \mu - \beta) |\alpha|^2 V_n - (\beta + \mu) (i\alpha \cdot V) (i\alpha \cdot n)}{(\lambda + 2\mu) (|\alpha|^2 - k^2)} \exp(-i\alpha \cdot x) ds \\ & - \int_{\partial D} V_n \exp(-i\alpha \cdot x) ds \end{aligned}$$

and

$$\begin{aligned} \tilde{v}_i^s = & i\alpha_i \int_{\partial D} \frac{-i\alpha \cdot t^g + (\lambda + \mu - \beta) |\alpha|^2 V_n - (\beta + \mu) (i\alpha \cdot V) (i\alpha \cdot n)}{\mu (1 + \mu / (\lambda + \mu)) (|\alpha|^2 - k^2) (|\alpha|^2 - k^{*2})} \exp(-i\alpha \cdot x) ds \\ & + \int_{\partial D} \frac{-t_i^g - (\lambda + \mu - \beta) i\alpha_i V_n - \beta (i\alpha \cdot V) n_i - \mu (i\alpha \cdot n) V_i}{\mu (|\alpha|^2 - k^{*2})} \exp(-i\alpha \cdot x) ds. \end{aligned} \quad (3.8)$$

The inverse Fourier transform formula

$$\begin{aligned} \mathcal{F}^{-1} \left[\frac{1}{|\alpha|^2 - k^{*2}} \right] &= \frac{\exp(ik^*|x|)}{4\pi|x|}, \\ \mathcal{F}^{-1} \left[\frac{1}{|\alpha|^2 - k^2} \right] &= \frac{\exp(ik|x|)}{4\pi|x|} \end{aligned}$$

and

$$\mathcal{F}^{-1} \left[\frac{1}{(|\alpha|^2 - k^2)(|\alpha|^2 - k^{*2})} \right] = \frac{\exp(ik|x|) - \exp(ik^*|x|)}{4\pi(k^2 - k^{*2})|x|}$$

are used to obtain

$$\left. \begin{aligned} V_i(x), & \quad x \in \mathbb{R}^3 \setminus \overline{D} \\ V_i(x)/2, & \quad x \in \partial D \\ 0, & \quad x \in D \end{aligned} \right\} = V_i^{\text{in}}(x) - \int_{\partial D} S_{ij}(x, x') t_j^g(x') ds' + \int_{\partial D} K_{ij}^g(x, x', n') V_j(x') ds'. \quad (3.9)$$

Here, we have also used the well-known property of the Fourier transform, which is that when recovering the original of a discontinuous function (across the ∂D surface), the inverse Fourier transform gives a function that has the values at the points of discontinuity equal to the average of the side limits of the original discontinuous function. Also,

$$S_{ij}(x, x') = \frac{\exp(ik^*|x-x'|)}{4\pi\mu|x-x'|} \delta_{ij} + \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\exp(ik|x-x'|) - \exp(ik^*|x-x'|)}{4\pi(k^2 - k^{*2})|x-x'|} \quad (3.10)$$

and

$$K_{ij}^g(x, x', n') = \left(\delta_{ij} \frac{\partial}{\partial n'} + \frac{\beta}{\mu} n'_i \frac{\partial}{\partial x'_j} + \frac{(\lambda+\mu-\beta)}{\lambda+2\mu} n'_j \frac{\partial}{\partial x'_i} \right) \frac{\exp(ik^*|x-x'|)}{4\pi|x-x'|} + \frac{(\lambda+\mu)(\beta+\mu)}{\mu(\lambda+2\mu)} \frac{\partial}{\partial n'} \frac{\partial^2}{\partial x'_i \partial x'_j} \frac{\exp(ik|x-x'|) - \exp(ik^*|x-x'|)}{4\pi(k^2 - k^{*2})|x-x'|} - \frac{(\lambda+\mu-\beta)(\lambda+\mu)}{\mu(\lambda+2\mu)} k^2 n'_j \frac{\partial}{\partial x'_i} \frac{\exp(ik|x-x'|) - \exp(ik^*|x-x'|)}{4\pi(k^2 - k^{*2})|x-x'|}. \quad (3.11)$$

Remark 3.1—The above proof assumes that the fields v^s and σ^g are continuously differentiable up to the boundary. These requirements can be weakened by the methods cited in remark 2.2. Finally, it can be shown that formula (3.9) proves true in the case $v^s \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ and is a solution of the basic equation.

Remark 3.2—In the case where $x \in \partial D$, the second integral in formula (3.9) is not convergent (as an improper integral); since it results by taking the limit when the radius of small excluded sphere ε approaches 0, it has to be considered a principal value integral.

(c) Viscous acoustic layer potentials

The general representation formula (3.9) for the solution of the basic equation (2.10) suggests the introduction of viscous acoustic layer potentials. These potentials will be defined in close analogy with acoustic layer potentials (Colton & Kress 1983; Kress 2001).

(i) Viscous acoustic single-layer potential—Given an integrable vector field φ , the vectorial integral $u(x)$ of components

$$u_i(x) := \int_{\partial D} S_{ij}(x, x') \varphi_j(x') ds', \quad x \in \mathbb{R}^3 \setminus \partial D \quad (3.12)$$

is called a *viscous acoustic single-layer potential* of density φ . It is a solution of the basic equation (2.10) in D and $\mathbb{R}^3 \setminus \overline{D}$ and is vanishing at infinity. Physically, the viscous acoustic single-layer potential gives the velocity field corresponding to a stress force (traction) along the surface ∂D given by the function $\varphi(x')$.

By using the formulae

$$\exp(ik^*|x|) = 1 + ik^*|x| + O(|x|^2) \quad (3.13)$$

and

$$\frac{\exp(ik|x|) - \exp(ik^*|x|)}{(k^2 - k^{*2})|x|} = \frac{i}{k+k^*} - \frac{|x|}{2} + O(|x|^2), \quad (3.14)$$

we can write

$$S_{ij}(x, x') = S_{ij}^0(x, x') + S_{ij}^R(x, x'),$$

where the singular part can be written as

$$S_{ij}^0(x, x') = \frac{1}{8\pi(\lambda+2\mu)|x-x'|} \left[(\lambda+3\mu)\delta_{ij} + (\lambda+\mu) \frac{(x_i - x'_i)(x_j - x'_j)}{|x-x'|^2} \right]$$

and the regular part is

$$S_{ij}^R(x, x') = \frac{\exp(ik^*|x-x'|) - 1}{4\pi|x-x'|} \delta_{ij} + \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{\exp(ik|x-x'|) - \exp(ik^*|x-x'|)}{(k^2 - k^{*2})|x-x'|} + \frac{|x-x'|}{2} \right].$$

The singular part $S_{ij}^0(x)$ has only integrable singularities of the same type as the simple-layer harmonic potential. Therefore, the regularity and jump relationships for the solutions for the viscous acoustic single-layer potential will be similar to those corresponding to the harmonic single-layer potential. For example, in the case $\varphi \in C^0(\partial D)$, the viscous acoustic single-layer is a continuous vector in the whole space.

Let now x be a point in space and consider a small area element having n as the direction of the normal. The generalized stress vector acting on this area element corresponding to the displacement field given by a generalized single-layer potential can be written in the form

$$t_i^g[u, n](x) \equiv n_j \sigma_{ij}^g[u] = \int_{\partial D} K_{ij}^{g*}(x, x', n) \varphi_j(x') ds',$$

where

$$K_{ij}^{g*}(x, x', n) = \left(\delta_{ij} \frac{\partial}{\partial n} + \frac{\beta}{\mu} n_j \frac{\partial}{\partial x_i} + \frac{(\lambda+\mu-\beta)}{\lambda+2\mu} n_i \frac{\partial}{\partial x_j} \right) \frac{\exp(ik^*|x-x'|)}{4\pi|x-x'|} + \frac{(\lambda+\mu)(\beta+\mu)}{\mu(\lambda+2\mu)} \frac{\partial}{\partial n} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\exp(ik|x-x'|) - \exp(ik^*|x-x'|)}{4\pi(k^2 - k^{*2})|x-x'|} - \frac{(\lambda+\mu-\beta)(\lambda+\mu)}{\mu(\lambda+2\mu)} k^2 n_i \frac{\partial}{\partial x_j} \frac{\exp(ik|x-x'|) - \exp(ik^*|x-x'|)}{4\pi(k^2 - k^{*2})|x-x'|}. \quad (3.15)$$

(ii) Viscous acoustic double-layer potential—The generalized viscous acoustic double-layer potential, of integrable density φ , is the vectorial field v^g having components

$$v_i^g = \int_{\partial D} K_{ij}^g(x, x', n') \varphi_j(x') ds', \quad x \in \mathbb{R}^3 \setminus \partial D. \quad (3.16)$$

Physically, in the case $\beta = \mu$, the double-layer potential represents the velocity field in the entire space produced by the concentrated moment $\varphi(x')$ on the surface ∂D with normal n' . The vectorial field v^g is a solution of the basic equation (2.10) in D and $\mathbb{R}^3 \setminus \bar{D}$ and is also vanishing at infinity.

Again, using formulae (3.13) and (3.14), we can separate the singular and regular components of the kernel

$$K_{ij}^g(x, x', n') = K_{ij}^{g0}(x, x', n') + K_{ij}^{gR}(x, x', n'),$$

where

$$K_{ij}^{g0}(x, x', n') = \frac{n'_i(x-x')}{4\pi|x-x'|^3} \left[\left(\frac{(\lambda+\mu)(\beta+\mu)}{2\mu(\lambda+2\mu)} - 1 \right) \delta_{ij} - \frac{3(\lambda+\mu)(\beta+\mu)}{2\mu(\lambda+2\mu)} \frac{(x_i-x'_i)(x_j-x'_j)}{|x-x'|^2} \right] + \frac{\beta(\lambda+3\mu)-\mu(\lambda+\mu)}{\mu(\lambda+2\mu)} \frac{n'_j(x_i-x'_i)n'_i(x_j-x'_j)}{8\pi|x-x'|^3} \quad (3.17)$$

and

$$K_{ij}^{gR}(x, x', n') = \left(\delta_{ij} \frac{\partial}{\partial n'} + \frac{\beta}{\mu} n'_i \frac{\partial}{\partial x'_j} + \frac{(\lambda+\mu-\beta)}{(\lambda+2\mu)} n'_j \frac{\partial}{\partial x'_i} \right) \times \frac{\exp(ik|x-x'|)-1}{4\pi|x-x'|} + \frac{(\lambda+\mu)(\beta+\mu)}{4\pi\mu(\lambda+2\mu)} \frac{\partial}{\partial n'} \frac{\partial^2}{\partial x'_i \partial x'_j} \times \left[\frac{\exp(ik|x-x'|)-\exp(ik^*|x-x'|)}{(k^2-k^{*2})|x-x'|} + \frac{|x-x'|}{2} \right] - \frac{(\lambda+\mu-\beta)(\lambda+\mu)}{\mu(\lambda+2\mu)} k^2 n'_j \frac{\partial}{\partial x'_i} \frac{\exp(ik|x-x'|)-\exp(ik^*|x-x'|)}{4\pi(k^2-k^{*2})|x-x'|} \quad (3.18)$$

In the case $\partial D \in C^2$, we have

$$|n' \cdot (x' - x)| \leq c|(x' - x)|^2,$$

for all $x', x \in \partial D$ and some positive constant c depending on ∂D (e.g. theorem 2.2 in Colton & Kress 1983). Consequently, the first term in (3.17) yields an integral operator with a weakly singular kernel. The second term, in the general case, has a non-integrable singularity for $x = x' \in \partial D$ and, correspondingly, the direct value of the generalized double-layer potential (appearing in the middle case in formula (3.9)) can be understood only as a principal value integral.

In the special case

$$\beta^p = \frac{\mu(\lambda+\mu)}{(\lambda+3\mu)}, \quad (3.19)$$

the non-integrable singularity in formula (3.17) disappears and the direct value of the double-layer potential on ∂D is a regular integral. The corresponding expression for the stress, traction

and all the other quantities will be called pseudostress, pseudotraction, etc. and will be, respectively, denoted by σ^p , t^p , etc., substituting the superscript g by p .

Now, the general representation formula (3.9) can be used for determining the limit values of the generalized double-layer potential on the point $x_0 \in \partial D$

$$\left. \begin{aligned} F_i(x_0) + \int_{\partial D} K_{ij}^g(x_0^+, x', n') V_j(x') ds &= V_i(x_0^+), & x_0^+ \in \mathbb{R}^3 \setminus \overline{D}, \\ F_i(x_0) + \int_{\partial D} K_{ij}^g(x_0, x', n') V_j(x') ds &= V_i(x_0^+)/2, & x_0 \in \partial D, \\ F_i(x_0) + \int_{\partial D} K_{ij}^g(x_0^-, x', n') V_j(x') ds &= 0, & x_0^- \in D, \end{aligned} \right\} \quad (3.20)$$

where we denoted

$$F_i(x_0) := V_i^{(in)}(x_0) - \int_{\partial D} S_{ij}(x_0, x') t_j^g(x') ds.$$

Here, the continuity of the incoming velocity field and of the single-layer potential across the surface ∂D has been used. By eliminating $F_i(x_0)$ in formulae (3.20), the following limit values of the generalized double-layer potential along the surface ∂D result:

$$v_{\pm}^g(x_0) = v_0^g(x_0) \pm \frac{1}{2} V(x_0), \quad x_0 \in \partial D, \quad (3.21)$$

where, by $v_0^g(x_0)$, the vector of components

$$v_{0i}^g(x_0) = \int_{\partial D} K_{ij}^g(x_0, x', n') V_j(x') ds,$$

has been denoted. In the general case, the integrals in this formula have to be considered as two-dimensional principal value integrals.

Remark 3.3: The above proof is valid in the case where the density $V(x_0)$ is the restriction to the surface ∂D of a function defined in $\mathbb{R}^3 \setminus D$. It is easy to extend the validity of these formulae to the case where the density $\psi(x')$ is a uniform Hölderian function of exponent γ , $0 < \gamma \leq 1$, defined on ∂D (Kupradze 1979).

(iii) Limiting values of the stress operator for a single-layer potential—The following relationship between the kernel of the double-layer potential (3.11) and the kernel of integral relationship giving the generalized stress operator applied to the single-layer potential (3.15) proves true:

$$K_{ij}^{g*}(x, x', n) = K_{ji}^g(x', x, n).$$

Since the matrix $\|K_{ij}^{g*}(x, x', n')\|$ is obtained from the matrix $\|K_{ji}^g(x, x', n)\|$ by interchanging the points x and x' and by transposing the elements, it will be called the conjugate to $\|K_{ij}^g\|$.

Let now $x_0 \in \partial D$, n_0 the normal unit vector (pointing outside domain D) at the point x_0 , and the points $x_0^+ \in \mathbb{R}^3 \setminus \overline{D}$ and $x_0^- \in D$ on the normal n_0 in close proximity to the surface ∂D . We can

consider the generalized stress vectors generated by the single-layer potential in the points $x_0^\pm$ and the direction n_0 . Then,

$$t_i^{g+}[u, n_0](x_0) = \lim_{x_0^+ \rightarrow x_0} \int_{\partial D} K_{ij}^{g*}(x_0^+, x', n_0) \varphi_j(x') ds'$$

and

$$t_i^{g-}[u, n_0](x_0) = \lim_{x_0^- \rightarrow x_0} \int_{\partial D} K_{ij}^{g*}(x_0^-, x', n_0) \varphi_j(x') ds'.$$

Owing to the above-noted conjugation, the proper value of the integral

$$t_i^g[u, n_0](x_0) = \int_{\partial D} K_{ij}^{g*}(x_0, x', n_0) \varphi_j(x') ds'$$

can be considered only as a principal value.

We can write

$$\begin{aligned} t_{+i}^g[u, n_0](x_0) = & \lim_{x_0^+ \rightarrow x_0} \int_{\partial D} \left[K_{ij}^{g*}(x_0^+, x', n_0) + K_{ij}^g(x_0^+, x', n') \right] \varphi_j(x') ds' \\ & - \lim_{x_0^+ \rightarrow x_0} \int_{\partial D} K_{ij}^g(x_0^+, x', n') \varphi_j(x') ds'. \end{aligned} \quad (3.22)$$

By introducing a local system of coordinates and using some estimates similar to those for a harmonic potential, it can be shown that the first integral in (3.22) is continuous for $x_0^+ \rightarrow x_0$. The second integral is a double-layer potential. Finally, we obtain the formulae

$$t_{\pm}^g[u, n_0](x_0) = t^g[u, n_0](x_0) \mp \frac{1}{2} \varphi(x_0), \quad x_0 \in \partial D.$$

(iv) Limiting values of the stress operator for a double-layer potential—The limiting values of the stress operator for a double-layer potential $t^{g+}[K^g, n_0](x_0)$ and $t^{g-}[K^g, n_0](x_0)$ are connected by the Lyapunov-Tauber theorem: *if the limiting value of the stress operator for a double-layer potential exists on one side of the surface ∂D , then the limiting value exists on the other side as well and these limiting values coincide*. A proof for this theorem can be found in Kupradze (1979); also see Natroshvili (1997).

(v) A summary of the jump relationships—We have

$$u_+ = u_- \quad \text{and} \quad t_+^g[u] - t_-^g[u] = -\varphi \quad \text{on} \quad \partial D, \quad (3.23)$$

for the viscous acoustic single-layer potential and

$$v_+^g - v_-^g = \varphi \quad \text{and} \quad t_+^g[v^g] = t_-^g[v^g], \quad \text{on} \quad \partial D, \quad (3.24)$$

for the generalized viscous acoustic double-layer potential.

(vi) Single- and double-layer operators on ∂D —We define the viscous acoustic single-layer operator $(S\varphi)(x)$ by its components

$$(S\varphi)_i(x) := \frac{1}{2} \int_{\partial D} S_{ij}(x, x') \varphi_j(x') ds', \quad x \in \partial D.$$

The corresponding generalized traction operator $(K^{\mathbb{g}*}\varphi)(x)$ has the components

$$(K^{\mathbb{g}*}\varphi)_i(x) := \frac{1}{2} \int_{\partial D} K_{ij}^{\mathbb{g}*}(x, x') \varphi_j(x') ds', \quad x \in \partial D.$$

Similarly, the generalized viscous acoustic double-layer operator $(K^{\mathbb{g}}\varphi)(x)$ has the components

$$(K^{\mathbb{g}}\varphi)_i(x) := \frac{1}{2} \int_{\partial D} K_{ij}^{\mathbb{g}}(x, x', n') \varphi_j(x') ds', \quad x \in \partial D.$$

In terms of these operators, the jump relations (3.23) and (3.24) yield, in the case of continuous densities φ ,

$$u_{\pm} = \frac{1}{2} S\varphi, \quad (t^{\mathbb{g}}u)_{\pm} = \frac{1}{2} K^{\mathbb{g}*}\varphi \mp \frac{1}{2}\varphi \quad \text{and} \quad v_{\pm}^{\mathbb{g}} = \frac{1}{2} K^{\mathbb{g}}\varphi \pm \frac{1}{2}\varphi. \quad (3.25)$$

By the same arguments as in Colton & Kress (1983), it can be shown that the operators S , $K^{\mathbb{P}}$ and $K^{\mathbb{P}*}$ are compact operators from $C(\partial D)$ into $C(\partial D)$ and from $C^{0,\gamma}(\partial D)$ into $C^{0,\gamma}(\partial D)$. However, the operators $K^{\mathbb{g}}$ and $K^{\mathbb{g}*}$ for $\beta \neq \beta_{\mathbb{P}}$ are singular operators.

By interchanging the order of integration results in the operator S being self-adjoint and the operator $K^{\mathbb{g}*}$ being the adjoint of $K^{\mathbb{g}}$ with respect to the L^2 vector bilinear form on $C(\partial D)$.

4. A regular integral equation for solving the direct viscous scattering problem

Inspired by the classical non-viscous case (Colton & Kress 1998; Kress 2001), we try to find the solution of the Dirichlet exterior problem in the form of a combined viscous acoustic double- and single-layer potential. If the physical stress and stress force (traction) are used, the resulting integral equation will be a singular one. For avoiding the appearance of the singular terms, consider the pseudostress operator resulting from the generalized operator for the value $\beta_{\mathbb{P}}$ given by (3.19)

$$\sigma_{ij}^{\mathbb{P}} = \sigma_{ij}^{\mathbb{P}}[v^{\mathbb{S}}] := \frac{(\lambda + \mu)(\lambda + 2\mu)}{(\lambda + 3\mu)} \left(\frac{\partial v_k^{\mathbb{S}}}{\partial x_k} \right) \delta_{ij} + \mu \frac{\partial v_i^{\mathbb{S}}}{\partial x_j} + \frac{\mu(\lambda + \mu)}{(\lambda + 3\mu)} \frac{\partial v_j^{\mathbb{S}}}{\partial x_i}.$$

Theorem 4.1

The exterior Dirichlet problem for equation (2.10) has a unique solution and the solution depends continuously on the boundary data with respect to the uniform convergence on the solution on $\mathbb{R}^3 \setminus \bar{D}$.

Proof—We take

$$v_i(x) := \int_{\partial D} \left\{ K_{ij}^p(x, x') + \eta S_{ij}(x, x') \right\} \varphi_j(x') ds', \quad x \in \mathbb{R}^3 \setminus \partial \overline{D}, \quad (4.1)$$

with the density $\varphi \in C(\partial D)$ and a positive coupling parameter η trying to find the solution in the form of a combined viscous acoustic double- and simple-layer potential. By using the definitions and properties of the viscous acoustic potentials of §3 results in the function v being the solution of the exterior Dirichlet problem, provided the density φ is a solution of the vector integral equation

$$\varphi + K^p \varphi + \eta S \varphi = 2f. \quad (4.2)$$

From the discussion at the end of §3, the operators $S, K^p: C(\partial D) \rightarrow C(\partial D)$ are compact operators and the existence of solutions to (4.2) can be proved by applying the Riesz-Fredholm theory for the second-kind equations with a compact operator. Therefore, for proving the existence of a solution to equation (4.2), we have to show that the homogeneous equation

$$\varphi + K^p \varphi + \eta S \varphi = 0 \quad (4.3)$$

has only the solution $\varphi = 0$. Let φ be a continuous solution of (4.3). The potential with the components (4.1) on all $\mathbb{R}^3$ satisfies the boundary condition $v_+ = 0$ on ∂D and equation (2.10) in $D \cup \mathbb{R}^3 \setminus \overline{D}$. According to the uniqueness solution for the exterior Dirichlet problem (proved in §3) follows $v = 0$ in $\mathbb{R}^3 \setminus D$. The jump relations (3.23) and (3.24) now yield

$$-v_- = \varphi \quad \text{and} \quad -t_-^p[v] = -\eta \varphi, \quad \text{on} \quad \partial D. \quad (4.4)$$

Then, we can write

$$\frac{\partial}{\partial x_j} (\overline{v_i} \sigma_{ij}^g[v]) = \overline{v_i} \frac{\partial \sigma_{ij}^g[v]}{\partial x_j} + \frac{\partial \overline{v_i}}{\partial x_j} \sigma_{ij}^g[v]. \quad (4.5)$$

But, according to equation (3.2), the first term on the r.h.s. of equation (4.5) becomes

$$\overline{v_i} \frac{\partial \sigma_{ij}^g[v]}{\partial x_j} = -i\omega \rho_0 |v|^2 \quad (4.6)$$

and the second term can be written as

$$\begin{aligned} \frac{\partial \overline{v_i}}{\partial x_j} \sigma_{ij}^g[v] = & (\lambda + \mu - \beta) |\operatorname{div} v|^2 + (\beta + \mu) \left| \frac{\partial v_i}{\partial x_i} \right|^2 \\ & + \frac{(\beta + \mu)}{4} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2 + \frac{\mu - \beta}{4} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right|^2. \end{aligned} \quad (4.7)$$

Particularly, for $\beta = \beta_p$, this formula becomes

$$\overline{\frac{\partial v_i}{\partial x_j}} \sigma_{ij}^p[v] = \frac{(\lambda+\mu)(\lambda+2\mu)}{(\lambda+3\mu)} |\operatorname{div} v|^2 + \frac{2\mu(\lambda+2\mu)}{(\lambda+3\mu)} \left| \frac{\partial v_i}{\partial x_i} \right|^2 + H(x),$$

where

$$H(x) \equiv H_1^2 + iH_2 = \frac{\mu(\lambda+2\mu)}{2(\lambda+3\mu)} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2 + \frac{2\mu^2}{(\lambda+3\mu)} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right|^2,$$

$$H_1^2(x) \geq 0.$$

Now, the intergral over D of relationship (4.5) gives

$$\begin{aligned} \int_{\partial D} t_i^p[v] \overline{v_i} dx = & \int_D -i\omega\rho_0 |v|^2 dx + \int_D \frac{(\lambda+\mu)(\lambda+2\mu)}{(\lambda+3\mu)} |\operatorname{div} v|^2 dx \\ & + \int_D \frac{2\mu(\lambda+2\mu)}{(\lambda+3\mu)} \left| \frac{\partial v_i}{\partial x_i} \right|^2 dx + \int_D H(x) dx. \end{aligned} \quad (4.8)$$

We substitute the expressions for velocity and traction given by formula (4.4) and also take the real part of formula (4.8),

$$\begin{aligned} \int_{\partial D} \eta |\varphi|^2 ds + \int_D H_1^2(x) dx + \frac{(a^2 - (2/3)\mu\lambda_2^2)}{|\lambda+3\mu|^2} \int_D |\operatorname{div} v|^2 dx \\ + \frac{(b^2 + 2\mu\lambda_2^2)}{|\lambda+3\mu|^2} \int_D \left| \frac{\partial v_i}{\partial x_i} \right|^2 dx = 0, \end{aligned} \quad (4.9)$$

where

$$a^2 = (\lambda_1 + \mu)(\lambda_1 + 2\mu)(\lambda_1 + 3\mu) + \mu_b \lambda_2^2 > 0$$

and

$$b^2 = 2\mu(\lambda_1 + 2\mu)(\lambda_1 + 3\mu) > 0.$$

By using the inequality

$$|z_1 + z_2 + z_3|^2 \leq 3(|z_1|^2 + |z_2|^2 + |z_3|^2),$$

we can write

$$\frac{(2/3)\mu\lambda_2^2}{|\lambda+3\mu|^2} \int_D |\operatorname{div} v|^2 dx - \frac{2\mu\lambda_2^2}{|\lambda+3\mu|^2} \int_D \left| \frac{\partial v_i}{\partial x_i} \right|^2 dx \leq 0. \quad (4.10)$$

Finally, by the addition of the relationships (4.9) and (4.10),

$$\int_{\partial D} \eta |\varphi|^2 ds + \int_D H_1^2(x) dx + \frac{a^2}{|\lambda + 3\mu|^2} \int_D |\operatorname{div} v|^2 dx + \frac{b^2}{|\lambda + 3\mu|^2} \int_D \left| \frac{\partial v_i}{\partial x_i} \right|^2 dx \leq 0.$$

Hence, $\varphi = 0$.

Remark 4.2

The inclusion of the viscous acoustic single-layer potential in the representation formula was essential. In the case $\eta = 0$, the corresponding homogeneous double-layer integral equation may have non-trivial solutions and the corresponding non-homogeneous integral equation is generally unsolvable.

Remark 4.3

The proved theorem also gives a method for obtaining the solution of the viscous scattering problem by solving the integral equation (4.2) and, afterwards, using formula (4.1) for determining the velocity field. The very important advantage of this approach is that it reduces a problem for an unbounded domain in $\mathbb{R}^3$ to one on a bounded surface. Thus, in the case when the above equation is solved numerically (e.g. by BEMs), this is a crucial advantage over other numerical methods.

Remark 4.4

The extension to the case of the viscous acoustic scattering by more obstacles is straightforward. This problem can be reduced also to a second-kind Fredholm regular integral equation that is uniquely solvable.

5. Conclusion

This paper presents a theoretical framework for the acoustic scattering problem in a viscous fluid. With the help of the potential method, the problem is reduced to the solution of a vectorial regular integral equation involving Fredholm integral operators of the second kind on the boundary of the domain. It is shown that the corresponding vectorial Fredholm operator is invertible in appropriate function spaces. This leads to the uniqueness and existence results for the original Dirichlet boundary-value problem in the class of regular vector functions.

The results presented here provide the foundation for a numerical solution procedure.

Owing to the success of the integral equation methods for direct obstacle scattering, we expect that integral equations also play an important role in the approximation of inverse scattering problems.

The conclusion of this work is that the influence of viscosity on the scattering of pressure waves by hard bodies can be investigated by using the same mathematical apparatus—regular boundary integral equations—as in the classical acoustical scattering.

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