Scattering calculation and image reconstruction using elevation-focused beams

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Pressure scattered by cylindrical and spherical objects with elevation-focused illumination and reception has been analytically calculated, and corresponding cross sections have been reconstructed with a two-dimensional algorithm. Elevation focusing was used to elucidate constraints on quantitative imaging of three-dimensional objects with two-dimensional algorithms. Focused illumination and reception are represented by angular spectra of plane waves that were efficiently computed using a Fourier interpolation method to maintain the same angles for all temporal frequencies. Reconstructions were formed using an eigenfunction method with multiple frequencies, phase compensation, and iteration. The results show that the scattered pressure reduces to a two-dimensional expression, and two-dimensional algorithms are applicable when the region of a three-dimensional object within an elevation-focused beam is approximately constant in elevation. The results also show that energy scattered out of the reception aperture by objects contained within the focused beam can result in the reconstructed values of attenuation slope being greater than true values at the boundary of the object. Reconstructed sound speed images, however, appear to be relatively unaffected by the loss in scattered energy. The broad conclusion that can be drawn from these results is that two-dimensional reconstructions require compensation to account for uncaptured three-dimensional scattering.

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I. INTRODUCTION

Ultrasonic *b*-scan imaging is a valuable tool for the detection and diagnosis of disease. Applications include imaging the heart, abdominal organs, and the developing fetus.^{1,2} Ultrasound has the advantages of being non-ionizing at intensities used for diagnostic imaging, showing motion in real time, and being inexpensive in comparison to other imaging modalities.

While *b*-scan imaging based on transmitting pulses and receiving echoes with focused beams is widespread, imaging based on inverse scattering is relatively undeveloped in medical applications. This lack of development can be attributed to a number of reasons. The most important of these is that a constructive solution of the inverse problem in two dimensions does not yet exist and current solutions now require *a priori* knowledge or iteration or both. Other important reasons are the complex apparatus to acquire data and the substantial computation needed to reconstruct images.

In x-ray tomography, an uncomplicated straight-line model facilitates data collection and fast reconstruction of images. In magnetic resonance imaging, a Fourier transform relation between the measured data and the image facilitates fast image reconstruction. In ultrasound tomography, however, acoustic scattering is a three-dimensional process that complicates data collection and image reconstruction. Nev-

ertheless, if two-dimensional data collection and algorithms are applicable, then imaging complexity is reduced from that required for the three-dimensional case.

Cross sections reconstructed using a two-dimensional algorithm can be influenced by scattering into the third dimension. This study examines the effects that three-dimensional scattering has on these two-dimensional reconstructions. The scattering objects used in this study are cylinders and spheres. However, the methods developed to examine these objects are also applicable to more complicated objects.

Analytic expressions for plane-wave scattering by a cylinder and by a sphere are extended to include elevation-focused illumination and reception using an angular spectrum representation of the focused beams. The extended expressions are employed to calculate scattering at multiple frequencies. The scattering data are used to reconstruct elevation-offset cross sections of a sphere. These reconstructions show variations in image accuracy that result from elevation focusing and scattering object variability in the width of the focus.

The choice of parameters in this study is motivated by the ring transducer system at the University of Rochester.³ The transducer in this system is a 150-mm diameter ring of 2048 equally spaced transducer elements each with an elevation of 25 mm. The elements are focused in elevation with an acoustic lens that creates a beam with an elevation *f*-number of 5. The focus of the beam is in the imaging plane at the

center of the ring. At the focus, the beam has a half-amplitude width of 4 mm in the elevation dimension and a 100-mm imaging-plane depth of field essentially centered around the focus. The transmit subsystem has 128 channels each with an individually programmable wave form. The receive subsystem has 16 channels each sampled at 20 MHz with 12 bits of resolution. Transmit and receive multiplexers are used to connect the transmit and receive channels to the transducer elements.

Ultrasound image reconstruction at a given temporal frequency is equivalent to finding a solution for the acoustic scattering potential in the Lippman–Schwinger equation given measurements of the incident pressure and the total pressure. Available reconstruction algorithms can be grouped in three classes. One class utilizes so-called time-of-flight approximations, another class finds a linearized solution (i.e., uses a Born or Rytov approximation), and the other class is based on iterative, nonlinear, full-wave methods to obtain a solution. Algorithms representative of the first type are those in Refs. 4 and 5. Algorithms representative of the second type are filtered backpropagation⁶ and Fourier-domain interpolation.⁷ Algorithms representative of the third type are nonlinear algebraic reconstruction techniques such as the Born iterative methods,^{8,9} the sinc-basis moment method,^{10,11} and a method using eigenfunctions of a scattering operator.¹²

The eigenfunction method is used in this paper to reconstruct images. The eigenfunction method was originally developed for single-frequency illumination,¹² later extended to use all the frequencies in an incident pulse and include adaptive demodulation that extends the validity of the Born approximation,¹³ and recently further extended to use iteration for imaging large-scale high-contrast objects.¹⁴ Although the eigenfunction method is also applicable in three dimensions, the ring transducer system available for measurements acquires data in two dimensions and, as noted, motivates the choice of parameters in this study.

The remainder of this paper is comprised of the following sections. Section II develops the theoretical basis for the scattering calculations. Section III describes the computational methods. Then, results of calculations are presented in Sec. IV for various elevation-focused scattering geometries and scattering objects. Section V discusses the results and implications of using elevation-focused beams and also discusses key assumptions and other steps used in the analysis and computations. Conclusions are presented in Sec. VI.

II. THEORY

The application of a two-dimensional reconstruction algorithm to scattering in three dimensions requires that the scattering in three dimensions be related to the scattering in two dimensions. For a specific temporal frequency f with time dependence $e^{-j2\pi ft}$, this is accomplished by expressing the three-dimensional scattered pressure p_s in a Born series as

$$p_s(\mathbf{x}, z, f) = p_0(\mathbf{x}, z, f) + p_1(\mathbf{x}, z, f) + p_2(\mathbf{x}, z, f) + \cdots, \quad (1)$$

where $\mathbf{x}$ has Cartesian components (x, y) in the x - y plane and z is the coordinate in the orthogonal (elevation) direction.

Each term in this series is determined recursively using the formula

$$p_{n+1}(\mathbf{x}, z, f) = k^2 \int_{z'} \int_{\mathbf{x}'} G_0^{(3D)}(\mathbf{x} - \mathbf{x}', z - z', f) \eta(\mathbf{x}', z') \times p_n(\mathbf{x}', z', f) d\mathbf{x}' dz', \quad (2)$$

where $\eta(\mathbf{x}, z)$ is the three-dimensional variation in the medium and $G_0^{(3D)}(\mathbf{x} - \mathbf{x}', z - z', f)$ is the three-dimensional free-space out-going Green's function given by

$$G_0^{(3D)}(\mathbf{x} - \mathbf{x}', z - z', f) = \frac{e^{j2\pi f \sqrt{\|\mathbf{x} - \mathbf{x}'\|^2 + (z - z')^2}/c}}{4\pi \sqrt{\|\mathbf{x} - \mathbf{x}'\|^2 + (z - z')^2}}, \quad (3)$$

in which c is the speed of sound in the background medium.

If the medium variations are essentially independent of elevation, i.e., $\eta(\mathbf{x}, z) \approx \eta(\mathbf{x})$, then Eq. (2) becomes

$$p_{n+1}(\mathbf{x}, z, f) = k^2 \int_{\mathbf{x}'} \eta(\mathbf{x}') \int_{z'} \frac{e^{j2\pi f \sqrt{\|\mathbf{x} - \mathbf{x}'\|^2 + (z - z')^2}/c}}{4\pi \sqrt{\|\mathbf{x} - \mathbf{x}'\|^2 + (z - z')^2}} \times p_n(\mathbf{x}', z', f) dz' d\mathbf{x}'. \quad (4)$$

Taking the Fourier transform with respect to the elevation coordinate z converts the convolution in the elevation coordinate into a product in the corresponding spatial-frequency coordinate ν . The result, as shown in the Appendix, is that Eq. (4) can be written^a as

$$p_{n+1}(\mathbf{x}, \nu, f) = k^2 \int_{\mathbf{x}'} G_0^{(2D)}(\mathbf{x} - \mathbf{x}', f_e) \eta(\mathbf{x}') p_n(\mathbf{x}', \nu, f) d\mathbf{x}', \quad (5)$$

where $G_0^{(2D)}(\mathbf{x} - \mathbf{x}', f_e)$ is the two-dimensional free-space out-going Green's function as a function of the effective temporal frequency $f_e = \sqrt{f^2 - c^2 \nu^2}$ and is given by

$$G_0^{(2D)}(\mathbf{x} - \mathbf{x}', f_e) = (j/4) H_0^{(1)}(\|\mathbf{x} - \mathbf{x}'\| 2\pi \sqrt{(f/c)^2 - \nu^2}). \quad (6)$$

$2\pi\nu$ is the wavenumber in the z coordinate, $2\pi\sqrt{(f/c)^2 - \nu^2}$ is the wavenumber in the x - y plane, and $p_n(\mathbf{x}', \nu, f)$ is a representation of the temporal-harmonic pressure at the n th iteration in terms of the spatial variable $\mathbf{x}'$ in the x - y plane and the spatial-frequency variable ν in the elevation dimension.

The foregoing analysis shows that scattering in three dimensions can be expressed as an uncoupled set of relations in two dimensions when the medium variations are independent of elevation. The scattering in three dimensions is then the sum of the two-dimensional scattering that results from each plane wave or spatial-frequency in the elevation dimension. The explicit relation is

$$p_n(\mathbf{x}, z, f) = \int_{\nu} p_n(\mathbf{x}, \nu, f) e^{j2\pi \nu z} d\nu. \quad (7)$$

Since the spatial-frequency ν in the elevation dimension is given as a function of the elevation angle ϑ by

$$\nu = \frac{f}{c} \cos \vartheta, \quad (8)$$

Eq. (7) can be expressed as

$$p_n(\mathbf{x}, z, f) = \int_{\vartheta} p_n(\mathbf{x}, (f \cos \vartheta/c), f) e^{j(2\pi f/c)z \cos \vartheta} \times d(f \cos \vartheta/c). \quad (9)$$

When $n=0$, the left side of this equation is the beam of illumination that is denoted as ψ_t in the next section and the integrand on the right side is the angular spectrum of plane waves that comprise the beam and is denoted as ϕ_t in that section. The angular spectrum form for the illumination pattern and also a corresponding form for reception sensitivity lead to convenient expressions for measured values of scattered pressure when the illumination pattern and reception sensitivity are focused in elevation.

Although the foregoing expressions and expressions in subsequent sections are for a single temporal frequency, each expression can be extended to the time domain by considering the temporal frequency to be a variable, weighting the frequency-domain expression with the spectrum of a temporal pulse such as the Gaussian-shaped bandpass pulse given by

$$g(t) = A_t e^{-t^2/2\sigma_t^2} \sin(2\pi f_0 t), \quad (10)$$

in which A_t is a constant amplitude scale factor, σ_t is a pulse-width parameter, and f_0 is the pulse center frequency, and then calculating the corresponding time-domain expression via an inverse Fourier transform of the weighted frequency-domain expression.

A. Focusing in elevation

The strength of acoustic illumination at a spatial location $(\mathbf{x}, z)$ is described by a beam that is focused in elevation and has the form of a plane wave in the x - y coordinates and a direction determined by the azimuthal angle ϕ_i . For monochromatic illumination with time dependence $e^{-j2\pi f t}$, this beam is defined by an angular spectrum of plane waves in the elevation coordinate by using the relation

$$\psi_t(\mathbf{x}, z) = \int_{\vartheta_i} \phi_t(\vartheta_i) e^{j(2\pi f/c)\mathbf{u}(\phi_i, \vartheta_i) \cdot (\mathbf{x}, z)} d\vartheta_i, \quad (11)$$

where $\phi_t(\vartheta_i)$ is the illumination angular spectrum as a function of the varying polar angle ϑ_i and $\mathbf{u}(\phi_i, \vartheta_i)$ is a unit vector in the plane-wave direction specified by a fixed azimuthal angle ϕ_i and the varying polar angle ϑ_i . The vector $\mathbf{u}(\phi_i, \vartheta_i)$ has an elevation-dimension projection of $\cos \vartheta_i$ and an x - y plane projection of $\sin \vartheta_i \mathbf{u}_{xy}(\phi_i)$, where $\mathbf{u}_{xy}(\phi_i)$ is a unit vector with angle ϕ_i in the x - y plane.

Although both the beam pattern ψ_t and the angular spectrum ϕ_t depend on frequency, the frequency in this section is assumed fixed so the frequency dependence of ψ_t and ϕ_t is left implicit to simplify notation.

The angular spectrum $\phi_t(\vartheta_i)$ of the illumination is determined in the x - y plane by the excitation of the ring transducer elements and in the elevation dimension by an acoustic lens that is the same for all elements in the ring. The excitation of the elements that are spaced evenly around the circumference of the ring $\{\mathbf{x}: \|\mathbf{x}\|=r_0\}$ is represented by a function $E(\mathbf{x}, \phi_i)$ for $\|\mathbf{x}\|=r_0$, and the variation in the elevation

dimension is represented by a function $\lambda(z)$ so that the field on the cylinder $\|\mathbf{x}\|=r_0$ at the ring radius r_0 is given by the product $\lambda(z)E(\mathbf{x}, \phi_i)$. Equating this field with Eq. (11) for points $(\mathbf{x}, z)$ on a portion of the cylinder $\|\mathbf{x}\|=r_0$ gives

$$\lambda(z)E(\mathbf{x}, \phi_i) = \int_{\vartheta_i} \phi_t(\vartheta_i) e^{j(2\pi f/c)\mathbf{u}(\phi_i, \vartheta_i) \cdot (\mathbf{x}, z)} d\vartheta_i. \quad (12)$$

To solve for ϕ_t the first step is to take the Fourier transform of Eq. (12) with respect to the z coordinate to obtain

$$\lambda(\nu)E(\mathbf{x}, \phi_i) = \int_{\vartheta_i} \phi_t(\vartheta_i) \left[\int_z e^{j(2\pi f/c)\mathbf{u}(\phi_i, \vartheta_i) \cdot (\mathbf{x}, z)} e^{-j2\pi \nu z} dz \right] d\vartheta_i. \quad (13)$$

Next, writing the spatial-frequency variable ν of the Fourier transform using Eq. (8) and expressing $\mathbf{u}(\phi_i, \vartheta_i)$ in $(\mathbf{x}, z)$ components as described after Eq. (11) gives

$$\lambda(f \cos \vartheta_i/c)E(\mathbf{x}, \phi_i) = \int_{\vartheta_i} \phi_t(\vartheta_i) e^{j(2\pi f/c) \sin \vartheta_i \mathbf{u}_{xy}(\phi_i) \cdot \mathbf{x}} \times \left[\int_z e^{j(2\pi f/c)(\cos \vartheta_i - \cos \vartheta_i)z} dz \right] d\vartheta_i. \quad (14)$$

Evaluation of the inner integral and the use of a small angle approximation in the exponential yields

$$\lambda(f \cos \vartheta_i/c)E(\mathbf{x}, \phi_i) = \frac{\phi_t(\vartheta_i)}{f \sin \vartheta_i/c} e^{j(2\pi f/c)\mathbf{u}_{xy}(\phi_i) \cdot \mathbf{x}}. \quad (15)$$

This implies that

$$\phi_t(\vartheta_i) = (f \sin \vartheta_i/c) \lambda(f \cos \vartheta_i/c) \quad (16)$$

and that

$$E(\mathbf{x}, \phi_i) = e^{j(2\pi f/c)\mathbf{u}_{xy}(\phi_i) \cdot \mathbf{x}}. \quad (17)$$

For calculations of multiple scattering in this paper, the lens factor

$$\lambda(z) = a(z) e^{-j(2\pi f/c)\ell(z)} \quad (18)$$

is used. In this expression, $a(z)$ is a Gaussian-weighted amplitude term,

$$a(z) = A_G e^{-z^2/2\sigma_a^2}, \quad (19)$$

and $\ell(z)$ is a length factor given by

$$\ell(z) = \sqrt{(r_0 + r_a)^2 + z^2} \quad (20)$$

for a geometric focus at $(r_a, 0, 0)$, where r_a is an adjustment chosen to produce a focal peak at $(0, 0, 0)$. The width parameter σ_a in Eq. (19) is real and produces a real-valued weight of the aperture, while the length factor ℓ produces a phase shift in the aperture.

A relation analogous to that between the illumination pattern and the angular spectrum of the illumination given in Eq. (11) can be defined between the reception sensitivity and the angular spectrum of the reception sensitivity. The relation is

$$\psi_r(\mathbf{x}, z) = \int_{\vartheta} \varphi_r(\vartheta) e^{j(2\pi f/c) \mathbf{u}(\phi, \vartheta) \cdot (\mathbf{x}, z)} d\vartheta, \quad (21)$$

where $\varphi_r(\vartheta)$ is the reception sensitivity angular spectrum expressed as a function of the varying polar angle ϑ and $\mathbf{u}(\phi, \vartheta)$ is a unit vector in the plane-wave direction specified by a fixed azimuthal angle ϕ and the varying polar receive angle ϑ . The vector $\mathbf{u}(\phi, \vartheta)$ has an elevation-dimension projection of $\cos \vartheta$ and an x - y plane projection of $\sin \vartheta \mathbf{u}_{xy}(\phi)$, where $\mathbf{u}_{xy}(\phi)$ is a unit vector with angle ϕ in the x - y plane.

Expressions (11) and (21) are special forms for illumination patterns and receiver sensitivities. These forms consist of plane-wave components that all advance in the same horizontal direction, i.e., have the same azimuthal angle. Illumination patterns given by Eq. (11) are produced by exciting ring transducer elements with amplitudes specified in Eq. (17). Receiver sensitivities given by Eq. (21) require that the temporal-frequency components of the signals from transducer elements be accumulated with weights also given by Eq. (17). These patterns are well suited for scattering measurements because each scattering measurement can be written as a linear combination of far-field values determined by a plane-wave component from the illumination pattern together with a plane-wave component from the receiver sensitivity. This is an efficient representation of scattering measurements when illumination patterns and receiver sensitivities have few plane-wave components.

More general expressions for illumination patterns and receiver sensitivities can also be formed. This is accomplished by applying the Rayleigh–Sommerfeld integral to field values on a portion of the cylindrical surface specified by $\|\mathbf{x}\| = r_0$ for excitations other than those given by Eq. (17). The result is

$$\psi_t(\mathbf{x}, z) = \int_{\zeta} \int_{z'} E(\mathbf{x}'(\zeta)) \lambda(z') \frac{\partial}{\partial r_{xx'}} \times G_0^{(3D)}(\mathbf{x} - \mathbf{x}'(\zeta), z - z', f) dz' d\zeta, \quad (22)$$

where $r_{xx'} = \|\mathbf{x} - \mathbf{x}'\|$, $\|\mathbf{x}'(\zeta)\| = r_0$, and ζ is the azimuthal angle over which the integration is performed in the x - y plane. After a change in variables and simplification, Eq. (22) becomes

$$\psi_t(\mathbf{x}, z) = \int_{\vartheta_i} (f \sin \vartheta_i / c) \lambda(f \sin \vartheta_i / c) e^{j(2\pi f/c) \cos \vartheta_i z} \times \left\{ -2 \int_{\|\mathbf{x}'\|=r_0} E(\mathbf{x}'(\zeta)) \frac{\partial}{\partial r_{xx'}} \times [G_0^{(2D)}(\mathbf{x} - \mathbf{x}'(\zeta), f \sin \vartheta_i)] d\zeta \right\} d\vartheta_i, \quad (23)$$

where the term in curly brackets may be identified as the two-dimensional Rayleigh–Sommerfeld integral for propagation of the ring element excitations $E(\mathbf{x}')$ at the temporal frequency $f \sin \vartheta_i$. If $E(\mathbf{x}')$ is chosen to have the plane-wave form $e^{j(2\pi f/c) \mathbf{u}_{xy}(\phi_i) \cdot \mathbf{x}'}$ in Eq. (17), then the propagated two-dimensional field given by the term in curly brackets may be approximated by the plane wave $e^{j(2\pi f/c) \sin \vartheta_i \mathbf{u}_{xy}(\phi_i) \cdot \mathbf{x}}$. Substitu-

FIG. 1. Scattering geometry. The incident wave has azimuthal angle ϕ_i and polar angle ϑ_i , and the scattered wave has azimuthal angle ϕ and polar angle ϑ .

tion of this approximation in Eq. (23) reproduces the expressions given by Eqs. (15)–(17) that represent the pattern of illumination.

The illumination pattern or receiver sensitivity of a single ring transducer element at location $\mathbf{x}'(\phi_0)$ is obtained by choosing $E(\mathbf{x}'(\zeta)) = \delta(\zeta - \phi_0)$. For example, making this substitution in Eq. (23) yields

$$\psi_t(\mathbf{x}, z, \phi_0) = \int_{\vartheta_i} (f \sin \vartheta_i / c) \lambda(f \sin \vartheta_i / c) e^{j(2\pi f/c) \cos \vartheta_i z} \times \left\{ -2 \frac{\partial}{\partial r_{xx'}} [G_0^{(2D)}(\mathbf{x} - \mathbf{x}'(\phi_0), f \sin \vartheta_i)] \right\} d\vartheta_i, \quad (24)$$

in which the dependence of the illumination pattern ψ_t on the azimuthal angle ϕ_0 of the incident illumination is explicitly indicated. Other illumination patterns can be formed by choosing element excitations $E(\mathbf{x}'(\zeta))$ to weight the single-element patterns given in Eq. (24) and using

$$\psi_t(\mathbf{x}, z, \phi_0) = \int_{\zeta} E(\mathbf{x}'(\zeta)) \psi_t(\mathbf{x}, z, \zeta) d\zeta. \quad (25)$$

Equations (24) and (25) simplify if the asymptotic form of the Hankel function¹⁶ in $G_0^{(2D)}$ is used.

The scattering geometry and notation used throughout this paper are summarized in Fig. 1. The elevation-focused illumination pattern and reception sensitivity are depicted in Fig. 2. A configuration of the ring transducer and an offset sphere treated in the analysis below is diagramed in Fig. 3.

B. Scattering from a cylinder with focused illumination and reception

Consider a fluid cylinder that has radius a_{cyl} , sound speed c_{cyl} , and density ρ_{cyl} and is positioned with its axis along the z axis in a Cartesian coordinate system in which the coordinates in the x - y plane and elevation dimension are denoted by the vector $\mathbf{x}$ and scalar z , respectively, and the corresponding coordinates in a cylindrical coordinate system are denoted as (r_{xy}, ϕ, z) . A cylinder is considered because it is an elementary two-dimensional object with an analytic solution for the scattered pressure, the cross section of a cylinder normal to its axis is circular, and a reconstruction of such a cross section can be compared to a reconstruction of

FIG. 2. Elevation-focused beams. The illumination pattern (upper panel) is focused in elevation by the lens factor $\lambda(z)$ and extends uniformly over the lateral extent of the scattering object as a result of the excitation $E(\mathbf{x})$ applied to the elements. The reception sensitivity (lower panel) shown for a single element is also focused in elevation by the same lens factor and has a lateral sensitivity (not shown) that is also essentially uniform over the extent of the scattering object.

any cross section of a sphere for which all cross sections are circular. Assume that the cylinder is illuminated by a monochromatic plane wave that has time dependence $e^{-j2\pi ft}$ and travels in a direction defined by a unit vector $\mathbf{u}(\phi_i, \vartheta_i)$ with components $\sin\vartheta_i \mathbf{u}_{xy}(\phi_i)$ in the x - y plane and $\cos\vartheta_i$ in the elevation coordinate. By matching the conditions of continuous pressure and normal component of particle velocity at the boundary of the cylinder, the pressure scattered from this cylinder in a lossless fluid background in which the sound speed is c and density is ρ can be expressed using an orthogonal function expansion in cylindrical coordinates as^{17,18}

$$p_s(r_{xy}, \phi, z, \phi_i, \vartheta_i) = e^{jk_z z} \sum_{n=-\infty}^{+\infty} j^n D_n H_n^{(1)}(k_{xy} r_{xy}) e^{jn(\phi - \phi_i)}, \quad r_{xy} > a_{cyl}. \quad (26)$$

In these expressions, r_{xy} is the radial distance in the x - y plane, ϕ is the azimuthal angle of the observation point, k_{xy} is the wavenumber of the incident wave component in the x - y plane, k_z is wavenumber of the incident wave component in the elevation coordinate, k_{cyl} is the wavenumber in the cylinder, D_n are coefficients depending on the conditions at the boundary of the cylinder, and $H_n^{(1)}$ is a cylindrical Hankel function of the first kind. The coefficients D_n in the orthogonal function expansion for the scattered pressure are found by matching boundary conditions and are the same as those in Ref. 17. The relation between the wavenumber k in the background and the wavenumber components k_{xy} and k_z in the background is

FIG. 3. A sphere with its center vertically offset from the center of the ring transducer imaging plane. Measurement of scattering in this configuration with an illumination pattern and reception sensitivity like those depicted in Fig. 2 is described.

$$k = \sqrt{k_{xy}^2 + k_z^2}. \quad (27)$$

The relation between the wavenumber component k_{xy} in the x - y plane and the polar angle ϑ_i is

$$k_{xy} = 2\pi \frac{f}{c} \sin\vartheta_i, \quad (28)$$

and the relation between the wavenumber component k_z in the elevation coordinate and the polar angle ϑ_i is

$$k_z = 2\pi \frac{f}{c} \cos\vartheta_i. \quad (29)$$

The relation between the wavenumber k_{cyl} in the cylinder and the wavenumber k in the background is

$$k_{cyl} = 2\pi \frac{f}{c} \sqrt{(c/c_{cyl})^2 + (\cos\vartheta_i)^2}. \quad (30)$$

The expression for D_n is also valid for a lossy cylinder with attenuation coefficient α_{cyl} in which case the effective planar wavenumber k_{cyl} corresponding to an incident plane wave with angle ϑ_i is

$$k_{cyl} = \sqrt{(2\pi f/c_{cyl} + j\alpha_{cyl})^2 - (2\pi f/c_{cyl})^2 \cos^2\vartheta_i}. \quad (31)$$

The development of an expression for the measurement of scattering from a cylindrical object as a far-field (i.e., plane-wave) sensitivity pattern summed with a sequence of weights requires care because a cylindrical object is unbounded. However, such an accumulation is justified here by the elevation limits imposed by the elevation focus of the illumination. The scattered pressure given in Eq. (26) is, therefore, written as the Herglotz wave,¹⁹

$$p_s(\mathbf{x}, z, \phi_i, \vartheta_i) = -k^2 \int_{z'} \int_{\|\mathbf{x}'\| < r_{cyl}} G_0^{(3D)}(\mathbf{x} - \mathbf{x}', z - z') \times \eta_{cyl} p(\mathbf{x}', z', \phi_i, \vartheta_i) d^2\mathbf{x}' dz', \quad (32)$$

where $\eta_{cyl} = 1 - (c/c_{cyl})^2$ and $p(\mathbf{x}', z', \phi_i, \vartheta_i)$ is the pressure inside the cylinder given by Eq. (26).

Since Eq. (32) gives the scattering produced by a single plane-wave component of the incident illumination, the scat-

tered pressure produced by an angular spectrum of plane waves is obtained by summing all the angular spectrum components. This yields

$$p_s(\mathbf{x}, z, \phi_i) = -k^2 \int_{z'} \int_{\|\mathbf{x}'\| < r_{\text{cyl}}} G_0^{(3D)}(\mathbf{x} - \mathbf{x}', z - z') \times \left[\int_{\vartheta_i} \varphi_t(\vartheta_i) \eta_{\text{cyl}} p(\mathbf{x}', z', \phi_i, \vartheta_i) d\vartheta_i \right] d^2 \mathbf{x}' dz' \quad (33)$$

The Green's function in this integral can be interpreted as the sensitivity of a point receiver at the position $(\mathbf{x}, z)$ to scattering that originates from a location $(\mathbf{x}', z')$ inside the scattering volume.

A measurement of scattering is the weighted accumulation of such point receivers with the angular spectrum sensitivity identified above. To model this measurement, the angular spectrum expansion for receiver sensitivity given in Eq. (21) is substituted for the Green's function in Eq. (33). This gives

$$M_{\text{cyl}}(\phi, \phi_i) = -k^2 \int_{\vartheta} \int_{\vartheta_i} \varphi_t(\vartheta_i) \varphi_r(\vartheta) \times \left[\int_{z'} e^{j(2\pi f \cos \vartheta/c + k_z) z'} dz' \right] \times \left[\int_{\|\mathbf{x}'\| < r_{\text{cyl}}} e^{j(2\pi f/c) \sin \vartheta \mathbf{u}_{xy}(\phi) \cdot \mathbf{x}'} \eta_{\text{cyl}} p(\mathbf{x}', 0, \phi_i, \vartheta_i) d^2 \mathbf{x}' \right] d\vartheta d\vartheta_i. \quad (35)$$

Since $k_z = 2\pi f \cos \vartheta_i/c$, the z integral evaluates to

$$\int_{z'} e^{j(2\pi f \cos \vartheta/c + k_z) z'} dz' = \delta(f(\cos \vartheta + \cos \vartheta_i)/c) = \frac{\delta(\vartheta - (\pi - \vartheta_i))}{f \sin \vartheta_i/c}. \quad (36)$$

When this result is substituted into Eq. (35), the ϑ variable is forced to assume the value $\pi - \vartheta_i$. However, symmetry in the angular spectra results in $\varphi_r(\pi - \vartheta_i) = \varphi_r(\vartheta_i)$ so that the measurement of scattered pressure with elevation-focused illumination and reception can be expressed as

$$M_{\text{cyl}}(\phi, \phi_i) = \int_{\vartheta_i} \frac{\varphi_t(\vartheta_i) \varphi_r(\vartheta_i)}{f \sin \vartheta_i/c} \times \left[-k^2 \int_{\|\mathbf{x}'\| < r_{\text{cyl}}} e^{j(2\pi f/c) \sin \vartheta_i \mathbf{u}_{xy}(\phi) \cdot \mathbf{x}'} \times \eta_{\text{cyl}} p(\mathbf{x}', 0, \phi_i, \vartheta_i) d^2 \mathbf{x}' \right] d\vartheta_i. \quad (37)$$

The integral over $\mathbf{x}'$ is the far-field evaluation of a Herglotz wave for two-dimensional scattering and, thus, can be equated to a radial limit of the expression for $p_s(\mathbf{x}, 0, \phi_i, \vartheta_i)$ given in Eq. (26) to obtain

$$M_{\text{cyl}}(\phi, \phi_i) = -k^2 \int_{z'} \int_{\|\mathbf{x}'\| < r_{\text{cyl}}} \left[\int_{\vartheta} \varphi_r(\vartheta) \times e^{j(2\pi f/c) \mathbf{u}(\phi, \vartheta) \cdot (\mathbf{x}', z')} d\vartheta \right] \times \left[\int_{\vartheta_i} \varphi_t(\vartheta_i) \eta_{\text{cyl}} p(\mathbf{x}', z', \phi_i, \vartheta_i) d\vartheta_i \right] d^2 \mathbf{x}' dz'. \quad (34)$$

Interchanging the order of integrations over ϑ_i and ϑ with the integrations over $\mathbf{x}'$ and z' , factoring out the z component of the exponential term, and noting from Eq. (26) that $p(\mathbf{x}', z', \phi_i, \vartheta_i) = e^{jk_z z'} p(\mathbf{x}', 0, \phi_i, \vartheta_i)$ allows the $\mathbf{x}'$ and z' integrals to be separated and the expression for the measurement to be written as

$$-k^2 \int_{\|\mathbf{x}'\| < r_{\text{cyl}}} e^{j(2\pi f/c) \sin \vartheta_i \mathbf{u}_{xy}(\phi) \cdot \mathbf{x}'} \eta_{\text{cyl}} p(\mathbf{x}', 0, \phi_i, \vartheta_i) d^2 \mathbf{x}' = \lim_{r \rightarrow \infty} \left\{ j \sqrt{8\pi j k_{xy}(\vartheta_i) r} e^{-jk_{xy}(\vartheta_i) r} \times \sum_n j^n D_n H_n^{(1)}(k_{xy}(\vartheta_i) r) e^{jn(\phi - \phi_i)} \right\} \quad (38)$$

in which the dependency of the x - y plane background wave-number k_{xy} on the polar angle ϑ_i is indicated explicitly. Use of the asymptotic expression for the Hankel function¹⁵ permits evaluation of the limit in Eq. (38). The result is

$$-k^2 \int_{\|\mathbf{x}'\| < r_{\text{cyl}}} e^{j(2\pi f/c) \sin \vartheta_i \mathbf{u}_{xy}(\phi) \cdot \mathbf{x}'} \eta_{\text{cyl}} p(\mathbf{x}', 0, \phi_i, \vartheta_i) d^2 \mathbf{x}' = 4j \sum_n D_n(\vartheta_i) e^{jn(\phi - \phi_i)} \quad (39)$$

in which the dependency of the orthogonal function expansion coefficients D_n on the incident wave polar angle is also indicated explicitly. Substitution of Eq. (39) into Eq. (37) gives

$$M_{\text{cyl}}(\phi, \phi_i) = \int_{\vartheta_i} \frac{\varphi_t(\vartheta_i) \varphi_r(\vartheta_i)}{f \sin \vartheta_i/c} \left[4j \sum_n D_n(\vartheta_i) e^{jn(\phi - \phi_i)} \right] d\vartheta_i. \quad (40)$$

To simplify the expression in Eq. (40) for the measured scattered pressure $M_{\text{cyl}}(\phi, \phi_i)$, the normalized far-field scattering pattern of a cylinder given by the right side of Eq. (39) is denoted using

$$A_{\text{cyl}}(\phi, \phi_i, \vartheta_i) = 4j \sum_n D_n(\vartheta_i) e^{jn(\phi - \phi_i)}. \quad (41)$$

The expression for $M_{\text{cyl}}(\phi, \phi_i)$ in Eq. (40) can then be compactly written as

$$M_{\text{cyl}}(\phi, \phi_i) = \int_{\vartheta_i} \frac{\varphi_t(\vartheta_i) \varphi_r(\vartheta_i)}{f \sin \vartheta_i / c} A_{\text{cyl}}(\phi, \phi_i, \vartheta_i) d\vartheta_i. \quad (42)$$

If the angular spectrum is concentrated in the vicinity of the polar angle $\vartheta_i = \pi/2$, then the small angle approximation $\sin \vartheta_i = \cos(\pi/2 - \vartheta_i) \approx 1$ may be used in Eq. (40). The result is

$$M_{\text{cyl}}(\phi, \phi_i) = \frac{1}{f/c} \left[\int_{\vartheta_i} \varphi_t(\vartheta_i) \varphi_r(\vartheta_i) d\vartheta_i \right] A_{\text{cyl}}(\phi, \phi_i, a_{\text{cyl}}) \quad (43)$$

in which the dependence of the far-field pattern A_{cyl} on the cylinder radius a_{cyl} is explicitly indicated. Thus, each scattering measurement is a product of a value from the two-dimensional far-field pattern for a cylinder of radius a_{cyl} and a weighting factor that is determined by the elevation overlap of the transmit focus and the receive focus.

The expression for $M_{\text{cyl}}(\phi, \phi_i)$ in Eq. (42) can be efficiently computed because $A_{\text{cyl}}(\phi, \phi_i, \vartheta_i)$, i.e., the bracketed quantity inside the integral in Eq. (40), is a Fourier series for which the coefficients can be obtained using a fast Fourier transform.

C. Scattering from a sphere with focused illumination and reception

Consider a fluid sphere that has radius a_{sph} , sound speed c_{sph} , and density ρ_{sph} and is positioned with its center at the origin of a Cartesian coordinate system in which, as in the case of the cylinder, the coordinates in the x - y plane and elevation dimension denoted by the vector $\mathbf{x}$ and scalar z , respectively, and the corresponding coordinates in a spherical coordinate system are denoted as (r, ϕ, ϑ) . A sphere is considered because it is an elementary three-dimensional object with an analytic solution for scattered pressure, has, as noted, a circular cross section like the cross section normal to the axis of a cylinder, and reconstructions of cross sections can, as also noted, be compared to reconstructions of cross sections of a cylinder normal to its axis. Assume that the sphere is illuminated by a monochromatic plane wave that has the same form as the illumination of the cylinder, i.e., has time dependence $e^{-j^2 \pi f t}$ and travels in a direction defined by a unit vector $\mathbf{u}(\phi_i, \vartheta_i)$ with components $\sin \vartheta_i \mathbf{u}_{xy}(\phi_i)$ in the x - y plane and $\cos \vartheta_i$ in the elevation dimension. By matching the conditions of continuous pressure and normal component of particle velocity at the boundary of the sphere, the pressure scattered from this sphere in a lossless fluid background that is the same as in the case of a cylinder, i.e., in which the

sound speed is c and the density is ρ , can be expressed using an orthogonal function expansion in spherical coordinates as^{16,17}

$$p_s(r, \phi, \vartheta, \phi_i, \vartheta_i) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} 4\pi j^n d_n h_n^{(1)}(kr) \times Y_n^m(\vartheta, \phi) Y_n^m(\vartheta_i, \phi_i)^*, \quad r > a_{\text{sph}}. \quad (44)$$

In these expressions, r is the radial distance from the origin to the point of observation, ϕ and ϑ are the azimuthal angle and polar angle, respectively, of the observation point, d_n are coefficients that depend on the conditions at the boundary of the sphere, k_{sph} is the wavenumber in the sphere, $h_n^{(1)}$ is a spherical Hankel function of the first kind, Y_n^m are spherical harmonics, and $*$ denotes complex conjugate. The coefficients d_n in the orthogonal function expansion for the scattered pressure are found by matching boundary conditions and are the same as those in Ref. 17. The spherical harmonics Y_n^m are defined in terms of associated Legendre polynomials P_n^m and are written using the same convention as in Ref. 18. The relation between the wavenumber k_{sph} in the sphere and the wavenumber k in the background is

$$k_{\text{sph}} = \frac{c}{c_{\text{sph}}} k. \quad (45)$$

The expressions for d_n is also valid for a lossy sphere with attenuation coefficient α_{sph} in which case the wavenumber k_{sph} is

$$k_{\text{sph}} = 2\pi f / c_{\text{sph}} + j\alpha_{\text{sph}}. \quad (46)$$

Development of an expression for the measurement of scattering from a spherical object as a far-field (i.e., plane-wave) sensitivity pattern summed with a sequence of weights is more straightforward than for a cylindrical object because a spherical object is bounded. If a sphere is illuminated with a plane wave traveling in the direction given by the unit vector $\mathbf{u}(\phi_i, \vartheta_i)$ and the scattered signal in direction given by the unit vector $\mathbf{u}(\phi, \vartheta)$ is measured with a receiver plane-wave sensitivity, then the measurement has a value that is the plane-wave sensitivity times the far-field pattern $A_{\text{sph}}(\phi, \vartheta, \phi_i, \vartheta_i)$ of the sphere. Measurements made with illumination patterns and receiver sensitivities that are described by angular spectra of plane waves can then be found by summing over the measurements for all illumination and receive plane-wave combinations to obtain

$$M_{\text{sph}}(\phi, \phi_i) = \int_{\vartheta} \int_{\vartheta_i} \varphi_t(\vartheta_i) \varphi_r(\vartheta) A_{\text{sph}}(\phi, \vartheta, \phi_i, \vartheta_i) d\vartheta_i d\vartheta. \quad (47)$$

The far-field pattern for the sphere results from taking the normalized radial limit of the scattered pressure given by

$$A_{\text{sph}}(\phi, \vartheta, \phi_i, \vartheta_i) = \lim_{r \rightarrow \infty} [-4\pi r e^{-jkr} p_s(r, \phi, \vartheta, \phi_i, \vartheta_i)]. \quad (48)$$

Substitution of the expression in Eq. (44) for the pressure scattered from a sphere illuminated by a plane wave traveling in the direction $\mathbf{u}(\phi_i, \vartheta_i)$ yields

$$A_{\text{sph}}(\phi, \vartheta, \phi_i, \vartheta_i) = \lim_{r \rightarrow \infty} \left[-4\pi r e^{-jkr} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} 4\pi j^n d_n \times h_n^{(1)}(kr) Y_n^m(\vartheta, \phi) Y_n^m(\vartheta_i, \phi_i)^* \right]. \quad (49)$$

The limit is found by using the asymptotic form for the spherical Hankel function.^{15,19} The result is

$$A_{\text{sph}}(\phi, \vartheta, \phi_i, \vartheta_i) = \frac{j8\pi}{f/c} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} d_n Y_n^m(\vartheta, \phi) Y_n^m(\vartheta_i, \phi_i)^*. \quad (50)$$

Expressing the spherical harmonics in terms of the Legendre polynomials by using in Eq. (50) yields the relation

$$A_{\text{sph}}(\phi, \vartheta, \phi_i, \vartheta_i) = \frac{j8\pi}{f/c} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} d_n \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \times P_n^m(\cos \vartheta) e^{jm\phi} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \times P_n^m(\cos \vartheta_i) e^{-jm\phi_i}, \quad (51)$$

and substituting this expression for $A_{\text{sph}}(\phi, \vartheta, \phi_i, \vartheta_i)$ in Eq. (47) for $M_{\text{sph}}(\phi, \phi_i)$ results in

$$M_{\text{sph}}(\phi, \phi_i) = \frac{j8\pi}{f/c} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} d_n T_n^m R_n^m e^{jm(\phi - \phi_i)}, \quad (52)$$

where the transmit sensitivity factor T_n^m is defined as

$$T_n^m = \int_{\vartheta_i} \varphi_t(\vartheta_i) \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos \vartheta_i) d\vartheta_i \quad (53)$$

and the receive sensitivity factor R_n^m is defined as

$$R_n^m = \int_{\vartheta} \varphi_r(\vartheta) \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos \vartheta) d\vartheta. \quad (54)$$

Interchanging the summation over n and the summation over m yields

$$M_{\text{sph}}(\phi, \phi_i) = \frac{j8\pi}{f/c} \sum_{m=-\infty}^{+\infty} \left[\sum_{n=|m|}^{\infty} d_n T_n^m R_n^m \right] e^{jm(\phi - \phi_i)}. \quad (55)$$

The expression for $M_{\text{sph}}(\phi, \phi_i)$ in Eq. (55) for a spherical object can, as in the case of $M_{\text{cyl}}(\phi, \phi_i)$ in Eq. (42) for a cylindrical object, be efficiently computed because the outer summation is a Fourier series for which the coefficients given by the bracketed quantity can be obtained using a fast Fourier transform. In this computation, additional efficiency is gained by evaluating the transmit sensitivity factors T_n^m and receive sensitivity factors R_n^m first and storing them. Also, since the sensitivity factors are independent of the sphere radius, once a set of factors have been computed, the factors may be used for a sphere with different acoustic properties (radius, sound speed, density, and position) as long as a sufficient number of orders for the different parameters have been computed.

The foregoing analysis has assumed that the sphere is located in the imaging plane at the center of the ring transducer, i.e., at (0,0,0). However, the analysis is easily modified to hold for a scattering object that is translated by an offset of $(\Delta \mathbf{x}, \Delta z)$ from the origin because translation only changes the object far-field scattering pattern A_{obj} by the phase factor $e^{j(2\pi f/c)[\mathbf{u}(\phi_i, \vartheta_i) + \mathbf{u}(\phi, \vartheta)] \cdot (\Delta \mathbf{x}, \Delta z)}$, i.e.,

$$\hat{A}_{\text{obj}}(\phi, \vartheta, \phi_i, \vartheta_i) = A_{\text{obj}}(\phi, \vartheta, \phi_i, \vartheta_i) \times e^{j(2\pi f/c)[\mathbf{u}(\phi_i, \vartheta_i) + \mathbf{u}(\phi, \vartheta)] \cdot (\Delta \mathbf{x}, \Delta z)} \quad (56)$$

in which $\hat{}$ denotes a shifted pattern. For a vertical offset from the center of the transducer ring, the phase factor assumes the simple form

$$e^{jk[\mathbf{u}(\phi_i, \vartheta_i) + \mathbf{u}(\phi, \vartheta)] \cdot (0, \Delta z)} = e^{jk(\cos \vartheta_i + \cos \vartheta) \Delta z}. \quad (57)$$

Introduction of the factor given by Eq. (57) in foregoing analysis for a spherical scattering object only changes the final results by altering the expressions for the coefficients T_n^m and R_n^m given above by Eqs. (53) and (54), respectively, to have the forms

$$T_n^m = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \int_{\vartheta_i} \varphi_t(\vartheta_i) e^{j(2\pi f/c) \Delta z \cos \vartheta_i} \times P_n^m(\cos \vartheta_i) d\vartheta_i \quad (58)$$

and

$$R_n^m = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \int_{\vartheta} \varphi_r(\vartheta) e^{j(2\pi f/c) \Delta z \cos \vartheta} \times P_n^m(\cos \vartheta) d\vartheta. \quad (59)$$

These expressions also describe the effect of vertically translating (by $-\Delta z$) the origin of the plane-wave components in the angular spectrum expansions for the illumination pattern and the sensitivity of reception when the center of the sphere is at the origin of the coordinate system.

D. Equal angles

When the scattered pressure is needed at a number of temporal frequencies, such as for the extended eigenfunction method of image reconstruction used in this paper, an angular spectrum calculation is required at multiple temporal harmonic frequencies. If the transmit angular spectrum is sampled at the Nyquist rate for each temporal frequency, a different set of angles can result in the angular spectrum. However, different angles for each temporal frequency creates a bottleneck that slows the calculation of scattering at multiple frequencies because the associated Legendre polynomials in the definition of the spherical harmonics must be recomputed for each temporal frequency. This recalculation is avoided by choosing the number of samples in the aperture to be sufficiently large to prevent spatial aliasing at the highest temporal frequency and by varying the sampling rate to permit use of the same angles for all temporal frequencies.

In the technique, the aperture is first spatially sampled at the Nyquist rate determined by the highest temporal frequency f_h . This spatial sampling rate ν_h is

$$\nu_h = 2 \frac{f_h}{c}. \quad (60)$$

Next, the number of points N_a in the spatial Fourier transform is chosen to avoid aliasing at the highest temporal frequency. For an aperture of length L , the number of points is

$$N_a = L\nu_h. \quad (61)$$

Then, in order to maintain the same set of angles for each temporal frequency, this number of samples is fixed and used to define the constant angular spacing $\Delta\vartheta$ given by

$$\Delta\vartheta = \arccos(2/N_a). \quad (62)$$

This results in the set of angles

$$\{\vartheta_n\} = \left\{ \arccos\left(n \frac{c}{L f_h}\right) \right\}, \quad n = 0, 1, \dots, N_a - 1 \quad (63)$$

that are the same for the calculation of elevation-focused illumination and reception at each temporal frequency.

This technique does not require an extraordinarily large number of points in the Fourier transform. For example, the extreme case in which $f_h = 5.0$ MHz, $L = 60$ mm, and $c = 1.5$ mm/ μ s results in $\nu_h = 6.7$ cycle/mm and $N_a = 400$ samples or Fourier transform points. Thus, although the aperture is always sampled at a rate determined by the highest temporal frequency, the so-called equal-angle approach still requires fewer computations than the approach of recomputing the associated Legendre polynomials at each temporal frequency.

E. Verification in the far field by using a weak-scattering approximation

The scattered pressure for elevation-focused illumination and reception calculated for a cylinder using Eq. (42) and for a sphere using Eq. (55) can be compared to an independent calculation of scattered pressure obtained under a weak-scattering approximation in the far field to instill confidence that the calculations are correct. For this comparison, the normalized limit used in Eq. (48) is applied to the Herglotz expression for scattering. This yields the far-field pattern

$$A(\phi, \vartheta, \phi_i, \vartheta_i) = k^2 \int_{\mathbf{x}} \int_z \eta(\mathbf{x}, z) p(\mathbf{x}, z, \phi_i, \vartheta_i) \times e^{j(2\pi f/c) \mathbf{u}(\phi, \vartheta) \cdot (\mathbf{x}, z)} d\mathbf{x} dz, \quad (64)$$

where $p(\mathbf{x}, z, \phi_i, \vartheta_i)$ is the total pressure and where the z integration is restricted to the range of elevations over which the product of the transmit focus and the receive focus is appreciable.

Using a weak-scattering approximation, i.e., $p(\mathbf{x}, z, \phi_i, \vartheta_i) \approx p_i(\mathbf{x}, z, \phi_i, \vartheta_i)$, in Eq. (64) gives

$$A(\phi, \vartheta, \phi_i, \vartheta_i) = k^2 \int_{\mathbf{x}} \int_z \eta(\mathbf{x}, z) \times e^{j(2\pi f/c) [\mathbf{u}(\phi_i, \vartheta_i) + \mathbf{u}(\phi, \vartheta)] \cdot (\mathbf{x}, z)} d\mathbf{x} dz. \quad (65)$$

The far-field scattered pressure with elevation-focused illu-

mination and reception described by angular spectra can then be written as

$$\tilde{A}(\phi, \vartheta, \phi_i, \vartheta_i) = \int_{\vartheta} \int_{\vartheta_i} A(\phi, \vartheta, \phi_i, \vartheta_i) \varphi_t(\vartheta_i) \varphi_r(\vartheta) d\vartheta_i d\vartheta \quad (66)$$

in which $\sim$ denotes focusing in elevation. Substitution of Eq. (65) into Eq. (66) for elevation-focused illumination and reception and use of Eqs. (11) and (21) for $\psi_t(\mathbf{x}, z)$ and $\psi_r(\mathbf{x}, z)$, respectively, yields

$$\tilde{A}(\phi, \vartheta, \phi_i, \vartheta_i) = k^2 \int_{\mathbf{x}} \int_z \eta(\mathbf{x}, z) \psi_t(\mathbf{x}, z) \psi_r(\mathbf{x}, z) dz d\mathbf{x}. \quad (67)$$

The expressions for $\psi_t(\mathbf{x}, z)$ and $\psi_r(\mathbf{x}, z)$ in Eq. (67) can be approximated in the scattering region by the product of an amplitude envelope and a plane wave propagating in the direction of the illumination for $\psi_t(\mathbf{x}, z)$ and in the direction of reception for $\psi_r(\mathbf{x}, z)$, i.e., by the expressions

$$\psi_t(\mathbf{x}, z) = |\psi_t(z)| e^{j(2\pi f/c) \mathbf{u}_{xy}(\phi_i) \cdot \mathbf{x}} \quad (68)$$

and

$$\psi_r(\mathbf{x}, z) = |\psi_r(z)| e^{j(2\pi f/c) \mathbf{u}_{xy}(\phi) \cdot \mathbf{x}}, \quad (69)$$

respectively, in which $(2\pi f/c) \mathbf{u}_{xy}(\phi_i)$ and $(2\pi f/c) \mathbf{u}_{xy}(\phi)$ are the wave vectors of the incident illumination and the scattered wave, respectively, in the x - y plane. Using these expressions in Eq. (67), the normalized far-field scattered pressure for elevation-focused illumination and reception under a weak-scattering approximation becomes

$$\tilde{A}(\phi, \phi_i) = k^2 \int_{\mathbf{x}} \eta_p(\mathbf{x}) e^{j(2\pi f/c) [\mathbf{u}_{xy}(\phi_i) + \mathbf{u}_{xy}(\phi)] \cdot \mathbf{x}} d\mathbf{x} \quad (70)$$

in which

$$\eta_p(\mathbf{x}) = \int_z \eta(\mathbf{x}, z) |\psi_t(z)| |\psi_r(z)| dz. \quad (71)$$

Equation (70) shows that the normalized far-field pattern in the presence of elevation-focused illumination and reception is the Fourier transform of the illuminated volume of $\eta(\mathbf{x}, z)$ weighted and projected onto the x - y plane according to Eq. (71). When the variation in the medium is essentially a two-dimensional quantity, i.e., $\eta(\mathbf{x}, z) \approx \eta(\mathbf{x})$, Eq. (70) simplifies to

$$\tilde{A}(\phi, \phi_i) = W k^2 \int_{\mathbf{x}} \eta(\mathbf{x}) e^{j(2\pi f/c) [\mathbf{u}_{xy}(\phi_i) + \mathbf{u}_{xy}(\phi)] \cdot \mathbf{x}} d\mathbf{x} \quad (72)$$

in which

$$W = \int_z |\psi_t(z)| |\psi_r(z)| dz \quad (73)$$

and is a scale factor that accounts for use of elevation-focused illumination and reception.

Equations (72) and (70) provide expressions to verify the scattered pressure given for a cylinder by Eq. (42) and for a sphere by Eq. (55), respectively, in the presence of elevation focusing.

TABLE I. Pulse, focus, ring, and background parameters used in calculations of scattering.

Parameter (symbol)	Value
Pulse-width parameter (σ_t)	0.220 μ s
Center frequency (f_0)	2.50 MHz
Aperture width parameter (σ_a)	4.231 mm
Ring radius (r_0)	75.0 mm
Geometric focus adjustment (r_a)	17.224 mm
Background sound speed (c)	1.590 mm/ μ s
Background density (ρ)	0.997 g/mm ³

III. COMPUTATIONAL METHODS

The temporal envelope of elevation-focused transmit-receive beam patterns was calculated for a Gaussian-shaped bandpass pulse given by Eq. (10) and transmit and receive apertures with identical Gaussian amplitude elevation weighting given by Eq. (19). In the calculations, the transmit and the receive beams given by Eqs. (16) and (18) were found at each temporal frequency in the pulse, corresponding temporal-frequency patterns were multiplied, and an inverse Fourier transform was taken to obtain the temporal beam as a function of time. Values of parameters in the calculations are given in Table I. The peak of the temporal envelope was then found in elevation cross sections to provide a description of the elevation-focused beams used in calculations of scattering by cylinders and by spheres.

To instill confidence that the scattered pressure for elevation-focused illumination and reception calculated using Eq. (40) for a cylinder is correct, the far-field scattered pressure under a weak-scattering approximation for a cylinder was computed using the Fourier transform relation given in Eq. (72) for a two-dimensional scattering object. To instill confidence also that the scattered pressure for elevation-focused illumination and reception calculated using Eq. (55) for a sphere is correct, the far-field scattered pressure under a weak-scattering approximation was computed for a sphere using the Fourier transform relation given in Eq. (70) for a projection of a three-dimensional scattering object and compared to the corresponding far-field normalized expressions used in the two-dimensional reconstruction process. Values of parameters in the two calculations are given in Table II.

The pressure scattered by a cylinder and by a sphere each with elevation-focused illumination and reception was computed at multiple temporal frequencies by using Eqs. (40) and (55), respectively, with the acoustic parameters of the spheres and cylinders having the values shown in Table III. In the computations, the axis of the cylinder was along

TABLE II. Scattering object parameters for comparison of analytic solution with weak-scattering far-field approximation.

Parameter (symbol)	Value
Radius (r_{cyl} , r_{sph})	10.0 mm
Sound speed (c_{cyl} , c_{sph})	1.50901 mm/ μ s
Attenuation slope (β_{cyl} , β_{sph})	0.0 dB/(cm MHz)
Density (ρ_{cyl} , ρ_{sph})	0.997 g/mm ³
Frequency (f)	1.40625 MHz

TABLE III. Cylinder and sphere parameters used in elevation-focused calculations of scattering.

Parameter	Value
Sound speed	1.570 mm/ μ s
Attenuation slope	0.462 dB/(cm MHz)
Density	0.997 g/mm ³

the z axis of the coordinate system, the center of the sphere was 0, 3, or 5 mm above the center of the focus that was at the origin of the coordinate system, the diameter of the sphere was 20 mm, and the diameter of the cylinder was the same as the diameter of the cross section of the sphere in the x - y plane at $z=0$; i.e., the cylinder diameter was 20, 19.08, and 17.32 mm, respectively. The computations were performed at 512 receive azimuthal angles and for 512 incident wave azimuthal angles each equally spaced between 0 and 2π around the circumference of the ring. This number of incident wave angles and receive angles was chosen to provide a sufficient number of angular samples for the size objects being reconstructed to avoid aliasing in the reconstruction process.¹³ The elevation-focused angular spectra that modeled the described elevation focusing in the ring transducer system were calculated using the equal-angle method to reduce computation time. The illumination and reception apertures included a Gaussian-shaped amplitude weight given by Eq. (19) and a length factor given by Eq. (20) to create a focus with a peak at the center of the 150-mm diameter ring transducer.

After the scattered pressure for elevation-focused illumination and reception was calculated at each temporal frequency, the pressure was normalized by the factor W given by Eq. (73) to compensate for the gain from elevation-focused illumination and reception. The resulting scattered pressure was then extrapolated to the far field using a two-dimensional extrapolation procedure described in Ref. 14 and normalized to remove the cylindrical-wave factor weighting the far-field pattern.

Reconstructions were obtained using the extended eigenfunction method described in Ref. 14. In the extension, scattering by a model object is calculated to form the difference between the scattering by the true object and the scattering by the model object. The scattering potential defined by this difference is expanded in a basis of products of acoustic fields. These fields are defined by eigenfunctions of the scattering operator associated with the estimate.

Computations that implement the reconstruction process were performed in five steps that are detailed in Ref. 14 and summarized here for convenient reference. In the first step, inner products needed for the computation of the coefficients in the expansion of the scattering potential are found by a two-dimensional Fourier transform of the normalized far-field pattern that is the difference between the calculated scattering by the true object and the calculated scattering by the model object. In the second step, the coefficients in the expansion were computed using symmetries that reduce the amount of computation for radial objects like those considered in this study. In the third step, a potential that is the

TABLE IV. Reconstruction parameters used in eigenfunction method of image reconstruction.

Parameter	Value
Initial frequency	1.40625 MHz
Frequency increment	78.125 kHz
Final frequency	3.4375 MHz

lowpass difference between the potential of the true scattering object and the model scattering object was found using a linearization that also exploits symmetries that exist in relations for radial scattering objects. In the fourth step, a weighted average of the potentials reconstructed at each temporal frequency was computed after the potentials were phase compensated using a phase that maximizes the coherence of the reconstructions as a function of frequency. In the fifth step, iteration was used to refine the estimate of the scattering potential until the difference potential between the model and the true object is sufficiently small.

Since the eigenfunction method utilizes *a priori* knowledge, an estimate of the scatterer shape and acoustic parameters is necessary for the reconstructions. A circular shape, i.e., the cross section of a sphere and the cross section of a cylinder normal to its axis, was used in the model for the reconstruction of a sphere and a cylinder. In the case of a cylinder, the radius of the cylinder was assumed known. In the case of a sphere, the estimated radius was the average radius of the volume isolated by the elevation-focused illumination and reception. The value of sound speed in the model for each iteration was the average of the sound speed in a 2-mm diameter region in the center of the previous estimate. The value of attenuation in the model was found frequency by frequency by minimizing the norm of the difference between the far-field scattering by the model and the far-field scattering obtained from the object over the range of

0° – 10° using elevation-focused transmit-receive beams. Since the calculations were designed to investigate the use of two-dimensional algorithms with three-dimensional objects, reconstructions of spheres using elevation-focused illumination and reception were compared to reconstructions of cylinders. For the comparisons, cross sections of a sphere were reconstructed for three offsets of the center of the sphere above the center of the focus. Parameters used in the reconstruction process are listed in Table IV. The calculations were repeated using a 4-mm diameter sphere that mostly fits within the half-amplitude (i.e., -6 dB) elevation width of the elevation-focused illumination and reception.

IV. NUMERICAL RESULTS

Cross sections and profiles of the peak temporal envelope of the elevation-focused transmit-receive beam patterns for the parameter values in Table I and depictions of the geometry for which the envelopes were computed are shown in Fig. 4. For these parameter values that were chosen to model the pulse and the focusing in the ring transducer system described in Ref. 3, the -3 dB width in elevation is 2.1 mm and remains the same as the transmit-receive angle varies, and the -3 dB focal length increases monotonically from 72.1 mm at a transmit-receive angle of 180° to 125.6 mm at a transmit-receive angle of 90° . The corresponding values for the -6 dB width in elevation and the range of the -6 dB focal length are 3.2 mm and 124.9–150.0 mm, respectively.

The analytic solution for scattering from a cylinder and from a sphere each with elevation-focused illumination and reception and the corresponding far-field weak-scattering approximations are shown in Fig. 5 for the parameter values in Table II. At the 1.40625 MHz frequency of the calculations, the combined illumination-reception elevation focusing resulted in a -6 -dB beam width of 5.4 mm in elevation. The

FIG. 4. (Color online) Peak temporal envelope of the elevation-focused transmit-receive beam pattern for transmit and receive apertures with identical Gaussian amplitude weighting in elevation and a bandpass pulse with a Gaussian envelope. The grayscale plots are cross sections of the envelope normalized to 1.0 at $(x,z)=(0,0)$ and shown on a linear scale. The Cartesian plots are profiles of the envelope in the cross section where the envelope in orthogonal grayscale-plot coordinate is a maximum (i.e., 0 dB) and is one-half the maximum (i.e., -6 dB relative to the maximum) in that coordinate. The three-dimensional diagrams show the spatial orientation of grayscale cross sections. For a transmit-receive angle of 180° , the cross sections are the same at every value of the coordinate orthogonal to the plane of the cross section. For a transmit-receive angle of 90° and the focus parameters in this study, the cross sections are essentially constant in the x - y plane when $-20 \leq x, y \leq 20$ mm.

FIG. 5. (Color online) Comparison of the analytic solution for scattering from a lossy cylinder with elevation-focused illumination and reception and the corresponding far-field weak-scattering approximation and comparison of the analytic solution for scattering from a lossy sphere with elevation-focused illumination and reception and the corresponding far-field weak-scattering approximation. The scattered pressure in each case is normalized to 1.0 in the forward (0° azimuthal angle) direction. The center of the sphere is at the center of the focus in elevation. —, far-field analytic solution; ···, far-field weak-scattering solution.

close agreement of the two plots for the cylinder shows that the two independent calculations of scattered pressure from a cylinder produce essentially the same values, and the corresponding close agreement of the two plots for the sphere shows that the two independent calculations of scattered pressure from a sphere produce essentially the same values. This agreement instills confidence that the calculations of scattering using elevation-focused illumination and reception are correct. The similarity in the scattering from the cylinder and from the sphere supports the expectation that scattering produced by elevation-focused illumination and measured using elevation-focused reception can be treated as two-dimensional when a three-dimensional scattering object has an essentially constant cross section throughout the extent of the elevation-focused beams as is the case in the calculations shown.

Scattering from a cylinder and from a sphere observed using elevation-focused illumination and reception and observed using plane-wave illumination and point reception is compared in Fig. 6 for the parameter values in Table III in which the offsets of the sphere are the vertical distance above the center of the focus. As noted above, the combined illumination-reception elevation focus at the frequency of 1.40625 MHz resulted in a 5.4-mm and 6-dB width in elevation. For the scattering object parameter values that were chosen to be clinically relevant for imaging two-dimensional and three-dimensional objects with the previously noted ring transducer system, the plot for elevation-focused illumination and elevation-focused reception is essentially identical to the plot for plane-wave illumination and point reception for the cylinder. The corresponding plots for a sphere show a difference that increases as the center of the sphere is more offset from the imaging plane and is due to the elevation-focused illumination and reception. For each of the three

FIG. 6. (Color online) Comparison of plane-wave illumination and point reception with elevation-focused illumination and reception for a lossy cylinder and for a lossy sphere. The scattered pressure in each case is evaluated in the far field and normalized to 1.0 in the forward (0° azimuthal angle) direction. In the calculations, the center of the sphere was vertically offset from the center of the focus by the distance given above each panel, and the radius of the cylinder was the radius of the circular intersection of the sphere with the ring plane through the center of the focus in elevation, i.e., 10, 9.54, and 8.66 mm in the upper, middle, and lower panels, respectively. —, cylinder: plane-wave illumination and point reception; - - -, cylinder: elevation-focused illumination and reception; —, sphere: plane-wave illumination and point reception; ···, sphere: elevation-focused illumination and reception.

offsets with elevation focusing, the high amplitude scattering from the sphere in the forward direction (i.e., at low azimuthal angles) is similar to the scattering amplitude from the corresponding cylinder, but the low amplitude scattering in the backward direction (i.e., high azimuthal angles) lacks similarity except for the zero-offset case. The similarity in the forward direction implies that low spatial-frequency components in reconstructions made for a spherical object by using elevation-focused illumination and elevation-focused reception are well represented by a two-dimensional model, while the lack of similarity in the backward direction implies that high spatial-frequency components in reconstructions made for a sphere by using elevation-focused illumination and reception may not be well represented by a two-dimensional model.

Radial profiles through reconstructions obtained by using transmit-receive elevation-focused beams and the two-dimensional form of eigenfunction method are shown in Fig. 7 for a 20-mm diameter sphere with an offset of the center 0, 3, and 5 mm from the imaging plane. The profiles of sound speed and attenuation slope show differences between the ideal radial profiles formed by intersection of the sphere and the imaging plane and the radial profiles in reconstructions obtained using transmit-receive elevation-focused beams and a two-dimensional reconstruction algorithm. The differences between the ideal and reconstructed profiles of sound speed and attenuation slope are larger close to the edges of the reconstructed spheres. The differences are small when the portion of the sphere located in the focused beam is relatively constant in elevation, but the differences increase as the center of the sphere is offset further from the plane of the

FIG. 7. (Color online) Radial profiles of sound speed and attenuation slope for a 20-diameter lossy sphere with center offset of 0, 3, or 5 mm from the imaging plane that is in the center of the elevation-focused transmit-receive beam. The solid lines are profiles through reconstructions using scattering from the sphere. The dashed lines are profiles through a cylinder with the diameter of the disk formed by the intersection of the sphere with the imaging plane. —, sphere; ---, cylinder.

focused beam. Large artifacts at the profile edges and large value differences in the center of the profiles indicate that three-dimensional scattering is not being adequately included in the elevation-focused beams.

Analogous radial profiles through reconstructions obtained using the same beams and two-dimensional form of the eigenfunction method are shown in Fig. 8 for a 4-mm diameter sphere with an offset of the center 0, 2, and 4 mm from the imaging plane. Because the -6 dB width of elevation focus in the calculations was 3.2 mm, the 4-mm diameter sphere with zero offset was essentially contained within the elevation-focused beam in zero-offset case, partially encompassed by the elevation-focused beam in the 2-mm offset case, and at the edge of the elevation-focused beam in the 4-mm offset case. These beam-object relations result in less effect on the overall propagation time of the incident wave and in substantial scattered energy outside the receive aperture. The consequence of the propagation time effect is that the sound speed profiles are generally lower than ideal values defined by a central cross section of the sphere, and the con-

FIG. 8. (Color online) Radial profiles of sound speed and attenuation slope for a 4-diameter lossy sphere with center offset of 0, 2, or 4 mm from the imaging plane that is in the center of the elevation-focused transmit-receive beam. The solid lines are profiles through reconstructions using scattering from the sphere. The dashed lines are profiles through a cylinder with the diameter of the sphere. —, sphere; ---, cylinder.

sequence of the unmeasured scattering energy is that the attenuation slope profiles can be much larger than ideal values again defined by a central cross section of the sphere.

V. DISCUSSION

The theoretical development that starts with a wave equation in three dimensions and yields expressions for measurements using elevation-focused illumination and reception has a special structure that merits note. The approach yields a two-dimensional wave equation with a term that describes the effect of scattering object variation in the width of the elevation-focused beams. The development uses expressions for the transmit beam pattern and the receive sensitivity pattern in forms that show the reciprocal nature of transmission and reception. The angular spectrum representation of transmit illumination and reception sensitivity facilitates the use of known relations for scattering by regular objects illuminated by monochromatic plane waves and for far-field reception of the scattering. Efficient calculation of scattering by elevation-focused transmit-receive beams temporal frequency by temporal frequency is enabled by varying

the number of samples in the aperture to allow use of the same angles for angular spectrum beam components at each temporal frequency. The consideration of cylindrical and spherical scattering objects that have the same cross sections permits direct comparison of images formed using elevation-focused beams and two-dimensional reconstruction methods. Overall, the theoretical formulation is general but facilitates computational elucidation of phenomena influencing the accuracy of using a two-dimensional reconstruction method to image objects having a variable cross section within the width of elevation-focused transmit-receive beams.

The two-dimensional reconstruction method used in these studies is an iterative multifrequency inverse scattering algorithm that relies on successive improvement of a model for the scattering object. The algorithm converges to an accurate representation of the object intrinsic parameters when the initial model is chosen to be a reasonably good approximation to the scattering object. A good approximation, however, does not necessarily mean that the model has intrinsic parameters approximating those of the scattering object but, rather, that the two-dimensional scattering by the model approximates the scattering from the cross section of the object in the elevation-focused transmit-receive beams. Good approximations in this study were obtained using model objects that include additional frequency-dependent attenuation to compensate for energy transported away from the plane of the cross section by vertical components of wave propagation.

The attenuation assigned to the homogeneous cylinders that were used to model the sphere cross sections can be divided into two additive components. One component is the intrinsic attenuation resulting from absorption of energy by the scattering medium and is an important characteristic of the medium. The other component is the extrinsic attenuation resulting from energy loss caused by spreading of the elevation-focused beam and by out-of-plane reflection and refraction at the surface of an object. Energy reflected into the plane of the cross section is included in the scattering measurements and, therefore, is not included in the energy loss associated with extrinsic attenuation. Extrinsic attenuation that is not uniform across an object with homogeneous intrinsic attenuation could be modeled more accurately by a spatially varying attenuation parameter. The model could also include boundary conditions that extend over a spatial region rather than being abrupt. However, these refinements increase computational cost and were not needed to secure convergence of the reconstructions in this study.

The attenuation for the two-dimensional model cylinders was chosen in this study frequency by frequency to minimize the norm of the residual error between the two-dimensional scattering from the cylindrical model and the scattering from the section of the sphere in the elevation-focused beam. Since the final rendering of each cross section is formed as a sum of the model parameters and the residual reconstruction, the extrinsic attenuation must be estimated and subtracted from the final attenuation in order to assign a value to the intrinsic attenuation. A portion of extrinsic attenuation that is due to the spreading of the focus may be conveniently estimated by simulation of elevation-focused scattering from a

three-dimensional cylinder with the same radius, sound speed, and attenuation as in the two-dimensional model. However, the contribution to extrinsic attenuation that is due to interaction with an object boundary that varies within the elevation-focused beam requires *a priori* knowledge of the boundary variation in elevation. Lack of this information may preclude accurate estimation of intrinsic attenuation but is not an obstacle to high-fidelity reconstruction of sound speed.

Out-of-plane reflection and refraction at the boundary of an object produce a loss even when the object has uniform intrinsic attenuation. A cylinder model is only able to compensate for this energy by means of a uniform increase in attenuation throughout the cylinder. However, the residual reconstruction in the final iteration of the inverse scattering computation depicts how this attenuation should be distributed. This residual term corrects the uniform attenuation of the model by modestly reducing attenuation in the interior region and substantially increasing attenuation near the boundary. Since extrinsic attenuation is small in regions far from the boundary, values of reconstructed attenuation approximate intrinsic attenuation near the center of the cross section.

The effect of unmeasured scattering on image reconstruction using a two-dimensional method is illustrated by calculations of scattering from which the reconstruction proceeds. First, however, the excellent agreement between the magnitude of the scattered amplitude obtained with a far-field weak-scattering approximation and the magnitude of the scattered amplitude obtained using a direct evaluation of the orthogonal function expansion for the scattered amplitude with elevation-focused beams in Fig. 5 is noted for both the scattering from a lossy cylinder and the scattering from a lossy sphere. This agreement gives confidence that other calculations of scattering using orthogonal function expansions with elevation-focused transmit-receive reception for a cylinder and for a sphere are correct. The comparison of plane-wave illumination and point reception with elevation-focused illumination and reception in Fig. 6 shows that calculations using plane-wave illumination and point reception can accurately describe illumination and reception with elevation-focused transmit-receive beams for a cylinder. The analogous comparison for a sphere shows an increasing lack of correspondence in the magnitude of scattered amplitude for a sphere as the center of the sphere is more offset from the imaging plane and the cross section of the sphere consequently varies more in the width of the elevation-focused beam. The comparison for a sphere centered in the imaging plane shows, however, that plane-wave illumination and point reception can reasonably approximate elevation-focused transmit-receive beams when the cross section of the sphere is relatively constant within the elevation beam width.

Features of sound speed and attenuation slope profiles in Fig. 7 for a 20-mm-diameter sphere with a center at different offsets from the imaging plane merit comment because the features also illustrate points noted above. The radial profile of sound speed in the reconstruction for no elevation offset closely matches the ideal profile of sound speed in the disk formed by the intersection of the sphere and the imaging

plane. The corresponding radial profile of attenuation slope also closely matches the ideal profile of attenuation slope in the disk formed by the intersection of the sphere and the imaging plane except at the edges. The profiles of sound speed and attenuation slope in the reconstruction for a 3-mm offset have similar but larger edge artifacts and larger differences of the central values from the corresponding ideal values again defined by the intersection of the sphere and the imaging plane. The profiles of sound speed and attenuation slope in the reconstruction for a 5-mm offset have still larger edge artifacts and differences of the central values from the corresponding ideal values defined by the intersection of the sphere and the imaging plane. Overall, the results show that profiles of sound speed can reasonably approximate ideal profiles of sound speed when the portion of the sphere in the elevation-focused beam is not constant, but profiles of attenuation slope can be larger than ideal profiles when the center of the sphere is offset from the imaging plane and appreciable scattering is not collected by the receive aperture.

Features of sound speed and attenuation slope profiles in Fig. 8 for a 4-mm diameter sphere with a center at different offsets from the imaging plane merit comment because the features are further illustrative. The profiles of sound speed in reconstructions of the 4-mm diameter sphere that was either mostly encompassed by, partially encompassed by, or at the edge of the elevation-focused beam are more smoothed and have smaller values than ideal values defined by a central cross section of the sphere. This is a result of partial volume effects. The profiles of attenuation slope have larger values than similarly defined ideal profiles. This is a result of energy scattered outside the area of the elevation-focused receive aperture dominating partial volume effects.

Insight about attenuation artifacts can be gained by expressing the three-dimensional wave equation as a wave equation in two dimensions plus a term in the third dimension. To begin, consider a pressure p that satisfies the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) p(\mathbf{x}, z) + k^2 p(\mathbf{x}, z) = k^2 \eta(\mathbf{x}, z) p(\mathbf{x}, z). \quad (74)$$

Integrating both sides of this equation in the elevation or z direction with weighting $w(z)$ gives

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz + \int_{-\infty}^{+\infty} \frac{\partial^2 p(\mathbf{x}, z)}{\partial z^2} w(z) dz \\ + k^2 \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz = k^2 \int_{-\infty}^{+\infty} \eta(\mathbf{x}, z) p(\mathbf{x}, z) w(z) dz. \end{aligned} \quad (75)$$

If $p(\mathbf{x}, z)$ slowly varies in the z direction, then the second integral on the left side may be neglected and Eq. (75) can be written as

$$\begin{aligned} (\Delta_{x,y} + k^2) \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz \\ = k^2 \int_{-\infty}^{+\infty} \eta(\mathbf{x}, z) p(\mathbf{x}, z) w(z) dz \end{aligned} \quad (76)$$

in which

$$\Delta_{x,y}(\cdot) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\cdot). \quad (77)$$

Finally, to complete the reduction to two dimensions, the integral on the right side is written as

$$\int_{-\infty}^{+\infty} \eta(\mathbf{x}, z) p(\mathbf{x}, z) w(z) dz = \bar{\eta}(\mathbf{x}) \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz, \quad (78)$$

where

$$\bar{\eta}(\mathbf{x}) = \int_{-\infty}^{+\infty} \eta(\mathbf{x}, z) p(\mathbf{x}, z) w(z) dz \bigg/ \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz. \quad (79)$$

If $\eta(\mathbf{x}, z) = \eta(\mathbf{x})$, i.e., the medium variations are independent of the elevation dimension z , then $\eta(\mathbf{x}) = \eta(\mathbf{x})$ regardless of the pressure characteristics. However, if $\eta(\mathbf{x}, z)$ varies in the z direction, then values of $\bar{\eta}(\mathbf{x})$ differ when the right side of Eq. (79) is evaluated for different pressures. To ensure a consistent value of $\bar{\eta}(\mathbf{x})$ in two-dimensional measurements of scattering, the pressure that is incident pressure under the Born approximation should have the same elevation profile throughout the range of z values over which $w(z)$ is appreciable. Then, Eq. (77) becomes

$$(\Delta_{x,y} + k^2) \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz = k^2 \bar{\eta}(\mathbf{x}) \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz, \quad (80)$$

i.e., a two-dimensional reduced wave equation for the two-dimensional pressure that is a weighted z -axis projection of the three-dimensional pressure. The weights $w(z)$ in the calculations reported here were chosen to be the complex amplitudes of the lens that produced the elevation focus in the ring transducer system because this choice ensures that the calculations correspond to scattering measurements made using the system.

Worthy of special comment is that the term

$$\int_{-\infty}^{+\infty} \frac{\partial^2 p(\mathbf{x}, z)}{\partial z^2} w(z) dz \quad (81)$$

was assumed negligible in the above analysis. To study this term, the pressure $p(\mathbf{x}, z)$ is factored into the product $p(\mathbf{x}, z) = \psi(\mathbf{x}) \zeta(z)$ so that the term in Eq. (81) can be written as

$$\int_{-\infty}^{+\infty} \frac{\partial^2 p(\mathbf{x}, z)}{\partial z^2} w(z) dz = \Delta \bar{\eta} k^2 \int_{-\infty}^{+\infty} p(\mathbf{x}, z) w(z) dz \quad (82)$$

in which

TABLE V. Power loss calculations and power loss ratios for various lossless and lossy cylinders and spheres illuminated by a plane wave. In the case of the cylinders, the plane wave travels normal to the axis of the cylinder.

	Radius (mm)	P_{loss} (nW)	P_{scat} (nW)	P_{abs} (nW)	$(P_{\text{abs}})/P_{\text{loss}}$ (%)
Lossless cylinder	10.00	10.2732	10.2732	0.8882×10^{-14}	0.864×10^{-13}
	9.54	8.1582	8.1582	0.3375×10^{-12}	-0.414×10^{-12}
	8.66	6.0006	6.0006	-0.3642×10^{-12}	-0.606×10^{-12}
Lossless sphere	10.00	0.1550	0.1550	0.4331×10^{-9}	2.79×10^{-7}
	9.54	0.1365	0.1365	0.2938×10^{-9}	2.15×10^{-7}
	8.66	0.1200	0.1200	0.3052×10^{-9}	2.54×10^{-7}
Lossy cylinder	10.00	10.8788	8.7190	2.1598	19.9
	9.54	9.0971	7.1125	1.9845	21.8
	8.66	7.1201	5.4543	1.6658	23.4
Lossy sphere	10.00	0.1661	0.1377	0.0284	17.1
	6.54	0.1478	0.1229	0.0249	16.8
	8.66	0.1279	0.0190	0.0189	14.8

$$\Delta \bar{\eta} = \int_{-\infty}^{+\infty} \frac{\partial^2 \zeta(z)}{\partial z^2} w(z) dz / k^2 \int_{-\infty}^{+\infty} \zeta(z) w(z) dz \quad (83)$$

and $\Delta \bar{\eta}$ represents a scattering potential variation when the pressure is not constant in elevation. A wave field with variations in elevation implies some of the wave is radiating out of the image plane and is not collected by the receive aperture. The resulting loss is manifested as additional attenuation in the reconstructions.

To quantify the observed loss in the reconstructions, a version of the so-called extinction theorem or optical theorem^{b,20,21} is used. A general expression of this theorem for a monochromatic acoustic pressure wave is

$$P_{\text{loss}} = \frac{k^2}{2\rho_0\omega} \text{Im} \left[\int_V \eta(\mathbf{x}) p_i^*(\mathbf{x}) p(\mathbf{x}) d^3\mathbf{x} \right], \quad (84)$$

where P_{loss} is the total power lost and p_i and p are the incident and total pressures, respectively. For plane-wave illumination, the total lost power is proportional to the imaginary part of the through-transmission normalized far-field pressure, i.e.,

$$P_{\text{loss}} = \frac{1}{2\rho_0\omega} \text{Im}[A(\mathbf{u}_i, \mathbf{u}_i)], \quad (85)$$

where A is given in three dimensions by Eq. (48) with the angular arguments represented using the unit vector $\mathbf{u}_i$ that is in the direction of the incident wave. Equations (84) and (85) can also be normalized to express the power loss in terms of a cross section by including a scale factor that results in units of area but are written here without a normalizing factor so the units are those of power. For the object sizes, the loss characteristics, and the wavelengths considered in this report, the various lost powers are on the order of nanowatts.

The power lost due to scattering alone can be calculated using the surface integral

$$P_{\text{scat}} = -\frac{1}{2\rho_0\omega} \text{Im} \left[\int_{\partial V} p_s^*(\mathbf{x}) \nabla p_s(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d\sigma(\mathbf{x}) \right] \quad (86)$$

in which $p_s(\mathbf{x})$ is the scattered pressure field and $\mathbf{n}(\mathbf{x})$ is a unit vector normal to the surface of the integration. This expression for the scattered power simplifies when the incident illumination is a plane wave and the integration surface is in the far field. In two dimensions, the simplified expression is

$$P_{\text{scat}}^{2d} = \frac{1}{16\pi k c \rho_0} \int |A(\mathbf{u}, \mathbf{u}_i)|^2 d\varsigma(\mathbf{u}), \quad (87)$$

where $\mathbf{u}$ is a unit vector in the direction of the observation line ς . In three dimensions, the corresponding expression is

$$P_{\text{scat}}^{3d} = \frac{1}{32\pi^2 c \rho_0} \int |A(\mathbf{u}, \mathbf{u}_i)|^2 d\sigma(\mathbf{u}), \quad (88)$$

where $\mathbf{u}$ is a unit vector in the direction of the observation surface σ . The total power lost due to absorption can then be found from the total lost power and the total scattered power by using

$$P_{\text{abs}} = P_{\text{loss}} - P_{\text{scat}} \quad (89)$$

in which P_{abs} and P_{scat} are the power lost due to absorption and scattering, respectively.

To give confidence that the calculations of lost power by using the extinction theorem and by using the surface integrals are correctly performed in both two and three dimensions, the lost power was computed for a lossless cylinder and a lossless sphere. The calculations were performed at 2.5 MHz and used the values of the background parameters in Table I and the values of object properties in Table III except that the attenuation slope was 0 dB/(cm MHz). The results of the calculations are shown in Table V. For comparison, the corresponding calculations for a lossy cylinder and sphere with an attenuation slope of 0.462 dB/(cm MHz) are also shown in Table V. The very small values of absorbed power for lossless objects are attributed to numerical

TABLE VI. Power scattered into a Gaussian-weighted band of an enclosing spherical surface and into the remainder of the enclosing surface by a lossy sphere centered in illumination by an elevation-focused beam.

Offset (mm)	P_{loss} (nW)	P_{scat} (nW)	$P_{\text{scat}}^{\text{in}}$ (nW)	$P_{\text{scat}}^{\text{out}}$ (nW)	$P_{\text{scat}}^{\text{out}}/P_{\text{loss}}$ (%)	P_{loss}^{2D} (nW)
0	0.1574	0.1256	0.0824	0.0432	27.5	10.5035
3	0.1342	0.1052	0.0621	0.0431	32.1	8.9585
5	0.1152	0.0915	0.0425	0.0490	42.5	7.6785

effects. The values of absorbed power for the lossy objects are, however, appreciable. The ratios of absorbed power to total lost power that appear in the last column of these tables are noteworthy because these ratios are dimensionless quantifications of the rates at which energy is absorbed. These ratios are dominated by the intrinsic attenuation of the scattering object although they are also influenced by the geometry and size of the scattering object as shown by the variations in this column of table entries. Since these ratios vanish for lossless objects, they cannot be used to assess extrinsic attenuation.

Two calculations were carried out to quantify loss due to extrinsic attenuation. In the first calculation, the total power lost from an elevation-focused beam illuminating a lossless sphere was found by direct application of the two-dimensional form of the extinction theorem. In the second calculation, the power losses due to scattering and absorption were found by separating the surface integral expression in Eq. (86) into an aperture surrounding the equatorial plane of the sphere and the complementary regions above and below the aperture. The power scattered into the aperture was weighted by a Gaussian that assumed a maximum value of 1 in the center of the aperture and declined with elevation above and below the aperture. The result of this calculation is a value $P_{\text{scat}}^{\text{in}}$ that measures the scattered energy in the aperture of the already noted ring transducer system. The absorbed power was then computed by the difference expression in Eq. (89). These computations were performed for a 20-mm diameter sphere with the same parameters as used in the reconstruction except that a lossless sphere was considered to eliminate the influence of absorption. In the calculations given by Eq. (86), a focused transmit beam and a 75-mm spherical collection surface were used to obtain power losses for the geometry of the measurements reported in Ref. 14.

The results of the calculations described in the above paragraph are given in Table VI. The ratios in the last two columns of the table are dimensionless quantities that are easiest to interpret. The ratios in the second-to-last column represent fractions of energy lost to measurement according to the surface integral calculation. The ratios in the last column of the table are determined from the two-dimensional extinction theorem. These show substantially smaller losses. This discrepancy is attributed to receive focusing that is included in two-dimensional extinction theorem calculations but is absent from the surface integral formulation. Receive focusing causes a reduction of both P_{loss} and P_{scat} , but the portion of power loss attributed to absorption is less because P_{loss} is reduced more.

TABLE VII. Power loss calculations and power loss ratios for a lossy and lossless 20-mm diameter sphere with elevation-focused illumination and reception having various offsets from the center of the sphere.

	Offset (mm)	P_{loss}^{2D} (nW)	P_{scat}^{2D} (nW)	P_{abs}^{2D} (nW)	$P_{\text{abs}}^{2D}/P_{\text{loss}}^{2D}$ (%)
Lossless sphere	0	9.8270	9.6081	0.2188	2.2
	3	8.0430	6.3658	1.6772	20.9
	5	6.7677	3.5179	3.2498	48.0
Lossy sphere	0	10.5035	8.2117	2.2918	21.8
	3	8.9585	5.7281	3.2304	36.1
	5	7.6785	3.5630	4.1155	53.6

The forgoing discussion suggests that power loss ratios obtained from the two-dimensional extinction theorem are mostly determined by extrinsic attenuation that appears in the two-dimensional reconstructions. Table VII gives values of this ratio for cross sections of a lossless and lossy 20-mm diameter sphere at offsets of 0, 3, and 5 mm. The loss fractions for the cross sections with 3 and 5 mm offsets are substantially greater than those for the cross section with no offset and are comparable for lossless and lossy media. This shows that the extrinsic attenuation in the two-dimensional reconstructions of offset cross sections can be greater than the intrinsic attenuation. The power distributions are visualized as grayscale variations over the surface of the sphere of observation in Fig. 9 for a lossless sphere.

Power loss fractions can also be computed for the two-dimensional cylindrical models used during the reconstruction process to adjust model cylinders that are unable to mimic the power loss fractions for offset cross sections because the abrupt boundary of the cylinder produces significant in-plane scattering across a wider range of scattering angles than the cross section of an object with a boundary that varies in the width of the elevation focus. The effective boundary of such varying cross sections is distributed over a range of radii and results in less in-plane scattering. This suggests that scattering from an elevation-varying cross section might be better modeled by a cylinder with a graduated boundary as may be produced by adding a matching layer to the perimeter of the cylinder. Then, comparison of power loss ratios might provide a numerical guide for adjusting these boundary conditions.

Cross-sectional imaging techniques that use two-dimensional methods to reconstruct sections of three-dimensional objects are more efficient and require less comprehensive measurement systems than reconstructions that are fully three-dimensional. However, significant artifacts and distortion can appear in reconstructions formed in this manner if three-dimensional effects are not taken into account. The presented simulations of three-dimensional scattering measurements using elevation-focused transmit-receive beams have explored these corrupting influences and have illustrated ways that these influences can be compensated. The simulations show that two-dimensional inverse scattering methods can be used successfully to reconstruct cross sections of three-dimensional scattering objects, provided that the residuals used in the reconstruction are made

FIG. 9. Intensity of scattering by a 20-mm diameter lossless sphere illuminated with an elevation-focused transmit beam offset of 0, 3, or 5 mm below the center of the sphere. The solid lines show the effective width of the receive aperture. The radius of the surface on which the intensity is shown is 75 mm. The dotted line denotes the center of the receive aperture. The dashed line is the intersection of the imaging plane with the sphere. The grayscale is logarithmic and spans a range of 60 dB.

sufficiently small by use of models that compensate for energy loss in the direction that is orthogonal to the plane of the cross section.

VI. CONCLUSION

A detailed analysis of using a two-dimensional method of image reconstruction with transmit-receive elevation-focused beams to image a cross section of a three-dimensional object has been performed. In the analysis, scattering in three dimensions is written as the sum of two-dimensional scattering that results from each plane wave or spatial-frequency in the elevation dimension. An angular spectrum form for the illumination pattern and a corresponding form for reception sensitivity lead to expressions for measured scattered pressure when the illumination pattern and receive sensitivity are focused in elevation. These expressions are used to calculate scattering by a cylinder and by a sphere with focused illumination and reception. The results show that two-dimensional reconstructions of scattering objects using elevation-focused beams can give accurate results when the scattering object is nearly the same throughout the width of the elevation focus. In situations where the three-dimensional scattering is not entirely captured by the receive aperture, two-dimensional reconstructions of sound speed are relatively unaffected, but corresponding reconstructions of attenuation slope can have elevated values at the object boundary. These artifacts can be compensated by endowing a background model with additional loss that accounts for unmeasured scattering.

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APPENDIX: FOURIER TRANSFORM IN ELEVATION OF THE THREE-DIMENSIONAL OUT-GOING FREE-SPACE GREEN'S FUNCTION

The Fourier transform in elevation of the three-dimensional out-going free-space Green's function is given by

$$G_0^{(3D)}(x, \nu, f) = \int_{-\infty}^{+\infty} G_0^{(3D)}(x, z, f) e^{-j2\pi\nu z} dz$$

$$= \int_{-\infty}^{+\infty} \frac{e^{j(2\pi f/c)\sqrt{r^2+z^2}}}{4\pi\sqrt{r^2+z^2}} e^{-j2\pi\nu z} dz. \quad (A1)$$

Since $G_0^{(3D)}(\mathbf{x}, z, f) = G_0^{(3D)}(\mathbf{x}, -z, f)$, the integration can be reduced to a half line and written as

$$G_0^{(3D)}(x, \nu, f) = \frac{1}{2\pi} \int_0^{\infty} \frac{e^{j(2\pi f/c)\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} \cos(2\pi\nu z) dz. \quad (A2)$$

This integral can be evaluated using Formulas 3.876.1 and 3.876.2 in Ref. 22 with $p=2\pi f/c$, $a=r$, and $b=2\pi\nu$. The result is

$$G_0^{(3D)}(x, \nu, f) = \begin{cases} \frac{j}{4} H_0(2\pi\sqrt{(f/c)^2 + \nu^2} r), & f/c > \nu \\ \frac{1}{2\pi} K_0(2\pi\sqrt{\nu^2 - (f/c)^2} r), & f/c < \nu. \end{cases} \quad (A3)$$

Since $K_0(p) = (\pi j/2) H_0(jp)$, the H_0 term on the right side can be used for all values of ν . This allows the Fourier transform in elevation of the three-dimensional free-space Green's function $G_0^{(3D)}(\mathbf{x}, \nu, f)$ to be expressed in terms of the two-dimensional free-space Green's function $G_0^{(3D)}(\mathbf{x}, f_e)$ as

$$G_0^{(3D)}(x, \nu, f) = G_0^{(2D)}(x, f_e) \quad (\text{A4})$$

in which f_e is the effective temporal frequency given by $\sqrt{f^2 - c^2 \nu^2}$.

^aIn Eq. (4) and in following analysis, the notation uses the quantum mechanics convention in which a symbol represents a conceptual object and the argument indicates the domain in which an object is evaluated. Thus, $p_n(\mathbf{x}, z, f)$ is the scattered pressure in the space domain as a function of the spatial-position vector $(\mathbf{x}, z)$ and $p_n(\mathbf{x}, \nu, f)$ is the one-dimensional Fourier transform as a function of the spatial-frequency ν .

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