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An Implementation of Calderón's Method for 3-D Limited-View EIT

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Abstract

Mathematical interest in electrical impedance tomography has been strong since the publication of Calderón's foundational paper. This paper introduced the idea of applying external voltage patterns to a medium such that, assuming that the medium is sufficiently close to a constant admittivity, the reconstruction can be accomplished directly by inverse Fourier transform. Motivated by Calderón's method, we have developed a variant of the algorithm which is applicable to the case of measurement on only a part of the boundary and on discrete electrodes. Here we determine voltage or current patterns to apply to the electrodes which optimally approximate Calderón's special functions in the interior. Furthermore, in three dimensions and higher, Calderón's method allows each point in Fourier space to be computed in a multiplicity of ways. We show that by making use of the inherent redundancy in our measurements, we can significantly improve the quality of the static images produced by our algorithm.

Index Terms

Breast cancer screening; direct reconstruction; electrode models; image reconstruction; impedance imaging

I. INTRODUCTION

In this paper, we expand upon a novel image reconstruction approach for electrical impedance tomography (EIT) introduced by Calderón in 1980 [1]. In Calderón's method, if we are able to apply a particular set of oscillatory and exponentially weighted potential patterns to the boundary of a medium, the image reconstruction reduces to an inverse Fourier transform. It is quite difficult in practice to apply such a set of patterns, but we can synthesize the currents that we would have measured had we applied these patterns. To date,

Calderón's method has been applied to 2-D EIT [2], but, to our knowledge, this paper represents the first application of the method to the reconstruction of 3-D experimental data. Analytical solutions to Calderón's inversion formula have also been found for a number of special cases [3], [4]. The effect on Calderón's method of having only finitely many electrodes and of limited measurement precision has also been studied [5].

In order to obtain a practical image reconstruction algorithm, it was necessary to modify Calderón's approach somewhat. In this work, we were motivated by the work of Ng [6], who explored the problem of limited-view EIT and showed the uniqueness of the inverse problem provided that the conductivity is constant in an arbitrarily small neighborhood of the boundary.

In particular, it was necessary to account for the effect of finite electrodes and to develop a modification of the algorithm for the realistic case where we measure on only part of the boundary. Our strategy, then, was to develop a set of functions which satisfied a particular set of boundary conditions, most importantly that the current density should zero on the part of the boundary not covered by electrodes. Our goal, then, was to optimally approximate Calderón's special functions in the interior of the medium while satisfying the additional boundary conditions.

We will consider two electrode models here: the ave-gap model [7] and the complete electrode model (CEM) [8]–[11]. In previous work, it has been suggested that the CEM predicts experimental measurements more accurately than the ave-gap model [7], [12] by requiring that the potential be constant over the exterior surface of each electrode and by modeling the electrode-medium interface as being, essentially, an infinitely thin resistive layer. This layer is intended to account for the electrochemical interaction between the electrodes and body. In using the CEM for Calderón's reconstruction method, we first estimate a set of electrode surface impedance values which best fit our data and then compute the reconstruction making use of these surface impedances to calculate our forward model data.

EIT is typically used in two modes: static imaging and difference imaging [13]. In difference imaging, we experimentally measure a medium in two states. For example, in pulmonary monitoring, we can measure the difference between the inspiration and expiration. Likewise, we can make measurements at various points in the cardiac cycle, using one of these points as a baseline measurement. In static imaging, in contrast, no baseline measurement is available and instead the baseline data are synthesized using a forward model. However, it is quite difficult to develop an accurate forward model for the propagation of electrical currents through the body, as many factors confound our measurements: uncertainty about the imaging geometry, possible anisotropy of tissues in the body, electrode contact, inaccuracy of the approximations used in our models (i.e., it is likely necessary to use Maxwell's equations rather than Laplace's equation at higher temporal frequencies). For these reasons, static imaging tends to be significantly more difficult than difference imaging, meaning that the images produced tend to be plagued by artifacts caused by modeling error. As our primary interest in this study is EIT for breast imaging, in which case static imaging is compulsory, we describe below a method for improving the quality of the static images that we are able to generate.

Intriguingly, Calderón's method leaves ambiguous exactly how the reconstruction for each point in $\mathbf{k}$ -space should be estimated. Specifically, in three dimensions and higher, we can obtain each point in $\mathbf{k}$ -space by applying a variety of patterns to the boundary. In some intuitive sense, the reason for this is that we are measuring 4-D data (currents and voltages on 2-D surfaces) and we are reconstructing a 3-D entity, the permittivity. Thus, intuitively,

we have a “redundant” dimension which we can exploit. We have found that we can significantly improve the static imaging performance of our algorithm by sampling the space of patterns which correspond to each point in $\mathbf{k}$ -space. In this way, we use the redundancy of the measured data to reduce the effect of modeling errors on the reconstruction.

Although Calderón’s method has remained more a tool for mathematical analysis of the EIT problem rather than a practical approach for imaging, it has a number of important advantages. First of all, there is no need to generate a reconstruction mesh and a corresponding Jacobian matrix. The data for each point in $\mathbf{k}$ -space can easily be computed by low-dimensional transformation of the measured data. Finally, as compared to Tikhonov regularization, the regularization parameters which we will employ have a clear, intuitive meaning. Specifically, our regularization parameters are the cutoff values in spatial frequency in the x , y , and z directions. In contrast, the meaning of a Tikhonov regularization operator is dependent on the singular value spectrum of a system’s Jacobian matrix.

While Calderón’s reconstruction approach has not found practical application until recently [2], it did motivate a great deal of mathematical research analyzing the uniqueness and stability of the EIT inverse problem. In particular, Sylvester and Uhlmann, using an approach inspired by Calderón’s method, showed that in three or more dimensions [14], the reconstruction problem could be solved with arbitrary accuracy if increasingly highly oscillatory patterns of voltages could be applied to the boundary. Due to the need to apply such highly oscillatory patterns, the Sylvester–Uhlmann algorithm cannot be directly implemented for experimental data, but, in future work, we plan to investigate whether a variation of their method can be made practical.

In two dimensions, there has been tremendous progress in the area of direct reconstructions for EIT. Nachman introduced a d-bar method to directly compute the solution of the full nonlinear inverse problem [15]. In this approach, the inverse conductivity problem is transformed into an inverse Schrodinger problem. Nachman then computes a nonlinear scattering transform of the boundary data, which can then be inverted by means of a particular well-posed d-bar equation. Isaacson, Mueller, and Siltanen have developed practical implementations of Nachman’s algorithm [16], [17], introducing a number of useful approximations for the scattering transform. Some suggestions for applying the d-bar method to 3-D reconstructions have recently been made [18]. The organization of this paper is as follows. We describe the physical model that governs electrical propagation in tissue in the low-frequency regime in Section II. The analytical solution of the forward model is discussed in Section III. We describe our new reconstruction algorithm, as well as the NOSER algorithm, for comparison in Section IV, and experimental results are shown in Section V.

II. PHYSICS OF EIT

In this paper, we will assume that the potential within a medium due to an applied external current satisfies the conductivity equation (Laplace’s equation if the conductivity is constant) with a Neumann boundary condition

$$\begin{cases} \nabla \cdot \gamma(\mathbf{x}) \nabla u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega \\ \gamma(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial \nu} = j(\mathbf{x}) & \mathbf{x} \in \partial\Omega \end{cases} \quad (1)$$

for a bounded region Ω with boundary $\partial\Omega$. The outward-pointing normal vector to the boundary is ν and the externally applied current density is $j(\mathbf{x})$. The potential, as a function of position, $\mathbf{x}$, is $u(\mathbf{x})$, and the admittivity distribution is $\gamma(\mathbf{x})$. We note that (1) specifies the

potential only up to a constant and an additional condition is needed to determine this constant.

We will consider two electrode models: the ave-gap model and the complete electrode model. In the ave-gap model, we assume that the current density is evenly distributed over the surface of each electrode and that the potential of each electrode is the mean of $u(\mathbf{x})$ over the area covered by the electrode

$$\begin{cases} j(\mathbf{x}) = \frac{1}{|e_\ell|} \mathbf{I}_\ell & \mathbf{x} \in e_\ell \\ j(\mathbf{x}) = 0 & \mathbf{x} \notin \cup_{\ell=1}^L e_\ell \end{cases} \quad (2)$$

$$U_\ell = \frac{1}{|e_\ell|} \int_{e_\ell} u(\mathbf{x}) dS \quad (3)$$

where e_ℓ is the spatial extent of electrode ℓ with area $|e_\ell|$, L is the total number of electrodes, and $\mathbf{I}_\ell$ U_ℓ are the currents and voltages applied to and measured on electrode ℓ respectively.

The complete electrode model introduces a number of realistic assumptions in modeling the electrodes. First of all, the potential is assumed to be uniform over the surface of each electrode. Furthermore, the electrode surface impedance is modeled as an infinitely thin resistive layer between the electrode and the body. The following equations specify the potential within the body given externally applied currents and have been shown to be uniquely solvable [8]

$$\begin{cases} \nabla \cdot \gamma(\mathbf{x}) \nabla u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega \\ \int_{e_\ell} \gamma \frac{\partial u(\mathbf{x})}{\partial \nu} dS = \mathbf{I}_\ell & \mathbf{x} \in e_\ell, \ell = 1, 2, \dots, L \\ \gamma \frac{\partial u(\mathbf{x})}{\partial \nu} = 0 & \mathbf{x} \notin \cup_{\ell=1}^L e_\ell \\ u(\mathbf{x}) + z_\ell \gamma \frac{\partial u(\mathbf{x})}{\partial \nu} = U_\ell & \mathbf{x} \in e_\ell, \ell = 1, 2, \dots, L. \end{cases} \quad (4)$$

where an electrode's surface impedance is z_ℓ . In order to ensure uniqueness, we also need the following conditions:

$$\begin{aligned} \sum_{\ell=1}^L \mathbf{I}_\ell &= 0 \\ \sum_{\ell=1}^L U_\ell &= 0. \end{aligned} \quad (5)$$

III. FORWARD MODELING

In our experiments, we consider a rectangular parallelepiped, with electrodes placed on its top and bottom surfaces only. This configuration, shown in Fig. 1, is intended to approximate the region of interest within a compressed breast and is being used in modeling our group's experiments combining EIT and mammography in human subjects.

In what follows, we derive analytical and semi-analytical expressions for computing the expected voltages on the electrodes due to all possible patterns of applied currents, the discrete Neumann-to-Dirichlet map. However, the Dirichlet-to-Neumann map can be computed using essentially the same approach. Here, we assume a homogeneous medium, with Neumann boundary conditions on the part of the boundary of the parallelepiped not in contact with the electrodes. In the ave-gap case, the solution to the forward problem can be

rapidly calculated using Fourier methods, while, in the case of the complete electrode model, the resulting voltages are calculated by inversion of a dense matrix. Further implementation details of the modeling and reconstruction using the ave-gap model in the compressed breast geometry can be found in [19] and [20]. The implementation of the CEM in this geometry, as well as a comparison of the ave-gap and complete electrode models for the 3-D limited-view case are described in [12].

A. Ave-Gap Model

In the case of the ave-gap forward model, we assume the following model for the rectangular parallelepiped:

$$\begin{cases} \nabla \cdot \gamma \nabla u(x, y, z) = 0 & 0 \leq x \leq h_1 \\ & 0 \leq y \leq h_2 \\ & -\frac{h_3}{2} \leq z \leq \frac{h_3}{2} \end{cases} \quad (6)$$

with the following boundary conditions:

$$\begin{cases} \gamma \frac{\partial u}{\partial \nu} = j^{\mathcal{T}}(x, y) & z = \frac{h_3}{2} \\ \gamma \frac{\partial u}{\partial \nu} = j^{\mathcal{B}}(x, y) & z = -\frac{h_3}{2} \\ \gamma \frac{\partial u}{\partial \nu} = 0 & x=0, x=h_1, y=0, y=h_2 \end{cases} \quad (7)$$

where $j^{\mathcal{T}}$ and $j^{\mathcal{B}}$ denote the current densities at the top and bottom planes, respectively, and ν is the unit outward normal to the body.

Using separation of variables and assuming that γ is constant, we see that the solution to (6) and (7) can be written as an infinite series

$$\begin{aligned} u(x, y, z) = & b_0 z \\ & + \sum_{n=0}^N \sum_{m=0}^N \cos(\lambda_n x) \cos(\lambda_m y) \\ & \times [a_{n,m} \cosh(\lambda_{n,m} z) + b_{n,m} \sinh(\lambda_{n,m} z)] \end{aligned} \quad (8)$$

where N is number of Fourier terms in our Fourier approximation, and

$$\begin{cases} \lambda_n = \frac{n\pi}{h_1} \\ \lambda_m = \frac{m\pi}{h_2} \\ \lambda_{n,m} = \sqrt{\lambda_n^2 + \lambda_m^2} \end{cases} \quad (9)$$

for $n, m = 0, 1, 2, \dots, N$.

Assuming the ave-gap model, we are able to specify $j^{\mathcal{T}}$ and $j^{\mathcal{B}}$ exactly and to solve for b_0 , $a_{n,m}$, and $b_{n,m}$ exactly for a given current pattern

$$a_{n,m} = \frac{j_{n,m}^{\mathcal{T}} + j_{n,m}^{\mathcal{B}}}{2\gamma \lambda_{n,m} \sinh\left(\lambda_{n,m} \frac{h_3}{2}\right)} \quad (10)$$

$$b_{n,m} = \frac{j_{n,m}^{\mathcal{T}} - j_{n,m}^{\mathcal{B}}}{2\gamma \lambda_{n,m} \cosh\left(\lambda_{n,m} \frac{h_3}{2}\right)} \quad (11)$$

$$b_0 = \frac{j_{0,0}^{\mathcal{T}}}{\gamma}. \quad (12)$$

We also assume that $b_{0,0} = 0$ and that $a_{0,0}$ is chosen such that $\sum_{\ell=1}^L \int_{e_\ell} u(\mathbf{x}) dS = 0$.

The Fourier coefficients $j_{n,m}^{\mathcal{T}}$ and $j_{n,m}^{\mathcal{B}}$ of the current density can be explicitly calculated as

$$j_{n,m}^* = \frac{c_{n,m}}{h_1 h_2} \int_0^{h_1} \int_0^{h_2} \cos(\lambda_n x) \cos(\lambda_m y) j^*(x, y) dx dy \quad (13)$$

where

$$\begin{cases} c_{n,m}=4 & n, m \neq 0 \\ c_{n,m}=2 & n=0 \text{ or } m=0 \\ c_{n,m}=1 & n=0 \text{ and } m=0 \end{cases} \quad (14)$$

and the superscript $*$ = $\mathbb{T}$ (top plane) or $*$ = $\mathbb{B}$ (bottom plane).

B. Complete Electrode Model

In the case of the CEM, we use the Ritz-Galerkin method to compute the currents and internal fields resulting from the application of specified voltages on the electrodes. Using the divergence theorem and applying the boundary conditions of the complete model, (4), we find

$$\begin{aligned} \int_{\Omega} \gamma \nabla u \cdot \nabla v dp + \sum_{\ell=1}^L \frac{1}{z_\ell} \left[\int_{e_\ell} u v dS - \frac{1}{|e_\ell|} \int_{e_\ell} u dS \int_{e_\ell} v dS \right] \\ = \sum_{\ell=1}^L \frac{I_\ell}{|e_\ell|} \int_{e_\ell} v dS \end{aligned} \quad (15)$$

for all test functions v . In order to obtain a discrete representation of (15), we suppose that both u and v can be written as a linear combination of a given set of orthonormal basis functions

$$u = \sum_{i=1}^P u_i \phi_i \quad (16)$$

$$v = \sum_{i=1}^P v_i \phi_i \quad (17)$$

where $P = 2(N+1)^2 - 1$. As the solution of the complete electrode model closely resembles that of the ave-gap model, we make use of the eigenvalues of the ave-gap model in (16) and 17. In the case of a rectangular parallelepiped, we use the following basis functions:

$$\left\{ \begin{array}{ll} \tilde{\phi}_i = z & i=1 \\ \phi_i = \cos(\lambda_n x) \cos(\lambda_m y) \cosh(\lambda_{n,m} z) & i=2 \dots (N+1)^2 \\ & 0 \leq n \leq N \\ & 0 \leq m \leq N \\ & n, m \neq 0 \\ \tilde{\phi}_i = \cos(\lambda_n x) \cos(\lambda_m y) \sinh(\lambda_{n,m} z) & i > (N+1)^2 \\ & 0 \leq n \leq N \\ & 0 \leq m \leq N \\ & n, m \neq 0 \end{array} \right. \quad (18)$$

We then obtain a basis which is orthonormal in the Sobolev space H^1 as follows:

$$\phi_i = \frac{1}{\sqrt{c_i}} \tilde{\phi}_i \quad (19)$$

$$c_i = \int_{x=0}^{h_x} \int_{y=0}^{h_y} \int_{z=-h_z/2}^{h_z/2} \nabla \tilde{\phi}_i \cdot \nabla \tilde{\phi}_i dx dy dz. \quad (20)$$

Discretizing (15) using the basis functions of (18) and (19) and applying the conditions specified by (5), we obtain the following linear system:

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C}^T & \mathbf{I} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{U} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \mathbf{e} \\ \mathbf{D} \mathbf{e} \\ \mathbf{0} \end{bmatrix} \quad (21)$$

$$\mathbf{A} = \mathbf{Q} + \mathbf{B} \quad (22)$$

and

$$\begin{aligned} Q_{p,q} &= \int_{\Omega} \gamma \nabla \phi_p \cdot \nabla \phi_q dV \\ B_{p,q} &= \sum_{i=1}^L \frac{1}{z_i} \int_{e_i} \phi_p \phi_q dS \\ C_{p,i} &= \frac{-1}{z_i} \int_{e_i} \phi_p dS \\ \mathbf{D} &= \text{diag} \left(\left[\frac{|e_1|}{z_1} \dots \frac{|e_L|}{z_L} \right] \right). \end{aligned} \quad (23)$$

Here, $\mathbf{u}$ represents basis function coefficients for the potential in the interior of the medium, $\mathbf{U}$ is an $L \times 1$ vector giving the measured potentials on the electrodes, and $\mathbf{e}$ is an $L \times 1$ vector specifying the currents on the electrodes. Also, $\mathbf{1}$ is an $1 \times L$ vector of all ones. The matrices $\mathbf{A}$ and $\mathbf{D}$ are of dimension $P \times P$, the matrix $\mathbf{C}$ is of dimension $P \times L$, and $\mathbf{I}$ is the $L \times L$ identity matrix. It can be shown that, if the admittivity is homogeneous, then $\mathbf{Q} = \gamma \mathbf{I}$, where $\mathbf{I}$ is the $P \times P$ identity matrix.

IV. RECONSTRUCTION ALGORITHMS

A. NOSER Method

In the NOSER method [21], we solve the linearized inverse problem by assuming that the medium is very nearly homogeneous and by discretizing the medium into a linear combination of nonoverlapping regions. We compute the Jacobian matrix analytically. Applying the divergence theorem to (1), we find that

$$\int_S \left[u_\alpha(\gamma_0) \gamma \frac{\partial u_\beta(\gamma)}{\partial \nu} - u_\beta(\gamma) \gamma_0 \frac{\partial u_\alpha(\gamma_0)}{\partial \nu} \right] dS \quad (24)$$

$$= \int_\Omega \eta \nabla u_\alpha(\gamma_0) \cdot \nabla u_\beta(\gamma) dp$$

where $\eta \equiv \gamma - \gamma_0$. The subscripts α and β denote pairs of current patterns. In order to reconstruct an image of the admittivity, we apply K linearly independent patterns, where it is generally true that $K = L - 1$. The term on the left-hand side of (24) represents the data matrix $\mathbf{D} \in \mathbb{C}^{K \times K}$, which can be computed as follows given the ave-gap model:

$$D_{\alpha,\beta} = \sum_{\ell=1}^L [I_{\ell,\alpha} U_{\ell,\beta}(\gamma_0) - I_{\ell,\beta} V_{\ell,\alpha}(\gamma)] \quad (25)$$

where $U_{\ell,\beta}(\gamma_0)$ is the voltage on electrode ℓ for current pattern β and a homogeneous complex admittivity γ_0 . Also, $V_{\ell,\alpha}(\gamma)$ is the voltage on electrode ℓ for current pattern α and for an in-homogeneous admittivity γ (i.e., with a target present). For difference imaging, we measure both $U_{\ell,\beta}(\gamma_0)$ and $V_{\ell,\alpha}(\gamma)$ for all current patterns, while for static imaging, $U_{\ell,\beta}(\gamma_0)$, the voltage expected to be measured on electrode ℓ for current pattern β in the absence of a target, is computed using an appropriate forward model (i.e., the ave-gap model or the complete electrode model). Since this calculation inevitably introduces modeling error, it is typically the case that difference imaging produces clearer images than those produced by static imaging, as we shall see in our experimental results.

Alternatively, we can write the data matrix as follows (and thus the forward model need only be computed once, for a homogeneous admittivity of 1 S/m)

$$D_{\alpha,\beta} = \sum_{\ell=1}^L \left[\frac{1}{\gamma_0} I_{\ell,\alpha} U_{\ell,\beta}(1) - I_{\ell,\beta} V_{\ell,\alpha}(\gamma) \right] \quad (26)$$

In the linearized reconstruction, we replace $\nabla u_\alpha(\gamma)$ with $\nabla u_\alpha(\gamma_0)$. Discretizing by setting

$\eta = \sum_{s=1}^{N_s} \eta_s \chi_s(\mathbf{x})$ where $\chi_s(\mathbf{x})$ is the characteristic function of η_s voxel, we then have

$$D_{\alpha,\beta} \approx \sum_{s=1}^{N_s} \eta_s J_{\alpha,\beta}^s \quad (27)$$

$$J_{\alpha,\beta}^s = \int_{\chi_s} \nabla u_\alpha(\gamma_0) \cdot \nabla u_\beta(\gamma_0) dV \quad (28)$$

where $\mathbf{J}$ is the Jacobian of the forward model with respect to a small perturbation in admittivity, Ω_s is the spatial extent of voxel s , and N_s is the total number of voxels.

We then obtain the solution of the regularized linear inverse problem as

$$\eta \approx [\mathbf{J}^T \mathbf{J} + r_1 \text{diag}(\mathbf{J}^T \mathbf{J}) + r_2 \mathbf{I}]^{-1} \mathbf{J}^T \mathbf{D} \quad (29)$$

where r_1 and r_2 are regularization parameters for NOSER-type and Tikhonov regularization, respectively, and we have reordered $D_{\alpha,\beta}$ and $J_{\alpha,\beta}^s$ into the vector $\mathbf{D}$ and the matrix $\mathbf{J}$ of sizes $K^2 \times 1$ and $K^2 \times N_s$, respectively. Finally, we note that our estimate of the homogeneous

admittivity, γ_0 is computed to minimize $\sum_{\alpha=1}^K \sum_{\beta=1}^K D_{\alpha,\beta}^2$, for which there is a simple

closed-form solution (obtained by differentiating the right-hand side of (26) with respect to $\rho_0 = 1/\gamma_0$).

B. Calderón's Method

In his seminal work on the subject of what is now called EIT, Calderón described a method for reconstructing the admittivity within a medium from complete knowledge of the Dirichlet-to-Neumann map, or the mapping from the set of all possible applied voltages within a particular function space to the corresponding measured currents. The Dirichlet-to-Neumann map, which we will denote by $\Lambda(\gamma)$, satisfies the following conditions:

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & u \in \Omega \\ u = g & u \in \partial\Omega \end{cases} \quad (30)$$

$$\begin{cases} \Lambda(\gamma)g = \gamma \frac{\partial u}{\partial \nu} & u \in \partial\Omega \end{cases} \quad (31)$$

where γ is an admittivity distribution, u is the potential, Ω , is a bounded region with boundary $\partial\Omega$, g is an applied potential distribution, and where ν is the unit outward normal direction to the boundary, $\partial\Omega$.

If the medium's admittivity is sufficiently close to a constant and we apply harmonic potentials, f and g , to the boundary, it can be shown that

$$\int_{\partial\Omega} f(\Lambda(\gamma) - \Lambda(\gamma_0))g \, dS \approx \int_{\Omega} (\gamma - \gamma_0) \nabla f \cdot \nabla g \, dV. \quad (32)$$

Calderón's innovation was to consider the following choices for f and g , which can easily be shown to be harmonic:

$$\begin{cases} f = e^{i\mathbf{k} \cdot \mathbf{x} + \mathbf{k}^\perp \cdot \mathbf{x}} \\ g = e^{i\mathbf{k} \cdot \mathbf{x} - \mathbf{k}^\perp \cdot \mathbf{x}} \end{cases} \quad (33)$$

where $\mathbf{k} \in \mathbb{R}^n$, $\mathbf{k}^\perp \in \mathbb{R}^n$, $\mathbf{k} \cdot \mathbf{k}^\perp = 0$, and $|\mathbf{k}| = |\mathbf{k}^\perp|$. We can then see that

$$\int_{\partial\Omega} f(\Lambda(\gamma) - \Lambda(\gamma_0))g \, dS \approx \int_{\Omega} -2|\mathbf{k}|^2 (\gamma - \gamma_0) e^{2i\mathbf{k} \cdot \mathbf{x}} \, dV. \quad (34)$$

Therefore, by applying a particular choice of potential to the boundary and measuring with respect to a particular choice of current pattern, we can obtain the Fourier transform of the perturbation in conductivity. The reconstruction can then be accomplished directly by inverse Fourier transform. In practice, however, it is not possible to physically apply extremely oscillatory voltage patterns to the boundary. Thus, we regularize by truncating the region in Fourier space in which we reconstruct (here the formula given is for $\mathbb{R}^3$)

$$\gamma - \gamma_0 \approx \int_{|\mathbf{k}_1| \leq M_1} \int_{|\mathbf{k}_2| \leq M_2} \int_{|\mathbf{k}_3| \leq M_3} \frac{-1}{2(2\pi)^3 |\mathbf{k}|^2} \times \mathcal{M}(\mathbf{k}) \mathcal{F}(\mathbf{k}) e^{-2\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \quad (35)$$

$$\mathcal{F}(\mathbf{k}) = \int_{\partial\Omega} e^{i\mathbf{k} \cdot \mathbf{x} + \mathbf{k}^\perp \cdot \mathbf{x}} (\Lambda(\gamma) - \Lambda(\gamma_0)) e^{i\mathbf{k} \cdot \mathbf{x} - \mathbf{k}^\perp \cdot \mathbf{x}} dS \quad (36)$$

where $\mathcal{M}(\mathbf{k})$ is a mollification function [2] and M_1 is the bound for the first element of $\mathbf{k}$, etc. In this paper, we use the following mollifier:

$$\mathcal{M}(\mathbf{k}) = e^{-|\mathbf{k}|^2/\sigma^2} \quad (37)$$

where we choose a value of σ to minimize the effect of Gibbs phenomena.

C. Generalization of Calderón's Method

Here we consider the problem of a Calderón-type reconstruction in three dimensions, with measurements made on only part of the boundary and using a discrete set of electrodes. We cover two electrode models here: the ave-gap model and the complete electrode model. In the case of the complete electrode model, we assume that we are able to estimate the electrode surface impedances exactly, using methods described in [22]. The problem of jointly estimating the body's admittivity distribution as well as the electrode surface impedances will be left for a future publication.

To accomplish the objective of modifying Calderón's method to allow for partial-view measurements, we approximate Calderón's special functions in the H^1 norm by applying currents only on the electrodes, as opposed to on the entire boundary, as in Calderón's original formulation. The inversion formula used is then

$$\gamma - \gamma_0 \approx \int_{|k_1| \leq M_1} \int_{|k_2| \leq M_2} \int_{|k_3| \leq M_3} \frac{-1}{2(2\pi)^3 |\mathbf{k}|^2} \times \mathcal{M}(\mathbf{k}) \mathcal{F}_\alpha(\mathbf{k}) e^{-2\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \quad (38)$$

$$\mathcal{F}_\alpha(\mathbf{k}) = \int_{\partial\Omega} \tilde{f}(\mathbf{k})(\Lambda(\gamma) - \Lambda(\gamma_0)) \tilde{g}(\mathbf{k}) dS. \quad (39)$$

In the case of the ave-gap model, $\tilde{f}(\mathbf{k})$ and $\tilde{g}(\mathbf{k})$ have the following properties:

$$\begin{cases} \frac{\partial \tilde{f}(\mathbf{k})}{\partial \nu} = \frac{I_\ell^{\tilde{f}(\mathbf{k})}}{|e_\ell|} & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{f}(\mathbf{k})}{\partial \nu} = 0 & \mathbf{x} \notin \bigcup_{\ell=1}^L e_\ell \\ \tilde{f}(\mathbf{k}) \approx e^{i\mathbf{k} \cdot \mathbf{x} + \mathbf{k}^\perp \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (40)$$

$$\begin{cases} \frac{\partial \tilde{g}(\mathbf{k})}{\partial \nu} = \frac{I_\ell^{\tilde{g}(\mathbf{k})}}{|e_\ell|} & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{g}(\mathbf{k})}{\partial \nu} = 0 & \mathbf{x} \notin \bigcup_{\ell=1}^L e_\ell \\ \tilde{g}(\mathbf{k}) \approx e^{i\mathbf{k} \cdot \mathbf{x} - \mathbf{k}^\perp \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (41)$$

where $|e_\ell|$ is the spatial extent of electrode ℓ , L is the total number of electrodes, and the applied currents on electrode ℓ used to generate $\tilde{f}$ and $\tilde{g}$ are $I_\ell^{\tilde{f}(\mathbf{k})}$ and $I_\ell^{\tilde{g}(\mathbf{k})}$, respectively.

Alternatively, for the complete electrode model, $\tilde{f}(\mathbf{k}, \mathbf{z})$ and $\tilde{g}(\mathbf{k}, \mathbf{z})$ are also dependent on the electrode surface impedances, $\mathbf{z}$, and have the following properties:

$$\begin{cases} \int_{e_\ell} \frac{\partial \tilde{f}(\mathbf{k}, \mathbf{z})}{\partial \nu} dS = I_\ell^{\tilde{f}(\mathbf{k}, \mathbf{z})} \\ \tilde{f}(\mathbf{k}, \mathbf{z}) + z_\ell \frac{\partial \tilde{f}(\mathbf{k}, \mathbf{z})}{\partial \nu} = U_\ell^{\tilde{f}(\mathbf{k}, \mathbf{z})} & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{f}(\mathbf{k}, \mathbf{z})}{\partial \nu} = 0 & \mathbf{x} \notin \bigcup_{\ell=1}^L e_\ell \\ \tilde{f}(\mathbf{k}, \mathbf{z}) \approx e^{i\mathbf{k} \cdot \mathbf{x} + \mathbf{k}^\perp \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (42)$$

$$\begin{cases} \int_{e_\ell} \frac{\partial \tilde{g}(\mathbf{k}, \mathbf{z})}{\partial \nu} dS = I_\ell^{\tilde{g}(\mathbf{k}, \mathbf{z})} \\ \tilde{g}(\mathbf{k}, \mathbf{z}) + z_\ell \frac{\partial \tilde{g}(\mathbf{k}, \mathbf{z})}{\partial \nu} = U_\ell^{\tilde{g}(\mathbf{k}, \mathbf{z})} & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{g}(\mathbf{k}, \mathbf{z})}{\partial \nu} = 0 & \mathbf{x} \notin \bigcup_{\ell=1}^L e_\ell \\ \tilde{g}(\mathbf{k}, \mathbf{z}) \approx e^{i\mathbf{k} \cdot \mathbf{x} - \mathbf{k}^\perp \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (43)$$

To approximate $\tilde{f}(\mathbf{k})$ and $\tilde{g}(\mathbf{k})$ for each value of $\mathbf{k}$ desired, we apply the following procedure. We first decompose the potential within the medium into a linear combination of basis functions which are orthonormal in H^1 . Specifically, we require that

$$\begin{cases} \langle \phi_i, \phi_j \rangle_{H^1} = \int_\Omega \nabla \phi_i^* \cdot \nabla \phi_j dV = I & i=j \\ \langle \phi_i, \phi_j \rangle_{H^1} = \int_\Omega \nabla \phi_i^* \cdot \nabla \phi_j dV = 0 & i \neq j \end{cases} \quad (44)$$

In the case of the rectangular parallelepiped depicted in Fig. 1, we make use of the basis functions of (18) and (19), which can be shown to be orthonormal in H^1 .

We assume that we apply a complete set of K voltage patterns on the electrodes and measure K corresponding current patterns, encoded as the $L \times K$ matrices $\mathbf{V}$ and $\mathbf{C}$, respectively. For example, $V_{\ell,q}$ gives the voltage applied to electrode ℓ for pattern q . For each pattern, q , we calculate the potential within the medium, assuming a homogeneous medium

$$u^q(\mathbf{x}) = \sum_{i=1}^P a_i^q \phi_i(\mathbf{x}). \quad (45)$$

Our strategy is to find coefficients: v_q for $q = 1 \dots K$ such that

$\langle e^{\tilde{\mathbf{k}} \cdot \mathbf{x}} - \sum_{q=1}^K v_q u^q(\mathbf{x}), e^{\tilde{\mathbf{k}} \cdot \mathbf{x}} - \sum_{q=1}^K v_q u^q(\mathbf{x}) \rangle_{H^1}$ is minimized for all complex $\tilde{\mathbf{k}}$ within our region of interest in $\mathbf{k}$ -space. We can find that the closed form solution for $\mathbf{v} = \{v_1 \dots v_K\}^T$ is

$$\mathbf{v} = (\mathbf{u}^* \mathbf{u})^{-1} \mathbf{u}^* \mathbf{y} \quad (46)$$

where $\mathbf{u}$ is an $P \times K$ matrix giving the coefficients of the potential for all current patterns and $\mathbf{y}$ is a vector giving the correlation in H^1 of $e^{\tilde{\mathbf{k}} \cdot \mathbf{x}}$ with all of our basis functions

$$y_i = \langle \phi_i, e^{\tilde{\mathbf{k}} \cdot \mathbf{x}} \rangle_{H^1}. \quad (47)$$

We note that $\mathbf{y}$ can be computed analytically.

In the case of the complete electrode model, we calculate the matrix $\mathbf{u}(\mathbf{z})$, which is additionally dependent on the surface impedances, $\mathbf{z}$, using (21). In this case, we compute the coefficients for the approximation of $e^{\tilde{\mathbf{k}} \cdot \mathbf{x}}$ as follows:

$$\mathbf{v} = (\mathbf{u}^*(\mathbf{z}) \mathbf{u}(\mathbf{z}))^{-1} \mathbf{u}^*(\mathbf{z}) \mathbf{y}. \quad (48)$$

We can then compute the data for each point in $\mathbf{k}$ -space as

$$\begin{aligned}\mathcal{F}_\alpha(\mathbf{k}) &= \int_{\partial\Omega} \tilde{f}(\Lambda(\gamma) - \Lambda(\gamma_0)) \tilde{g} dS \\ &= \left(\mathbf{f}^T(\mathbf{k}) \mathbf{V}(\gamma)^T \mathbf{C}(\gamma) \mathbf{g}(\mathbf{k}) \right. \\ &\quad \left. - \mathbf{f}^T(\mathbf{k}) \mathbf{V}(\gamma_0)^T \mathbf{C}(\gamma_0) \mathbf{g}(\mathbf{k}) \right)\end{aligned}\quad (49)$$

where $\mathbf{f}(\mathbf{k})$ and $\mathbf{g}(\mathbf{k})$ are coefficients for the approximations of $\tilde{f}(\mathbf{k})$ and $\tilde{g}(\mathbf{k})$, respectively. Here, if we use the ave-gap model, then (46) is employed, and, if we use the complete electrode model, then (48) is employed. In the case of the complete model, the dependence of $\mathbf{f}(\mathbf{k})$, $\mathbf{V}(\gamma)$, $\mathbf{V}(\gamma_0)$, $\mathbf{C}(\gamma)$, $\mathbf{C}(\gamma_0)$, and $\mathbf{g}(\mathbf{k})$ on the surface impedances is assumed to be implicit. For static imaging, we generate $\mathbf{V}(\gamma_0)$ and $\mathbf{C}(\gamma_0)$ computationally ($\mathbf{V}(\gamma_0)$ gives the voltages on the electrodes for a homogeneous medium), while, for difference imaging, these matrices are actually measured, without the presence of an imaging target.

Interestingly, we note that for each value of $\mathbf{k}$, there are infinitely many choices for $\mathbf{k}^\perp$ such that $\mathbf{k} \cdot \mathbf{k}^\perp = 0$ and $|\mathbf{k}| = |\mathbf{k}^\perp|$. In fact, these choices lie in a circle with radius $|\mathbf{k}|$. This reflects the observation that the data that we measure is, in some sense, four-dimensional, while we wish to reconstruct a three-dimensional medium. We thus endeavor to make use of this redundant information to improve our reconstruction by modifying the algorithm as follows:

$$\gamma - \gamma_0 \approx \int_{|\mathbf{k}_1| \leq M_1} \int_{|\mathbf{k}_2| \leq M_2} \int_{|\mathbf{k}_3| \leq M_3} \frac{-1}{2(2\pi)^3 |\mathbf{k}|^2} \times \mathcal{M}(\mathbf{k}) \mathcal{F}_R(\mathbf{k}) e^{-2\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \quad (50)$$

$$\mathcal{F}_R(\mathbf{k}) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_{\partial\Omega} \tilde{f}(\mathbf{k}, \theta) (\Lambda(\gamma) - \Lambda(\gamma_0)) \tilde{g}(\mathbf{k}, \theta) dS d\theta \quad (51)$$

where, assuming the ave-gap model, $\tilde{f}(\mathbf{k}, \theta)$ and $\tilde{g}(\mathbf{k}, \theta)$ have the following properties:

$$\begin{cases} \frac{\partial \tilde{f}(\mathbf{k}, \theta)}{\partial \nu} = \mathbf{I}_\ell^T \tilde{f}(\mathbf{k}, \theta) & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{f}(\mathbf{k}, \theta)}{\partial \nu} = 0 & \mathbf{x} \notin \bigcup_{\ell=1}^L e_\ell \\ \tilde{f}(\mathbf{k}, \theta) \approx e^{i\mathbf{k} \cdot \mathbf{x} + (\cos(\theta)\mathbf{k}^\perp + \sin(\theta)\mathbf{k}^{\perp\perp}) \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (52)$$

$$\begin{cases} \frac{\partial \tilde{g}(\mathbf{k}, \theta)}{\partial \nu} = \mathbf{I}_\ell^T \tilde{g}(\mathbf{k}, \theta) & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{g}(\mathbf{k}, \theta)}{\partial \nu} = 0 & \mathbf{x} \notin \bigcup_{\ell=1}^L e_\ell \\ \tilde{g}(\mathbf{k}, \theta) \approx e^{i\mathbf{k} \cdot \mathbf{x} - (\cos(\theta)\mathbf{k}^\perp + \sin(\theta)\mathbf{k}^{\perp\perp}) \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (53)$$

and $\mathbf{k}^{\perp\perp}$ is a vector in $\mathcal{R}^3$ orthogonal to both $\mathbf{k}$ and $\mathbf{k}^\perp$ with $|\mathbf{k}^{\perp\perp}| = |\mathbf{k}|$.

For the complete electrode model, $\tilde{f}(\mathbf{k}, \theta, \mathbf{z})$ and $\tilde{g}(\mathbf{k}, \theta, \mathbf{z})$ are also dependent on the electrode surface impedances, $\mathbf{z}$, and have the following properties:

$$\begin{cases} \int_{e_\ell} \frac{\partial \tilde{f}(\mathbf{k}, \theta, \mathbf{z})}{\partial \nu} dS = \mathbf{I}_\ell^T \tilde{f}(\mathbf{k}, \theta, \mathbf{z}) \\ \tilde{f}(\mathbf{k}, \theta, \mathbf{z}) + z_\ell \frac{\partial \tilde{f}(\mathbf{k}, \theta, \mathbf{z})}{\partial \nu} = U_\ell^T \tilde{f}(\mathbf{k}, \theta, \mathbf{z}) & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{f}(\mathbf{k}, \theta, \mathbf{z})}{\partial \nu} = 0 & \mathbf{x} \notin \bigcup_{\ell=1}^L e_\ell \\ \tilde{f}(\mathbf{k}, \theta, \mathbf{z}) \approx e^{i\mathbf{k} \cdot \mathbf{x} + (\cos(\theta)\mathbf{k}^\perp + \sin(\theta)\mathbf{k}^{\perp\perp}) \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (54)$$

$$\begin{cases} \int_{e_\ell} \frac{\partial \tilde{g}(\mathbf{k}, \theta, \mathbf{z})}{\partial v} dS = \tilde{I}_\ell^{\tilde{g}(\mathbf{k}, \theta, \mathbf{z})} \\ \tilde{g}(\mathbf{k}, \theta, \mathbf{z}) + z_\ell \frac{\partial \tilde{g}(\mathbf{k}, \theta, \mathbf{z})}{\partial v} = U_\ell^{\tilde{g}(\mathbf{k}, \theta, \mathbf{z})} & \mathbf{x} \in e_\ell \\ \frac{\partial \tilde{g}(\mathbf{k}, \theta, \mathbf{z})}{\partial v} = 0 & \mathbf{x} \notin \cup_{\ell=1}^L e_\ell \\ \tilde{g}(\mathbf{k}, \theta, \mathbf{z}) \approx e^{i\mathbf{k} \cdot \mathbf{x} - (\cos(\theta)\mathbf{k}^\perp + \sin(\theta)\mathbf{k}^{\perp\perp}) \cdot \mathbf{x}} & \mathbf{x} \in \Omega \end{cases} \quad (55)$$

The discrete approximation of this algorithm is then

$$\mathcal{F}_R(\mathbf{k}) \approx \frac{1}{N_\theta} \sum_{t=1}^{N_\theta} \int_{\partial\Omega} \tilde{f}\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) (\Lambda(\gamma) - \Lambda(\gamma_0)) \tilde{g}\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) dS \quad (56)$$

where

$$\begin{aligned} & \int_{\partial\Omega} \tilde{f}\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) (\Lambda(\gamma) - \Lambda(\gamma_0)) \tilde{g}\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) dS \\ &= \left(\mathbf{f}^T\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) \mathbf{V}(\gamma)^T \mathbf{C}(\gamma) \mathbf{g}\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) - \mathbf{f}^T\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) \mathbf{V}(\gamma_0)^T \mathbf{C}(\gamma_0) \mathbf{g}\left(\mathbf{k}, \frac{2\pi t}{N_\theta}\right) \right) \end{aligned} \quad (57)$$

and N_θ is the number of discrete points in θ used to calculate each point in $\mathbf{k}$ -space. Here, again, we calculate the coefficients of $\tilde{f}(\mathbf{k}, \theta)$ using (46) for the ave-gap model and using (48) for the complete electrode model. In the latter case, we also compute $\mathbf{V}(\gamma_0)^T$ and $\mathbf{C}(\gamma_0)$ using the complete electrode model when performing static imaging. In the case of difference imaging, again, these matrices are measured experimentally.

V. RESULTS AND DISCUSSION

We present reconstructions using the Calderón-type method described earlier, with partial-view measurements. Specifically, the reconstructions were generated using (56). Here, we made use of the ACT 4 instrument [23], operating at 10 kHz. The experimental setup for these experiments is shown in Fig. 2, with the breast-shaped saline tank depicted in Fig. 2(a) and (b) and the geometry employed in the reconstruction shown in Fig. 2(c). The location of the copper conducting target is shown in Fig. 2(d). We placed the target as shown in Fig. 2(d) in the x - y plane and varied the distance of the target from the bottom electrode array. Here, the reconstructions were computed using difference imaging, meaning that $\mathbf{V}(\gamma_0)^T \mathbf{C}(\gamma_0)$ in (56) is generated using data taken from a homogeneous tank (i.e., not containing a target), as opposed to being computed using an appropriate forward model (i.e., ave-gap or complete electrode model). The reconstruction results, are displayed in Fig. 3. Here, we used regularization parameters (cutoff values in $\mathbf{k}$) of $3\pi/h_x$, $3\pi/h_y$, and $2\pi/h_z$ in the x , y , and z directions, respectively, where h_x , h_y , h_z were the dimensions of the phantom. The integral in (56) was approximated using the trapezoidal rule, where $p_1(\mathbf{k}, 2\pi t/N_\theta)(\Lambda(\gamma) - \Lambda(\gamma_0))p_2(\mathbf{k}, 2\pi t/N_\theta)$ was discretized on a mesh with dimensions $20 \times 20 \times 10$ in the interval $(-1.03k_{\max}, k_{\max})$ for each dimension such that $(0,0,0)$ was not a node of the mesh.

Each column of the figure depicts a horizontal slice through the reconstructed 3-D image at various depths. In each row of the figure, the actual depth of the target was changed, with the target moving progressively away from the bottom electrode array. As expected, when we placed electrodes on only the top and bottom of the phantom, our depth resolution was compromised, but Fig. 3 demonstrates that a significant degree of depth resolution remains.

Next, we examine the performance of our algorithm for static imaging, for the two electrode models that we have described, and varying N_θ . Here again, the reconstructions are computed using (56), but, in static imaging, we compute $\mathbf{V}(\gamma_0)^T \mathbf{C}(\gamma_0)$ using a forward model. The same regularization parameters (cutoff values in $\mathbf{k}$) were selected as for the reconstructions in Fig. 3. The reconstructions, where N_θ was 1, are shown in Fig. 4(a) and (b). The ave-gap model was used in Fig. 4(a), and the complete electrode model was employed in Fig. 4(b), in which a nonlinear optimization procedure was first used to estimate the electrode surface impedances. The shape of the target seems to be less distorted in Fig. 4(b) and we also observe fewer image artifacts in the reconstruction with the complete electrode model.

Most dramatically, we observe a significant improvement in the reconstructed images if we increase N_θ , essentially making use of redundant information in our data to estimate each point in $\mathbf{k}$ -space. The reconstructions with $N_\theta = 8$ for the ave-gap model and the complete electrode model, respectively, are shown in Fig. 4(c) and (d). We note that the quality of the reconstructed images approaches the quality of the difference images presented in Fig. 3. Interestingly, the difference images did not seem to be sensitive to N_θ , with very little difference observed in increasing N_θ from 1 to 8. We suggest that the improvement in the static images is obtained by means of using the redundant data to reduce the effect of forward modeling errors on the reconstruction.

Finally, we compare the images produced by the Calderón-type method with those produced by the NOSER algorithm. For the same data as for Fig. 3 and Fig. 4, the corresponding difference and static images generated using the NOSER one-step algorithm are shown in Fig. 5(a) and (b). A mesh with dimensions $16 \times 16 \times 11$ was generated, along with its Jacobian matrix, but only the center nine slices are displayed. Tikhonov and NOSER regularization parameters of $0.1 \max(\text{diag}(\mathbf{J}^T \mathbf{J}))$ and 0.1 were used, found empirically. The difference images, depicted in Fig. 5(a) closely resemble Fig. 3, although the reconstructed target seems to become more blurry as the target is moved towards the interior of the tank. The reason for this is that, unlike for the Calderón-type method, the regularization here is spatially variant. The static reconstructions for the same data are shown in Fig. 5(b). In this case, due to presence of image artifacts, the target can hardly be seen beyond a distance of approximately 10 cm from the bottom electrode array. The reason for this is likely the spatially-variant sensitivity of the NOSER algorithm. In addition, here we do not make any use of the redundant information present in the data.

In comparing Fig. 5 with Fig. 3 and Fig. 4, it does appear that the target is resolved with greater clarity in the reconstructions using the NOSER algorithm. This may be due to the particular choices of regularization parameters for the two algorithms, although another possibility is that, as the reconstruction basis used in the NOSER algorithm is composed of discrete voxels, while that used in Calderón's method is composed of functions which are smooth in the entire region of interest, the NOSER algorithm should in principle have better performance for compactly supported targets. The negative artifacts seen in Fig. 3 and Fig. 4 are most likely due to Gibbs phenomena, which are reduced but not eliminated by the mollifier introduced into the Fourier inversion.

In terms of computation time, the most expensive step of our algorithm is the computation of the solution in Fourier space, (56), whereas the transformation from the Fourier domain to the spatial domain in (50) is nearly instantaneous. In our current Matlab implementation, using a 2.4 GHz Intel Pentium Core 2 Duo processor, computation of $\mathfrak{F}_R(\mathbf{k})$ for a $20 \times 20 \times 10$ grid in Fourier space currently takes approximately 10 minutes for $N_\theta = 1$, with computation time scaling linearly with N_θ . Nearly all of the computation time is currently used for analytical computation of the coefficients in (47). In principle, these coefficients

could be precomputed, or, at any rate, this computation is embarrassingly parallel. In contrast, the reconstruction using the NOSER algorithm for a $16 \times 16 \times 11$ grid, implemented using Matlab, and using the same processor, currently computes in approximately 10 s. However, the Jacobian matrix with respect to a discrete voxel grid, which can also be precomputed and stored, currently requires approximately two hours of computation time to generate when 2049 basis functions are used, although this calculation is also embarrassingly parallel.

VI. CONCLUSION AND FUTURE WORK

To recapitulate, we have developed and implemented a variation of Calderón's method in three dimensions, in which measurements are made on only a limited portion of a medium's body. We have described our algorithm in detail for the ave-gap and complete electrode model and we have presented experimental results for both static and difference imaging, also comparing the images obtained to those produced by the NOSER method. We furthermore show that the quality of the static images generated can be improved by exploiting the redundancy in our measurements. In future work, we plan to address the fully nonlinear image reconstruction problem.

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Fig. 1.
Rectangular parallelepiped measurement configuration.

(a)

(b)

(c)

Fig. 2. Experimental setup for 3D imaging tests. (a) Breast-shaped saline tank. (b) Schematic of the breast-shaped tank. (c) Location of the inclusion near electrode 8.

Fig. 3.

Difference imaging reconstructions, using a Calderón-type method, of a target near electrode 8 in the x - y plane, with the target at various distances from the bottom electrode array. The horizontal axis denotes the depth of each slice, while the vertical axis denotes the actual position of the target.

Fig. 4.

Static imaging reconstructions of a target near electrode 8 in the x - y plane, with the target at various distances from the bottom electrode array. The horizontal axis denotes the depth of each slice, while the vertical axis denotes the actual position of the target. (a) $N_{\theta} = 1$ and using the ave-gap model. (b) $N_{\theta} = 1$ and using the complete electrode model. (c) $N_{\theta} = 8$ and using the ave-gap model. (d) $N_{\theta} = 8$ and using the complete electrode model.

Fig. 5. Static and difference imaging reconstructions, using the NOSER algorithm, of a target near electrode 8 in the x - y plane, with the target at various distances from the bottom electrode array. Horizontal axis denotes the depth of each slice, while the vertical axis denotes the actual position of the target. (a) NOSER difference imaging reconstruction. (b) NOSER static imaging reconstruction.