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Reconstruction of Vectorial Acoustic Sources in Time-Domain Tomography

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Abstract

A new theory is proposed for the reconstruction of curl-free vector field, whose divergence serves as acoustic source. The theory is applied to reconstruct vector acoustic sources from the scalar acoustic signals measured on a surface enclosing the source area. It is shown that, under certain conditions, the scalar acoustic measurements can be vectorized according to the known measurement geometry and subsequently be used to reconstruct the original vector field. Theoretically, this method extends the application domain of the existing acoustic reciprocity principle from a scalar field to a vector field, indicating that the stimulating vectorial source and the transmitted acoustic pressure vector (acoustic pressure vectorized according to certain measurement geometry) are interchangeable. Computer simulation studies were conducted to evaluate the proposed theory, and the numerical results suggest that reconstruction of a vector field using the proposed theory is not sensitive to variation in the detecting distance. The present theory may be applied to magnetoacoustic tomography with magnetic induction (MAT-MI) for reconstructing current distribution from acoustic measurements. A simulation on MAT-MI shows that, compared to existing methods, the present method can give an accurate estimation on the source current distribution and a better conductivity reconstruction.

Index Terms

Magnetoacoustic imaging; electrical impedance imaging; current density imaging; reconstruction; tomography

I. INTRODUCTION

Acoustic tomography is widely used to reconstruct acoustic property distributions through measurements of diffractive acoustic waves. In medical imaging, for example, tissue acoustic properties such as acoustic speed, acoustic attenuation and acoustic scattering coefficients as well as other nonlinear parameters are of interest in various applications of acoustic tomography. In other applications, however, reconstruction of an acoustic source field can provide information other than acoustic properties of tissue. A specific acoustic source field can be related to the distribution of certain physical parameters such as the conductivity and permeability and the distribution of the electromagnetic absorption coefficient [1–4]. Thus, imaging the acoustic source field can provide important information which is not available from the acoustic properties of the tissue.

Noninvasive investigation of the conductivity and permittivity distribution of tissues has drawn much attention, due to its sensitivity to physiological and pathological conditions of

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living system [5]. Electrical impedance tomography (EIT) [6] has been developed since mid 1980s using current injection and noninvasive surface voltage measurements to reconstruct tissue impedance images. EIT has the benefits of low cost, real-time speed and safety. Its major limitations include low spatial resolution and degraded sensitivity in the center of an object especially in regions surrounded by low conductive tissues, namely the shielding effect [7]. Magnetic induction tomography (MIT) [8] uses the interaction of an oscillating magnetic field with conductive media, and collects the signals by small coils arranged around the object. It avoids the shielding effect and the need of contact measurements, but the spatial resolution of current MIT method is still quite limited. In magnetic resonance electrical impedance tomography (MREIT) [9], the magnetic field disturbance induced by the current injection is measured by a magnetic resonance imaging system. High spatial resolution conductivity image can be obtained by MREIT method, but MREIT is currently limited by its requirement of high level current injection to obtain acceptable signal-to-noise (SNR) level.

Recently, magnetoacoustic tomography with magnetic induction (MAT-MI) [1, 10–14] was proposed to achieve high spatial resolution noninvasive electrical impedance imaging by integrating magnetic induction and ultrasonic measurements. In MAT-MI, the object is placed in a static magnetic field and a time-varying magnetic field. The time-varying magnetic field induces eddy current in the object volume. Following that, the eddy current in the static magnetic field causes vibrations due to the Lorentz force and emits ultrasound waves. These ultrasound waves are collected by transducers surrounding the object and the reconstruction of the conductivity properties is obtained with a spatial resolution close to that of sonography.

The driving source field in MAT-MI is a vector field [15], which is the Lorentz force acting on the induced electrical current. The divergence of this vector field serves as the conventional conceived acoustic source. Reconstruction of this vector field of Lorentz force directly from scalar acoustic measurements would thus greatly aid in solving the MAT-MI imaging problem. However, currently available algorithms for reconstructing an acoustic source field are limited to scalar source field reconstruction such as those used in photoacoustic or thermoacoustic tomography [2, 3], and the former scalar reconstruction algorithm of MAT-MI [1]. Algorithms [16, 17] were also proposed to reconstruct two-dimensional (2D) vector field with measurements of weighted line integration, but it can only be applied to non-divergence vector field reconstruction.

In the present study, we derive an analytical algorithm for reconstruction of a vector field, whose divergence acts as acoustic source, from scalar acoustic measurements. Theoretically, the present method expands the application domain of the existing acoustic reciprocity principle from a scalar field to a vector field. As stated in the traditional acoustic reciprocity principle, the acoustic source and the transmitted acoustic wave at the receiver are interchangeable. Similarly, we will show that the stimulating vector source and the transmitted acoustic pressure vector (acoustic pressure vectorized according to certain measurement geometry) are also interchangeable. The present method can be applied to reconstruct a vector field such as the Lorentz force field employed in magnetoacoustic tomography or other vector field with its divergence serving as the acoustic source in acoustic tomography. The proposed method is verified by numerical experiments and simulation results of applying this method on MAT-MI demonstrate its merits in reconstructing the induced current density distribution and electrical conductivity distribution.

II. THERORY

In a weakly scattering medium, the general form of the wave equation can be written as

$$\nabla^2 p(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(\mathbf{r}, t) = A(\mathbf{r}, t) \quad (1)$$

where $p(\mathbf{r}, t)$ is the acoustic pressure, c is the acoustic speed and $A(\mathbf{r}, t)$ is a scalar acoustic source field. Here we consider a broadband diffraction tomography problem while the scattering is sufficiently weak for the Born approximation to hold. Using Green's function technique, the acoustic source term on the right hand side of (1) which is generally related to the object heterogeneity can be reconstructed from acoustic measurements $p(\mathbf{r}_0, t)$ with certain aperture geometry [1–2, 18–19]. If the acoustic vibration or displacement is driven by a force field $\mathbf{F}(\mathbf{r}, t)$, as shown in Fig. 1, and at the same time the force source in time domain was designed as a short positive pulse (less than $0.5\mu\text{s}$) which can be approximated as a delta function, the wave equation in Eq. (1) then becomes [20,21]

$$\nabla^2 p(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(\mathbf{r}, t) = (\nabla \cdot \mathbf{F})\delta(t) \quad (2)$$

Here $(\nabla \cdot \mathbf{F})\delta(t)$ is a scalar field and takes the same role as $A(\mathbf{r}, t)$ in Eq. (1). $\nabla \cdot \mathbf{F}$ can be similarly reconstructed from a set of data $p(\mathbf{r}_0, t)$ measured at positions $\mathbf{r}_0$ on a surface S_0 that encloses the force field. However, it is generally not easy to solve the driving force field $\mathbf{F}$ with only the knowledge of $\nabla \cdot \mathbf{F}$ and employment of other conditions is necessary. In the following, we will show that under the far-field condition, a rigorous reconstruction of the driving force field $\mathbf{F}$ directly from the acoustic measurements $p(\mathbf{r}_0, t)$ is available by employing Green's function in the time domain [2, 15].

By taking the Fourier transform of Eq. (2) on variable $\bar{t} = tc$ and $k = \omega/c$, we obtain

$$(\nabla^2 + k^2)\tilde{p}(\mathbf{r}, k) = (\nabla \cdot \mathbf{F})c \quad (3)$$

Here the following Fourier transform pair is used,

$$\begin{cases} \tilde{p}(\mathbf{r}, k) = \int_{-\infty}^{+\infty} p(\mathbf{r}, \bar{t}) \exp(jk\bar{t}) d\bar{t} \\ p(\mathbf{r}, \bar{t}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{p}(\mathbf{r}, k) \exp(-jk\bar{t}) dk \end{cases} \quad (4)$$

According to Green's function, the acoustic pressure at $\mathbf{r}_0$ is the summation from all the elementary acoustic sources plus the waves reflected by the boundaries. If the reflective sources on the boundaries vanish as stated in the three conditions in Ref [22], the pressure signal measured at position $\mathbf{r}_0$ on a surface S_0 that encloses the source field volume can be expressed as

$$\tilde{p}(\mathbf{r}_0, k) = -c \iiint_V dv (\nabla \cdot \mathbf{F}) G_k(\mathbf{r}, \mathbf{r}_0) \quad (5)$$

where $G_k(\mathbf{r}, \mathbf{r}_0) = \exp(ik|\mathbf{r}_0 - \mathbf{r}|)/(4\pi |\mathbf{r}_0 - \mathbf{r}|)$. The inverse problem is to reconstruct the force distribution from a set of data $\tilde{p}(\mathbf{r}_0, k)$. Because the divergence of curl of a vector constantly equals zero, the force field is not uniquely defined by its divergence, except with the condition that the curl component of the force vector equals zero. In general, the curl of $\mathbf{F}$ does not go to zero. Only under certain circumstances will this curl-free condition be met.

For example, in MAT-MI, if the object is homogeneous in z direction, and the static and dynamic magnetic field are also in this direction, i.e. the whole problem is z independent, this curl-free condition can be met. The curl of $\mathbf{F}$, according to elasticity, may serve as a shear wave source. If the particle velocity in a media is v , the acoustic pressure of

longitudinal wave is governed by $\nabla p = -\rho_0 \frac{\partial v}{\partial t}$, while the pressure of shear wave can be

written as $p = -\eta \frac{\partial v}{\partial z}$, where η is the viscosity factor [15]. Most biological soft tissue has resistance to compression, but not to shear deformation [23]. For this reason the shear wave in soft tissue is much smaller than the longitudinal wave. In addition, the attenuation of shear wave in tissue is generally two to three magnitudes larger than that of longitudinal wave which makes the shear wave hard to detect in practice.

In the right hand side of Eq. (5), because $G_k \nabla \cdot \mathbf{F} = -\mathbf{F} \cdot \nabla G_k + \nabla \cdot (\mathbf{F} G_k)$, and the force is zero outside the source region, according to the divergence theorem the integral of $G_k \nabla \cdot \mathbf{F}$ over the whole source region is equal to the integral of $-\mathbf{F} \cdot \nabla G_k$ over the same region. Thus we can rewrite Eq. (5) as follows

$$\tilde{p}(\mathbf{r}_0, k) = c \iiint_V dv (\mathbf{F} \cdot \nabla G_k(\mathbf{r}, \mathbf{r}_0)) \quad (6)$$

where $\nabla G_k(\mathbf{r}, \mathbf{r}_0) = -(jk - \frac{I}{|\mathbf{r}_0 - \mathbf{r}|}) G_k \mathbf{n}_{(r, r_0)}$, with $\mathbf{n}_{(r, r_0)}$ denoting the unit vector in the direction of $\mathbf{r}_0 - \mathbf{r}$. Substituting ∇G_k into Eq. (6), we get

$$\tilde{p}(\mathbf{r}_0, k) = -c \iiint_V dv (jk - \frac{I}{|\mathbf{r}_0 - \mathbf{r}|}) (\mathbf{F} \cdot \mathbf{n}_{(r, r_0)}) G_k(\mathbf{r}, \mathbf{r}_0) \quad (7)$$

For a relatively large measurement geometry with appropriate acoustic frequency, (ie. if $|\mathbf{r} - \mathbf{r}_0| = 10\text{cm}$ and frequency $f=500\text{kHz}$, $|\mathbf{r} - \mathbf{r}_0|k \approx 200 \gg 1$) the $1/|\mathbf{r}_0 - \mathbf{r}|$ term in Eq. (7) can be ignored. Meanwhile, as shown in Fig. 1, if $r_0 \gg r$, as stated in the far field condition, $\mathbf{F} \cdot \mathbf{n}_{(r, r_0)}$ can be approximated as $\mathbf{F} \cdot \mathbf{n}_{r_0}$, where $\mathbf{n}_{r_0}$ is the unit vector in the direction of $\mathbf{r}_0$ and can also be considered as a unit normal vector of a normalized surface Ω centered at the origin.

Multiplying r_0 and taking gradient on both hand sides of Eq. (7) with respect to $\mathbf{r}_0$, the right hand side becomes

$$\begin{aligned} & -(jk)c \iiint_V dv \nabla (G_k(\mathbf{F}(\rho, \theta, \varphi) \cdot \mathbf{r}_0)) \\ & = jk \tilde{p}(\mathbf{r}_0, k) \mathbf{r}_0 - (jk)c \iiint_V dv \mathbf{F}(\rho, \theta, \varphi) G_k - (jk)c \iiint_V dv \left(\frac{\partial G_k}{\partial \theta_0} \nabla \theta_0 + \frac{\partial G_k}{\partial \varphi_0} \nabla \varphi_0 \right) (\mathbf{F}(\rho, \theta, \varphi) \cdot \mathbf{r}_0) \end{aligned} \quad (8)$$

And the left hand side becomes

$$\nabla (\tilde{p}(\mathbf{r}_0, k) \cdot \mathbf{r}_0) = jk \tilde{p}(\mathbf{r}_0, k) \mathbf{r}_0 + \tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} - (jk)c \iiint_V dv \left(\frac{\partial G_k}{\partial \theta_0} \nabla \theta_0 + \frac{\partial G_k}{\partial \varphi_0} \nabla \varphi_0 \right) (\mathbf{F}(\rho, \theta, \varphi) \cdot \mathbf{r}_0) \quad (9)$$

From (8) and (9), we get a vector equation,

$$\tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} = -jk c \iiint_V dV \mathbf{F}(\rho, \theta, \varphi) G_k \quad (10)$$

Multiplying both hand sides of Eq. (10) by $Y_l^n(\theta_0, \varphi_0)$ and integrating over the normalized spherical surface Ω_0 , we have

$$\iint_{\Omega_0} \tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} Y_l^n(\theta_0, \varphi_0) d\Omega_0 = - (jk) c \iiint_V dV \iint_{\Omega_0} d\Omega_0 G_k \mathbf{F}(\rho, \theta, \varphi) Y_l^n(\theta_0, \varphi_0) \quad (11)$$

where $Y_l^n(\theta_0, \varphi_0)$ is the spherical harmonics function, $d\Omega_0 = \sin(\theta_0) d\theta_0 d\varphi_0$, the source point and measurement point are represented in spherical coordinate at (ρ, θ, φ) and $(\rho_0, \theta_0, \varphi_0)$, respectively. In spherical coordinates, Green's function can be expressed as [2]:

$$G_k(\mathbf{r}, \mathbf{r}_0) = jk \sum_{l=0}^{+\infty} \sum_{m=-l}^l Y_l^{m*}(\theta_0, \varphi_0) Y_l^m(\theta, \varphi) \cdot H_l^{(1)}(k\rho_0) \cdot J_l(k\rho),$$

where the asterisk denotes the complex conjugate and J_l and H_l are the spherical Bessel and Hankel functions of the first kind, respectively. Substituting Green's function $G_k(\mathbf{r}, \mathbf{r}_0)$ into Eq. (11), we get,

$$\begin{aligned} & \iint_{\Omega_0} \tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} Y_l^n(\theta_0, \varphi_0) d\Omega_0 \\ &= k^2 c \iiint_V dV \sum_{l=0}^{+\infty} H_l^{(1)}(k\rho_0) \cdot J_l(k\rho) \cdot \mathbf{F}(\rho, \theta, \varphi) \sum_{m=-l}^l Y_l^m(\theta, \varphi) \iint_{\Omega_0} d\Omega_0 Y_l^{m*}(\theta_0, \varphi_0) Y_l^n(\theta_0, \varphi_0) \end{aligned} \quad (12)$$

Employing the orthogonal relationship of spherical harmonics

$$\iint_{\Omega_0} d\Omega_0 Y_l^{m*}(\theta_0, \varphi_0) Y_l^n(\theta_0, \varphi_0) = \delta(l, l') \delta(m, n),$$

we can then derive

$$\iint_{\Omega_0} d\Omega_0 \tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} Y_l^n(\theta_0, \varphi_0) = k^2 c \iiint_V dV J_l(k\rho) H_l^{(1)}(k\rho_0) Y_l^n(\theta, \varphi) \cdot \mathbf{F}(\rho, \theta, \varphi) \quad (13)$$

By dropping the primes, multiplying both hand sides of Eq. (13) by $J_l(k\rho')/H_l^{(1)}(k\rho_0)$, integrating k from 0 to $+\infty$ and then multiplying both sides by $Y_l^{n*}(\theta', \varphi')$ and summing n from $-l$ to $+l$, and l from 0 to $+\infty$, one can get

$$\begin{aligned} & \int_0^{+\infty} dk \cdot \iint_{\Omega_0} d\Omega_0 \tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} \sum_{l=0}^{+\infty} \sum_{n=-l}^l Y_l^n(\theta_0, \varphi_0) Y_l^{n*}(\theta', \varphi') / H_l^{(1)}(k\rho_0) \cdot J_l(k\rho') \\ &= c \iiint_V dV \cdot \mathbf{F} \cdot \int_0^{+\infty} dk \cdot k^2 J_l(k\rho) J_l(k\rho') \sum_{l=0}^{+\infty} \sum_{n=-l}^l Y_l^n(\theta, \varphi) Y_l^{n*}(\theta', \varphi') \\ &= c \iiint_V dV \cdot \mathbf{F} \cdot \frac{\pi}{2\rho^2} \delta(\rho - \rho') \cdot \frac{\delta(\theta - \theta') \delta(\varphi - \varphi')}{\sin \theta} \\ &= \frac{\pi}{2} c \cdot \mathbf{F}(\rho', \theta', \varphi') \end{aligned} \quad (14)$$

In the derivation of Eq. (14), the identity of the Bessel function

$$\int_0^{+\infty} dk \cdot k^2 J_l(k\rho) J_l(k\rho') = \frac{\pi}{2\rho^2} \delta(\rho - \rho') \quad (15)$$

and the identity of the spherical harmonics

$$\sum_{l=0}^{+\infty} \sum_{n=-l}^l Y_l^n(\theta, \varphi) Y_l^{n*}(\theta', \varphi') = \frac{\delta(\theta - \theta') \delta(\varphi - \varphi')}{\sin \theta} \quad (16)$$

were used. Because of the far field approximation [2], $1/H_n^{(1)}(k\rho_0) \approx (k\rho_0)^2 H_n^{(2)}(k\rho_0)$, considering the complex conjugate of Green's function

$$G_k^*(\mathbf{r}, \mathbf{r}_0) = -jk \sum_{l=0}^{+\infty} \sum_{n=-l}^l Y_l^n(\theta_0, \varphi_0) Y_l^{n*}(\theta, \varphi) \cdot H_l^{(2)}(k\rho_0) \cdot J_l(k\rho), \quad \text{and dropping the primes in Eq. (14), we can get}$$

$$\begin{aligned} \mathbf{F}(\rho, \theta, \varphi) &= \frac{2}{\pi c} \int_0^{+\infty} dk \iint_{\Omega_0} \rho_0^2 d\Omega_0 \tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} \cdot k^2 \sum_{l=0}^{+\infty} \sum_{n=-l}^l Y_l^n(\theta_0, \varphi_0) Y_l^{n*}(\theta, \varphi) \cdot H_l^{(2)}(k\rho_0) \cdot J_l(k\rho) \\ &= \frac{2}{\pi c} \iint_{S_0} dS_0 \cdot \int_0^{+\infty} \tilde{p}(\mathbf{r}_0, k) \mathbf{n}_{r_0} dk (jk) \cdot \frac{\exp(-ik|\mathbf{r}_0 - \mathbf{r}|)}{4\pi|\mathbf{r}_0 - \mathbf{r}|} \end{aligned} \quad (17)$$

where S_0 is a spherical surface with radius ρ_0 . Because $\tilde{p}(\mathbf{r}_0, t)$ is a real function, the integral of k can be extended to $-\infty$ together with dividing the whole integral by two to hold the Eq. (17). We can then expand the vectors in Eq. (17) to the Cartesian coordinate system after applying the inverse Fourier transformation and the final reconstruction formula can be written as

$$\begin{aligned} &|\mathbf{F}(\rho, \theta, \varphi)| (\sin(\theta_F) \cos(\varphi_F) \mathbf{i} + \sin(\theta_F) \sin(\varphi_F) \mathbf{j} + \cos(\theta_F) \mathbf{k}) \\ &= -\frac{1}{2\pi c^3} \iint_{S_0} dS_0 \frac{1}{r} \frac{\partial p(\mathbf{r}_0, t)}{\partial t} \Big|_{t=|\mathbf{r}_0 - \mathbf{r}|/c} \cdot (\sin(\theta_0) \cos(\varphi_0) \mathbf{i} + \sin(\theta_0) \sin(\varphi_0) \mathbf{j} + \cos(\theta_0) \mathbf{k}) \end{aligned} \quad (18)$$

where (θ_F, φ_F) and (θ_0, φ_0) are the zenith and azimuth of vector $\mathbf{F}$ and $\mathbf{n}_{r_0}$ respectively, and $|\mathbf{F}|$ is the amplitude of the force vector $\mathbf{F}$. As indicated in Eq. (17), we can reconstruct the vectorial acoustic source field by time reversing the vectorized acoustic pressure measurements.

If we rewrite Eq. (18) in its vector form and replace the time item by Green's function, we can get Eq. (19) as shown below.

$$\mathbf{F}(\rho, \theta, \varphi) = -\frac{2}{c^2} \iint_{S_0} dS_0 \frac{\partial \mathbf{p}(\mathbf{r}_0, t)}{\partial t} G_k^*(\mathbf{r}, \mathbf{r}_0) \quad (19)$$

Correspondingly Eq. (5) in time-domain can be expressed as follows,

$$p(\mathbf{r}_0, t) = -c \iiint_V dv \cdot \nabla \cdot \mathbf{F} \cdot G(\mathbf{r}, \mathbf{r}_0) \quad (20)$$

Comparing Eq. (19) and Eq. (20), they are similar in form. It can be interpreted in the similar way as the principle of acoustic reciprocity. The acoustic reciprocity principle states that an acoustic response remains the same when the source and receiver are interchanged.

In the view of classical principle of point source, integrating all of the acoustic point sources and multiplying by their corresponding Green's function can be used to describe the transmitted acoustic pressure at the receiver. Likewise, the acoustic pressure at each receiving point multiplied by the conjugate of the Green's function can be used to reconstruct the acoustic pressure at each source point. Correspondingly, applying acoustic reciprocity to explain Eq. (18), we can consider the driving force field and the vectorized

pressure item $\frac{\partial \mathbf{p}(\mathbf{r}_0, t)}{\partial t}$ as two counterparts in the principle of acoustic reciprocity. Integrating the divergence of the force sources leads to the transmitted acoustic pressure at the receivers. Correspondingly integrating the time derivative of the vectorized acoustic pressure can be used to reconstruct the vector force field at each source point. In theory, the present reconstruction method expands the application domain of the existing principle of acoustic reciprocity.

III. SIMULATION

To demonstrate the validity of the proposed reconstruction method, we conducted a three-dimensional numerical simulation study, with the assumption that the wave is omnidirectional (i.e. spherical wave) and the media is acoustically homogeneous. The forward and inverse simulations were carried out mainly according to equation (5) and equation (18), which were basically forward propagation and filtered back projection algorithms respectively. The finite difference method was used to calculate the divergence of the vector field.

As shown in Fig. 2 (a), the original force field has two spherical regions. One region is centered at (7, 0, 3) mm and has a radius of 10 mm while the other region is centered at (-10, -2, -2) mm and has a radius of 6 mm. Each region has a force distribution along the radial direction from the center to the periphery. The force amplitude at each spherical center is zero and increases linearly along the radius. The simulation was done using an imaging area of 38.4mm × 38.4mm × 38.4mm with calculation grids of 128 × 128 × 128. The acoustic signals driven by the force field were computed at 7080 detection positions that were evenly distributed on a spherical surface with a radius of 128 mm. Here we used idealized measurement surface or aperture to verify the proposed reconstruction algorithm. Finite and limited aperture would, as expected, give imperfect reconstruction and the corresponding effects will be evaluated in future investigations. As shown in Fig. 2 (b), the reconstructed distribution of force vector is well-correlated with the actual distribution. In order to give a more clear comparison, Fig. 2(c) and Fig. 2(d) illustrate the original and reconstructed force amplitude in the Z=0 plane. Fig. 2(e) displays the profile of the force amplitude along the axis of Z=0 and Y=0. It can be seen from these figures that the reconstructed force field agrees well with the original field. Only near the edge of the spherical object there are some smooth effects, which are caused by numerical computation.

In the derivation procedure of Eq. (18), approximations were applied while evaluating the gradient of Green's function, complex conjugate of Green's function and unit vector direction of $\mathbf{n}_{(r, r_0)}$. All of them are introduced under the far-field condition where $r_0 \gg r$. In order to get a better understanding on how big the difference between r_0 and r is sufficient to make the reconstruction algorithm valid, we performed a simulation study with measurement radii varying from 1.1 to 10 times the radius of the imaging area. The correlation coefficient (CC), relative error (RE) and average angular error (AAE) were used to evaluate the reconstruction quality. Fig. 3(a) is the curve of CC between the reconstructed force field and the original field as a function of the ratio between detecting radius and the radius of the imaging area. Fig. 3(b) shows the curves of amplitude $\underline{RE}_x$, $\underline{RE}_y$, $\underline{RE}_z$, green, blue, and black color lines represent the corresponding amplitude REs of F_x, F_y, F_z and F ,

respectively. The AAE, which is defined as the average of all the deviation angles between the reconstructed vectors and original vectors, was less than 0.5° and varied by only 0.03° , as shown in Fig. 3 (c). The CCs between the x, y, z components or the amplitude of the original and reconstructed force field were close to 98%. The corresponding REs were approximately 18%. The CCs and REs both varied less than 1% as the measuring radius became larger. This simulation study suggests that the reconstruction of the vectorial source field using the proposed method is not sensitive to changes in the measuring radius. It also suggests that the proposed reconstruction method can provide a fairly good reconstruction of the force field if the observing radius is larger than twice the radius of the imaging area.

The present method for the reconstruction of a vector force source field described above can be easily applied to the magneto-acoustic technique [1,10–12] that has been developed to image the conductivity distribution of biological tissue. In the magneto-acoustic imaging with magnetic induction approach, the Lorentz force acting on the induced current is the primary driving force field. In these approaches, the Lorentz force is always perpendicular to the static magnetic field. Taking the direction of the static magnetic field as the Z axis in a cylindrical coordinate system and applying a similar derivation procedure in cylindrical coordinates, the reconstruction formula of the Lorentz force field can be derived as

$$|\mathbf{F}(\rho, \varphi, z)| \exp(i\theta_F) = -\frac{1}{2\pi c^3} \iint_{S_0} dS_0 \sqrt{1 - \frac{(z_0 - z)^2}{|\mathbf{r} - \mathbf{r}_0|^2}} \frac{1}{t} \frac{\partial p(\mathbf{r}_0, t)}{\partial t} \bigg|_{t=|\mathbf{r}-\mathbf{r}_0|/c} \exp(i\theta_0) \quad (21)$$

where θ_0 and θ_F are the azimuth of the vectors $\mathbf{r}_0$ and $\mathbf{F}$, respectively. Based on the reconstructed Lorentz force field, the induced current in biological tissue can then be calculated.

In order to demonstrate the merits of the present method as applied to magnetoacoustic tomography, a simulation study using the concentric spherical model of MAT-MI in Ref [12] was conducted. The conductivity values referring to different concentric spherical layers from inside to outside are 0.6, 0.2 and 0 S/m. Simulated pressure signals from 6600 transducer sites distributed on a cylindrical surface with radius of 38.4 mm were used to feed into the proposed reconstruction algorithm. It can be shown that in this model the induced eddy current are mainly constrained in the XY plane. If $\mathbf{B}_0$ is uniform and in the z

direction, then $\nabla \times \mathbf{F} = B_0 \frac{\partial \mathbf{J}}{\partial z}$. If electric current changes slowly in Z direction, the curl component of $\mathbf{F}$ can be ignored, and then the proposed algorithm can reconstruct the distribution of force uniquely. Actually, in the concentric spherical model, the $\nabla \times \mathbf{F}$ is non-zero only at those conductivity boundaries and that is where the main reconstruction error occurs. Using the proposed method here, a current density distribution can be reconstructed without employing complicated experiment setup as mentioned in Ref [1]. Based on the reconstructed current density distribution and Ohm's law $\mathbf{J} = \sigma \mathbf{E}$, assuming the electric field $\mathbf{E}$ is similar to that induced in a uniform conductive media by the magnetic stimulation, a better conductivity reconstruction was shown as Fig. 4. Among them Fig. 4(a) is the modeled conductivity image slice at $Z=0$ plane. The corresponding induced current density distribution is shown in Fig. 4(c). Using Eq. (21), the current density can be reconstructed and it is quite similar to the original current density distribution as shown in Fig. 4(d). The CC and RE between the original and reconstructed current density amplitudes in the whole volume are 97.8% and 22.6%, respectively. Fig. 4(b) shows the reconstructed conductivity image slice at $Z=0$ plane. The CC and RE between the conductivity image slices shown in Figs. 4 (a) and (b) are 99.23% and 16.46%, respectively.

As compared to the results in Ref [12], the present method enables reconstruction of the source current density using a closed-form solution, leading to much better performance in conductivity reconstruction.

IV. DISCUSSION

Imaging tissue electrical impedance properties with non-invasive measurements is an important research topic in the field of medical imaging. The magnetoacoustic tomography with magnetic induction (MAT-MI), as a new imaging modality of imaging electrical impedance [1, 10–12], has been proposed to achieve non-invasive tissue electrical impedance imaging with high spatial resolution.

In the previous reconstruction algorithms of MAT-MI, the inverse problem can be divided into two steps. In the first step, divergence of Lorentz force $\nabla \cdot (\mathbf{J} \times \mathbf{B}_0)$ was reconstructed from acoustic pressure measured around an object. In the second step, by assuming the object is piecewise homogeneous, its conductivity distribution can be reconstructed from divergence of Lorentz force. In the second step, the reconstruction on the conductivity boundaries represents a challenge. As comparison, the proposed new method gives an analytical formula to obtain $\mathbf{J} \times \mathbf{B}_0$ instead of $\nabla \cdot (\mathbf{J} \times \mathbf{B}_0)$. This can avoid the need to use median filter and it can give an accurate estimation on the *Lorentz* force distribution and a better conductivity reconstruction. As shown in the present simulation study, the proposed new method can increase the correlation coefficient of the reconstruction by about 10%, and decrease relative error by about 10%.

Like any other algorithm, the proposed method has its own limitation. According to Helmholtz theorem, any vector can be written as a combination of an irrotational part and a solenoidal part as: $\mathbf{V} = \nabla \phi + \nabla \times \Psi$, where ϕ and Ψ are scalar and vector potential. Generally, in order to determine a vector, three types of data are required: divergence of the vector, curl of the vector and the corresponding boundary conditions. If the vector field is divergenceless or curl-free in the domain of reconstruction field, it may be possible to reconstruct the vector field with just one kind of radiation data, such as the weighted line-integral data in the radiation direction as in the reconstruction of fluid velocity field [16, 17]. For example, if the velocity field $\mathbf{v}(\mathbf{r})$ is divergenceless, by solving Laplace's equation, the irrotational component $\nabla \phi$ is determined uniquely by the normal component of $\mathbf{v}(\mathbf{r})$ on the boundary and the solenoidal component $\nabla \times \Psi$ can be reconstructed uniquely by a complete set of line integrals of $\mathbf{v}(\mathbf{r})$ through the radiation direction [16, 24]. In the present paper, the method we proposed is to solve the irrotational component $\nabla \phi$ under the condition that the vector field $\mathbf{v}(\mathbf{r})$ is curl-free, e.t. $\nabla \times \Psi = \mathbf{0}$. In Eq. (5), as the divergence of vector force field $\mathbf{F}$ serves as the source in the wave equation, we can not determine the vector force field $\mathbf{F}$ uniquely, unless we have the precondition that the curl component of $\mathbf{F}$ is zero or small enough to be ignored. As we discussed above, this assumption could be met in the MAT-MI setup under certain conditions. In our simulation shown in Fig. 2, all the force sources we pre-set were vertical to their boundary surfaces, which satisfy the condition that the solenoidal component $\nabla \times \Psi = \mathbf{0}$, otherwise the reconstructed image only represents the irrotational component of vector force.

Besides the condition of curl-free field, the approximations used in our derivation should be discussed here to clarify its rationale. The first approximation we used is the far-field approximation. This is a widely used approximation in acoustic field computation, when the radius of observing point is much larger than the size of acoustic source. In order to estimate the possible maximum error, it is necessary to set some parameters, such as radius of imaging region $|\mathbf{r}| = 5\text{cm}$, radius of acoustic measurement point $\mathbf{r}_0 = 25\text{cm}$ and working frequency $f=500\text{kHz}$. In our derivation, the first place where we use the far field

approximation is in Eq. (7). Here we ignore the $1/|\mathbf{r}_0 - \mathbf{r}|$ term in the gradient of Green's function, because the $1/|\mathbf{r}_0 - \mathbf{r}|$ term is much smaller than j/k , the maximum relative error is $1/(k|\mathbf{r}_0 - \mathbf{r}|)$, only 0.25% due to this ignoring. This approximation was also used in Eq. (8) and Eq. (9), so the total error caused by this approximation should be three times of 0.25%, which is 0.75%. Another far field approximation was used in using the complex conjugate of Green's function. Its' error is at the level of the square of $(1/k|\mathbf{r}_0 - \mathbf{r}|)^2$ [2], so it should be negligible. The core approximation in the derivation is where $\mathbf{n}_{(r,r_0)}$ is approximated by $\mathbf{n}_{r_0}$ after Eq. (7). Here the worst angular error $\Delta\theta$, which happens at some boundary points with $\mathbf{r}$ vertical to $\mathbf{r}_0$, should be equal to $ac, \tan(r/r_0)$, and it can reach 11.3° . The relative error

caused by this approximation is $\left|1 - \frac{F \cos(\theta)}{F \cos(\theta + \Delta\theta)}\right|$, where the θ is the angle between $\mathbf{F}$ and $\mathbf{r}_0$, and $\Delta\theta$ is the angle between $\mathbf{n}_{(r,r_0)}$ and $\mathbf{n}_{r_0}$. Because each reconstructed point in tomography is a weighted average of all the measurement data, the maximum absolute error should be integrated over spherical surface and the relative error should be written as $|1 - 1/\cos(\Delta\theta)|$. As the maximum $\Delta\theta$ is 11.3° , the relative error of this direction approximation is about 2%. It is obvious that the main error of this algorithm comes from the approximation of $\mathbf{n}_{(r,r_0)}$ by $\mathbf{n}_{r_0}$. If the ratio of the observation radius to the imaging area radius is set to 2, the maximum $\Delta\theta$ will be 26.6° and the relative error will become 12%. Some simulation results of the effect of far field approximation were shown in Fig. 3.

In the present derivation, we also assumed that the medium is acoustically homogenous, and the effect of acoustic heterogeneity in soft tissues is negligible. The acoustic heterogeneity of soft tissue is less than 10%, and the acoustic pathway in MAT-MI is about half the length of conventional diagnostic ultrasound. Furthermore, the effect of acoustic heterogeneity in MAT-MI is quite similar as that in photoacoustic tomography (PAT), which has been demonstrated negligible both theoretically [25] and experimentally [26]. Normally the acoustic heterogeneity could be considered as a secondary scattering acoustic source. If the amplitude of these secondary acoustic signals is small, acoustic signals generated by them can be taken as background noise. Corresponding random noise model may be used to investigate how acoustic heterogeneity influences the proposed reconstruction algorithm, but it is beyond the scope of this paper.

In summary, we have developed a theory for reconstruction of the vector acoustic source field using only acoustic signals measured on a spherical surface enclosing the imaged source in 3-dimensional space. The present theory in an acoustically homogeneous media is verified by computer simulations. Numerical results of the reconstruction with different observation distances show that the present method is not sensitive to the observation radius and the ratio of the observation radius to the imaging area radius should be set to be larger than 2. In theory, the present method expands the existing acoustic reciprocity principle from a scalar field to a vector field. Application of the present theory to MAT-MI imaging suggests the possibility relating the current source field with acoustic measurements over a surface enclosing the current sources. The present computer simulation results demonstrate the promise of the proposed method in imaging current source distribution and electrical conductivity distribution from acoustic measurements. The present study should be of interest not only to MAT-MI researchers but also to those working on any magnetoacoustic tomography method replying on Lorentz force generated sources [4, 7, 20, 21], and investigators working on bioimpedance imaging.

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Fig 1. Schematic diagram of acoustic tomography with pulsed driving force field $\mathbf{F}$ as acoustic sources. $\mathbf{r}$ is the position vector of a source point, and $\mathbf{r}_0$ is the position vector of a receiving point. The object is shaded, and the transducers scan on a spherical surface.

Fig 2.

(a) The original force vector distribution, and (b) the corresponding reconstructed vector distribution of force; (c) and (d) are images of original and reconstructed amplitude of force distribution at $Z=0$ slice, respectively. (e) is the profile of force amplitude along the horizontal central line at ($Z=0, Y=0$) of (c) and (d). The red line is for reconstructed amplitude value, and the blue one is for original value.

Fig 3.

(a) The curve of correlation coefficient between the reconstructed force field and the original force field as a function of the ratio between the detecting radius and the imaging area radius. (b) The corresponding curve of Relative Error. (c) The corresponding curve of average angular error.

Fig 4.

(a) and (b) are the images of original conductivity and reconstructed conductivity distribution at $Z=0$ slice of a spherical model in the MAT-MI simulation; (c) and (d) are the corresponding original and reconstructed current density distributions. See text for details.