Published in final edited form as:

Automatica (Oxf). 2008 December 1; 44(12): 3061–3069. doi:10.1016/j.automatica.2008.05.024.

Multivariable Harmonic Balance for Central Pattern Generators★

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Abstract

The central pattern generator (CPG) is a nonlinear oscillator formed by a group of neurons, providing a fundamental control mechanism underlying rhythmic movements in animal locomotion. We consider a class of CPGs modeled by a set of interconnected identical neurons. Based on the idea of multivariable harmonic balance, we show how the oscillation profile is related to the connectivity matrix that specifies the architecture and strengths of the interconnections. Specifically, the frequency, amplitudes, and phases are essentially encoded in terms of a pair of eigenvalue and eigenvector. This basic principle is used to estimate the oscillation profile of a given CPG model. Moreover, a systematic method is proposed for designing a CPG-based nonlinear oscillator that achieves a prescribed oscillation profile.

Keywords

Oscillators; Pattern generation; Neural networks; Harmonic balance; Nonlinear systems

1 Introduction

Rhythmic movements in animal locomotion are controlled by a neuronal circuit called the central pattern generator (CPG). A CPG is a group of neurons interconnected in a specific manner so that their membrane potentials autonomously oscillate with particular phase relations, generating a “pattern” to be used for muscle activation. To uncover the biological control mechanism, CPGs have been extensively studied in the neuroscience literature (Cohen et al., 1988; Orlovsky et al., 1999). Individual neurons participating rhythmic pattern generation have been identified and their connections determined from physiological experiments. The knowledge on the mechanism of CPGs thus generated has been utilized in engineering applications. For instance, the basic architectures of CPGs are exploited in robotics literature to design feedback controllers that achieve stable limit cycles with desired phase coordination properties (Fukuoka et al., 2003; Creps et al., 2005). A CPG with sensory feedback may have a potential to provide a new paradigm for nonlinear control theory and applications where the control objective is to achieve robust and adaptable oscillations.

Various theoretical analysis methods for CPGs, or nonlinear oscillators in general, have been developed in the literature, including the Poincaré-Bendixson theorem (Khalil, 1996), Hopf bifurcation theorem (Marsden and McCracken, 1976), perturbation theory and averaging

★This paper was not presented at any IFAC meeting.

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(Guckenheimer and Holmes, 1983), integral quadratic constraints (Jonsson et al., 2002), and the Malkin theorem for phase coupled oscillators (Izhikevich, 2006). Most of these results are effective for analysis of dynamical systems, but not directly useful for the problem of our interest — design of (artificial) CPGs to achieve oscillations with a desired profile (frequency, amplitudes, and phases). A rigorous theoretical result has been obtained by Lohmiller and Slotine (1998) using contraction analysis for global convergence, but for a slightly different problem — design of coupled oscillators to achieve prescribed phases (Pham and Slotine, 2007).

The objective of this paper is to develop a systematic method for the analysis and synthesis of CPGs. We consider the class of CPGs modeled by a group of *identical* neuron models. The analysis problem addressed here is to determine whether there is a stable oscillatory trajectory, and if so, predict the oscillation profile. The synthesis problem is to determine appropriate neuronal connections so that the resulting circuit achieves a stable oscillation with a prescribed profile. Our main result shows that the oscillation frequency and overall amplitude are encoded in the “maximal” eigenvalue, while the relative amplitudes and phase information are embedded in the corresponding eigenvector. Thus, both analysis and synthesis of CPGs are essentially reduced to simple eigenvalue problems.

The approach we employed to arrive at these results is the multivariable harmonic balance. The harmonic balance is a classical technique that detects the presence and estimates the profile of oscillations, provided that the shape of the oscillatory signal is close to a sinusoid (Mees, 1981; Khalil, 1996). The method has been applied in various engineering fields, including control designs (Berns et al., 1998; Tesi et al., 1996), but these results focus on the amplitudes and frequency of oscillations. On the other hand, our focus here is the pattern generation and hence the phase is the most important property of oscillation. To our knowledge, the potential of the harmonic balance idea has not been fully explored in the context of CPG analysis and synthesis. The method is approximate in nature but numerous examples demonstrate that our method works reasonably well for CPGs.

We use the following notation. For a matrix $M \in \mathbb{C}^{n \times n}$, let $\rho(M) \in \mathbb{C}$ be the eigenvalue of M with the largest imaginary part among those eigenvalues with the largest real part. We call $\rho(M)$ the maximal eigenvalue. Let $M^\dagger$ be the Moore-Penrose inverse of M . Denote by $\mathbf{U}$ the set of vectors $u \in \mathbb{C}^n$ such that $|u_i| = 1$ for all $i = 1, \dots, n$.

2 The CPG model and problem formulation

Let the electrical activity of a neuron be modeled by the dynamical mapping from input u to output v :

$$v = \psi(q), \quad q = f(s)u$$

where ψ is a static nonlinear function and $f(s)$ is a transfer function. The output v is either the cell membrane potential of the neuron or the spike frequency of the membrane potential. The input u is taken as the weighted sum of the output of the presynaptic neurons. The nonlinear function $\psi(q)$ captures the threshold and saturation properties. The transfer function $f(s)$ represents the linear time-invariant part of the neuronal dynamics. Typical choices for $f(s)$ include

$$f_a(s) = \frac{\omega_o}{s + \omega_o}, \quad f_b(s) = \frac{(\omega_1 + \omega_2)s}{(s + \omega_1)(s + \omega_2)}, \quad (1)$$

where $\omega_i > 0$ with $i = 0, 1, 2$. The former $f_a(s)$ is a low-pass filter representing the cell membrane time constant $1/\omega_0$, and is one of the simplest and standard choices in many engineering applications of neural network (Hunt et al., 1992). The latter $f_b(s)$ is a band-pass filter with pass band $\omega_1 < \omega < \omega_2$, capturing the adaptation property (impulse adaptation and/or synaptic fatigue) in addition to the time lag. The adaptation is known to be important in the pattern generation of certain CPGs (Matsuoka, 1985; Iwasaki and Zheng, 2006; Futakata and Iwasaki, 2008).

A CPG is a group of neurons interconnected in a specific manner to generate a desired phase pattern. Let us consider a CPG consisting of n neurons where the dynamical behavior of the i^{th} neuron is described by

$$v_i = \psi_i(q_i), \quad q_i = f_i(s)u_i, \quad u_i = \sum_{j=1}^n \mu_{ij}(s)v_j$$

where $\mu_{ij}(s)$ is the transfer function of the synaptic connection from the j^{th} neuron. If $\mu_{ij}(0)$ is negative/positive, then the connection is said to be inhibitory/excitatory. In the vector form, the CPG can be written

$$v = \Psi(q), \quad q = M(s)\Psi(q) \quad (2)$$

$$\begin{aligned} \Psi &:= \text{diag}(\psi_i), \quad F(s) := \text{diag}(f_i(s)), \\ M(s) &:= F(s)M(s), \end{aligned} \quad (3)$$

where $M(s)$ is the transfer matrix whose (i, j) entry is $\mu_{ij}(s)$. We consider the following.

Assumption 1

The CPG described by (2) and (3) consists of identical neurons with static interconnections:

$$\begin{aligned} M(s) &= M, \quad \Psi = \psi I, \\ F(s) &= f(s)I, \quad f(s) = f_a(s) \text{ or } f_b(s), \end{aligned} \quad (4)$$

where I is the $n \times n$ identity matrix, the second equation means that the i^{th} entry of $\Psi(q)$ is $\psi(q_i)$, and $f_a(s)$ and $f_b(s)$ are defined in (1) with positive real parameters ω_0 , ω_1 , and ω_2 . Moreover, the nonlinear function $\psi(x)$ is continuous, monotonically increasing, bounded, odd, strictly concave on $x > 0$, and has a unity slope at the origin.

We will propose a method for analyzing the general CPG model in the next section, and then apply the method to the special case under Assumption 1 in the sections that follow. The hyperbolic tangent function $\psi(x) := \tanh(x)$ satisfies the properties indicated in Assumption 1, and will be used for all numerical examples presented later.

In this paper, we will address the following:

Analysis Problem: Determine whether the CPG in (2) has an oscillatory trajectory, and if so, estimate the oscillation profile (frequency, amplitudes, phase) without actually simulating the differential equations.

Synthesis Problem: Given a desired oscillation profile, find the connectivity matrix M in (2) so that the resulting CPG achieves the profile.

Modeling Problem: Given an observed oscillation profile, find the connectivity matrix M in (2) so that the resulting CPG exhibits the profile.

The behavior of a CPG can be analyzed by simulations, but a solution to the analysis problem can be more advantageous in certain aspects. First, it could save time for analysis, especially when the CPG consists of a large number of neurons and its simulation is time consuming. More importantly, theoretical analysis would provide more insights into the pattern generation mechanism; for instance, it would uncover how the neuronal interconnection structure relates to the resulting phase relations. Such result would be useful in neuroscience for understanding the biological control mechanism. The synthesis problem is of importance for engineering applications where artificial CPGs are used to control robotic locomotion systems, whereas the modeling problem is important in neuroscience to understand the neuronal mechanism for generating a particular oscillation pattern. The synthesis and modeling problems are mathematically equivalent and hence we discuss the former only.

3 General framework

In this section, we consider the class of systems described by a feedback connection of general transfer function $\mathcal{M}(s)$ and nonlinear function Ψ as in (2). Throughout this section, $\mathcal{M}(s)$ is an arbitrary stable transfer function matrix of dimension $n \times n$ unless otherwise noted, and Ψ is a diagonal nonlinearity with ψ_i on the i^{th} diagonal, where ψ_i satisfies the properties imposed on ψ in Assumption 1. We will first discuss a method for estimating the profile of oscillation, assuming existence of a periodic orbit. We will then provide a condition for the existence of an oscillatory trajectory.

3.1 Multivariable harmonic balance

We shall develop a multivariable harmonic balance (MHB) equation to characterize the profile of oscillation for (2). The development is based on the approximations of the periodic solution of (2) by a sinusoidal signal, and of the nonlinearity Ψ by its describing function. Suppose (2) has a periodic solution. Then, by the Fourier series expansion, we have

$$q(t) = \sum_{k=0}^{\infty} a_k \sin(\omega k t) + b_k \cos(\omega k t)$$

for some real vectors a_k and b_k and frequency ω . Suppose that ω is sufficiently high in comparison with the pass band of $\mathcal{M}(s)$ so that the second and higher order harmonics can safely be neglected. We further assume that the variable q has been defined so that the average value of oscillation is zero. This assumption holds if $f_i(s) = f_b(s)$ since the static gain of $f_b(s)$ is zero. The signal $q(t)$ can then be written as

$$q_i(t) \cong \alpha_i \sin(\omega t + \phi_i), \quad i=1, \dots, n, \quad (5)$$

where α_i and ϕ_i are the amplitude and phase, respectively. Let ψ_i be approximated by its describing function κ_i :

$$\psi_i(q_i) \cong \kappa_i(\alpha_i)q_i, \quad \kappa_i(x) := \frac{2}{\pi x} \int_0^\pi \psi_i(x \sin \theta) \sin \theta d\theta.$$

The describing function $\kappa_i(x)$ is the gain of ψ_i when the input is a sinusoid of amplitude x and the output is approximated by its first harmonic (Khalil, 1996). For $\kappa(x)$ satisfying Assumption 1, its describing function $\kappa(x)$ is a monotonically decreasing function with $\kappa(0) = 1$ and $\kappa(\infty) = 0$. With these approximations, the dynamical equation for the CPG reduces to

$$\begin{aligned} (\mathcal{M}(j\omega)\mathcal{K}(\alpha) - I)q &= 0, \\ q_i &:= \alpha_i e^{j\phi_i}, \quad \mathcal{K}(\alpha) := \text{diag}(\kappa_i(\alpha_i)). \end{aligned} \quad (6)$$

This is the MHB equation for the CPG. It is expected that the frequency, amplitude, and phase satisfy these conditions approximately. In particular, we can obtain estimates of the oscillation profile by solving the MHB equation for (ω, α, ϕ) . The solutions are not unique in general, and we need to identify the most likely estimate for stable oscillations. The stability property of the predicted oscillation with profile (ω, α, ϕ) can be examined through a standard argument (Glad and Ljung, 2000) as described below.

Consider the linear system obtained by replacing the nonlinear function Ψ in the CPG with a constant gain K :

$$v = Kq, \quad q = \mathcal{M}(s)Kq. \quad (7)$$

Let us denote this system by L_K . From the describing function perspective, L_K with $K = \mathcal{K}(\alpha)$ may be considered to be an approximate linear model that contains some information on the behavior of the CPG when its trajectory is nearly sinusoidal with amplitudes α . The oscillation (ω, α, ϕ) is expected stable if L_K with $K = \mathcal{K}(\alpha)$ is stable/unstable when α is perturbed slightly in the larger/smaller direction so that the perturbed orbit tends to return to the original. In summary, the key ideas of the MHB analysis are the following:

- For the CPG (2), the profile (frequency, amplitudes, and phases) of oscillations, if any, can be estimated by solving (6) for ω and q where ω is the frequency and the phasor q contains the information on the amplitudes and phases.
- The predicted oscillation (ω, q) is expected stable if the corresponding linear system L_K in (7) with $K := \mathcal{K}(|q|)$ is marginally stable with a pair of poles $s = \pm j\omega$ on the imaginary axis and the rest in the open left half plane.

Based on these ideas, the profile of stable oscillation, if any, may be estimated by solving (6) for $\omega \in \mathbb{R}$ and $q \in \mathbb{C}^n$ such that the linear system (7) with $K = \mathcal{K}(|q|)$ is marginally stable. Once ω and q are found, an oscillation is expected as in (5) where $\alpha_i, \phi_i \in \mathbb{R}$ are found from $q_i = \alpha_i e^{j\phi_i}$.

3.2 Linear stability analysis

The MHB method described in the previous section is based on the linear system L_K in (7) associated with a particular oscillation. Stability of the oscillation is related to the marginal stability of the linear system. In this section, we study the stability property of L_K under Assumption 1.

The transfer functions $f_a(s)$ and $f_b(s)$ have some special properties as summarized below. The result can be verified in a straightforward manner and hence the proof is omitted.

Lemma 1—Let $f(s) = f_a(s)$ or $f_b(s)$ in (1) with $\omega_0, \omega_1, \omega_2 > 0$. Then we have the following, where in (iii) and (iv) is defined in (ii).

- i. The transformation $\lambda = 1/f(s)$ from s to λ maps the open left half plane to the open half plane to the left of $\Re(\lambda) = 1$, i.e., $\Re(\lambda) < 1$ if and only if $\Re(s) < 0$.
- ii. For each $\omega \in \mathbb{R}$, we have $1/f(j\omega) = 1 + j\varpi$ for some $\varpi \in \mathbb{R}$. In the domain $\omega > 0$, ϖ is a monotonically increasing, continuous function of ω . Moreover, the function is invertible on $\varpi \geq 0$.
- iii. The transfer function $f_a(s)$ is a low pass filter with pass band $\omega \leq \omega_0$. When $f(s) = f_a(s)$, we have $\omega > \omega_0$ if and only if $\varpi > 1$.
- iv. The transfer function $f_b(s)$ is a band pass filter with pass band $\omega_1 \leq \omega \leq \omega_2$ and the center frequency $\omega_c := \sqrt{\omega_1 \omega_2}$. When $f(s) = f_b(s)$, we have $\omega \geq \omega_c$ if and only if $\varpi \geq 0$.

Property (i) is useful to prove the following stability result.

Lemma 2—Consider the linear system L_K in (7) where $\mathcal{M}(s)$ is given by (3). Suppose Assumption 1 holds. Then the linear system L_K is stable if and only if every eigenvalue λ of MK satisfies $\Re(\lambda) < 1$.

Proof: The system is stable if and only if all the roots s of the characteristic equation

$$\det(MK - \lambda I) = 0, \quad \lambda := 1/f(s), \quad (8)$$

satisfy $\Re(s) < 0$. From Lemma 1, all solutions s to (8) satisfy $\Re(s) < 0$ if and only if all the eigenvalues λ of MK satisfy $\Re(\lambda) < 1$.

3.3 Condition for existence of oscillation

With a solution to the MHB equation (6), the system (2) is *expected* to have an oscillation. However, we cannot draw a definite conclusion due to the approximate nature of the MHB method. In this section, we shall provide a condition under which existence of oscillation is definitely guaranteed. We will not restrict our attention to periodic trajectories, but instead, we will deal with the more general notion of Y-oscillations, named after Yakubovich (1973). An autonomous system $\dot{x} = f(x)$ with $x(t) \in \mathbb{R}^m$ is said to be *Y-oscillatory* (Pogromsky et al., 1999) if each solution $x(t)$ is bounded and there exists a state variable x_i such that

$$\lim_{t \rightarrow +\infty} x_i(t) < \overline{\lim_{t \rightarrow +\infty} x_i(t)}$$

for almost all initial states $x(0)$. This type of notions have been used for oscillation analyses by Pogromsky and Nijmeijer (2001); Efimov and Fradkov (2005). The following result provides a condition for system (2) to be Y-oscillatory.

Proposition 1—Consider the CPG described by (2) and (3), where Assumption 1 holds. Suppose that the origin is the unique equilibrium point of the system and that M has at least

one eigenvalue with its real part greater than 1 and has no eigenvalue with its real part equal to 1. Then, the CPG is Y-oscillatory.

Proof: We prove the result as a special case of Theorem 1 in (Pogromsky et al., 1999). First note that the solutions of (2) are uniformly ultimately bounded since the CPG is a feedback connection of the stable transfer function $\mathcal{M}(s)$ and the bounded nonlinearity Ψ . Second, we claim that the origin is the unique hyperbolic, unstable equilibrium point. Note that the linearization of the CPG around the origin is given by L_K with $K = I$. Hence, from statement (i) of Lemma 2, we lead to the above claim under the eigenvalue conditions imposed on M . Finally, the uniform ultimate boundedness of trajectories and hyperbolicity/instability of every equilibrium imply that the system is Y-oscillatory.

The key ideas behind the oscillatory behavior are the boundedness of every trajectory that prevents divergence and the instability of every equilibrium that prevents convergence to a constant state. Strictly speaking, the trajectory is not guaranteed to converge to a *periodic* orbit, but to an oscillatory trajectory. However, our numerical experience suggests that the difference carries little significance for practical purposes, as illustrated by a number of examples later.

We remark on the uniqueness of the equilibrium point. Let (A, B, C) be a state space realization of $\mathcal{M}(s)$. The state equation for (2) and its equilibrium point x_o are given by

$$\dot{x} = Ax + B\Psi(Cx), \quad x_o = -A^{-1}B\Psi(Cx_o).$$

Multiplying C from the left and defining $q_o := Cx_o$,

$$q_o = \mathcal{M}(0)\Psi(q_o). \quad (9)$$

Since the nonlinearities pass through the origin, i.e. $\psi_i(0) = 0$, we have $q_o = 0$ as a solution to this equation. Hence, the origin is an equilibrium point. If $\mathcal{M}(s)$ has the structure as in (3) with $f_i(s)$ being of the form $f_b(s)$ in (1), then we have $\mathcal{M}(0) = 0$ due to $f_b(0) = 0$, implying that $q_o = 0$ is the only solution. Thus, the neuronal adaptation property (zero of $f_b(s)$ at the origin) guarantees uniqueness of the equilibrium point at the origin. On the other hand, if $f(s) = f_a(s)$, we have $f(0) = 1$ and hence $\mathcal{M}(0) = M$. In this case, it is difficult to characterize the uniqueness of the equilibrium point at the origin in general. For a special case where M is a circulant matrix, Chen and Iwasaki (2008) provides a sufficient condition for the uniqueness.

4 CPG analysis

In this section, we apply the general idea of the MHB method to the class of CPGs described by (2) and (3). Throughout this section, we impose Assumption 1.

4.1 Uniform amplitude case

Let us first consider a simple case to gain some insight. We assume that the CPG has a certain symmetry so that the oscillation amplitudes of q_i are equal. This assumption greatly simplifies our analysis and allows us to see how the connectivity matrix M determines the oscillation profile. In particular, with $\alpha_i = \alpha$ for all i , (6) reduces to

$$(M - \lambda I)q = 0, \quad q_i := e^{j\phi_i}, \quad \lambda := 1/(\kappa(\alpha)f(j\omega)) \quad (10)$$

where κ is the describing function of ψ . Clearly, λ and q are an eigenvalue and the corresponding eigenvector of M . Thus, an eigenvalue of M sets the amplitude α and frequency ω while an eigenvector sets the phases ϕ_i .

The uniform amplitude assumption is consistent with the MHB equation if M has the structure such that each eigenvector q belongs to $\mathbb{U}$, i.e., $|q_i| = 1$ for all i . Conversely, if M has no eigenvectors in $\mathbb{U}$, the harmonic balance analysis predicts that the amplitudes of oscillation, if any, will not be uniform. The classes of M whose eigenvectors belong to $\mathbb{U}$ include the circulant and skew-circulant matrices.

We now consider the problem of estimating the oscillation profile (frequency, amplitude, phase) for a given matrix M . This problem is addressed by finding a solution to the harmonic balance equation in (10). Let (λ, q) be an eigenvalue/eigenvector pair of M and suppose $q \in \mathbb{U}$. First, the phase $\phi \in \mathbb{R}^n$ can be estimated from $q_i = e^{j\phi_i}$ where q_i and ϕ_i are the i^{th} entries of q and ϕ , respectively. Next, the frequency and amplitude are estimated by solving

$$\lambda \kappa(\alpha) = 1 / f(j\omega) = 1 + j\varpi, \quad (11)$$

$$\varpi := \begin{cases} \omega / \omega_o & \text{if } f(s) = f_a(s) \\ \frac{\omega - \omega_1 \omega_2 / \omega}{\omega_1 + \omega_2} & \text{if } f(s) = f_b(s) \end{cases} \quad (12)$$

for positive parameters ω and α . In particular, they are calculated from

$$\Re(\lambda) \kappa(\alpha) = 1, \quad \tan(\angle \lambda) = \varpi. \quad (13)$$

The describing function $\kappa(\alpha)$ for $\psi(x)$ satisfying Assumption 1 is monotonically decreasing from one to zero. Hence, a (unique) nonzero solution α to the first equation in (13) exists if and only if $\Re(\lambda) > 1$, which is exactly the condition for oscillation stated in Proposition 1. When $f(s) = f_a(s)$, the second equation is trivially solved for ω . When $f(s) = f_b(s)$, there are two values of ω satisfying the second equation in general and one of them is always positive.

By the above procedure, we may obtain multiple estimates for the oscillation profile based on different eigenvalues of M . The remaining task is to determine which oscillations are stable. As described earlier, an oscillation is expected stable if the associated linear system L_K with $K := \kappa(\alpha)I$ is marginally stable. According to Lemma 2, L_K is marginally stable when $\Re[\lambda_k(\alpha)] \leq 1$ holds for all eigenvalues λ_k of M . Since $\Re[\lambda \kappa(\alpha)] = 1$, a stable oscillation is characterized by an eigenvalue of M with the largest real part.

Finally, the eigenvalues of M with the largest real part are not unique in general. We argue that the best estimate of oscillation results if we use the *maximal eigenvalue* $\rho(M)$, which is defined to be the eigenvalue with the largest imaginary part among those with the largest real part. If ω turns out to be in or below the pass band of $f(s)$, then the higher order harmonics of $q(t)$ may not be well attenuated and the approximation of $q(t)$ by its first harmonic term would be erroneous. In this sense, it would be the safest to use the eigenvalue that results in the highest frequency. Since ϖ is a monotonically increasing function of $\omega > 0$ for both $f_a(s)$ and $f_b(s)$, the eigenvalue that provides the most reasonable estimate is the one with the largest imaginary part. This conclusion will be backed up by examples later.

In summary, we have the following.

Proposition 2—Given a matrix $M \in \mathbb{R}^{n \times n}$, transfer function $f(s)$, and nonlinear function ϕ , consider the CPG described by (2) and (3) under Assumption 1. Let $\lambda := \rho(M)$ and q be the corresponding eigenvector. Suppose $\Re(\lambda) > 1$ and $q \in \mathbb{U}$. Then the existence of an oscillatory trajectory $q(t)$ is expected at frequency ω , with amplitude α and phase ϕ :

$$q_i(t) \cong \alpha \sin(\omega t + \phi_i) \quad (14)$$

where (ω, α, ϕ) are specified by $q_i = e^{j\phi_i}$, $\omega > 0$, and (13), where ϖ is defined in (12).

Example 1—Consider the CPG described by (2)–(4) with

$$M = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ -1 & 0 & 2 & 0 & 1 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix}, \quad \begin{aligned} f(s) &= f_a(s), \\ \omega_o &= 1. \end{aligned}$$

This matrix is of the form $M = S + aI$ with $S + S^T = 0$, and hence all its eigenvalues have the same real part $a = 2$; in fact, they are 2 , $2 \pm 1.90j$, and $2 \pm 1.18j$. Due to the skew-circulant structure, all the eigenvectors belong to $\mathbb{U}$. Figure 1 shows a diagram of the neuronal connections. Each circle represents a neuron, and the number inside a circle indicates the phase angle calculated by simulation. For each line, the “ $\perp$ ” and “ $\bullet$ ” endings indicate excitatory and inhibitory connections, respectively.

An estimate for the oscillation profile is obtained from the maximal eigenvalue $\rho(M) = 2 + 1.9j$, and is compared with the actual values observed in simulation where the phases are measured with respect to the time instant at which $q_i(t)$ becomes zero. The result is summarized in Table 1. The estimates are reasonably close to the actual values. In general, the oscillations are close to sinusoids if $\Re[\rho(M)]$ is close to one, resulting in better estimates of the oscillation profile by the MHB method. Figure 2 shows the simulated time course of $q(t)$ in the steady state, where each curve corresponds to $q_i(t)$, with the peaks occurring first for q_5 , then for q_4 , and so on. Clearly, the amplitudes are uniform and the phases are equally spaced as predicted.

4.2 General amplitude case

We now present a method for estimating the oscillation profile without assuming that amplitudes are uniform. Under Assumption 1, the MHB equation (6) becomes

$$\begin{aligned} (M\mathcal{K}(\alpha) - \lambda I)q &= 0, \quad \lambda := 1/f(j\omega), \\ q_i &:= \alpha_i e^{j\phi_i}, \quad \mathcal{K}(\alpha) := \text{diag}(\kappa_i(\alpha_i)). \end{aligned} \quad (15)$$

which indicates that $1/f(j\omega)$ is an eigenvalue of $M\mathcal{K}(\alpha)$ and q is the corresponding eigenvector. The associated linear system L_K with $K = \mathcal{K}(\alpha)$ must be marginally stable. Thus, in view of Lemma 1, the MHB equation should be solved with the additional constraint that every eigenvalue of $M\mathcal{K}(\alpha)$ has real part smaller than or equal to one. Satisfaction of (15) ensures that $M\mathcal{K}(\alpha)$ has a pair of eigenvalues with unity real part since $\Re(1/f(j\omega)) = 1$. Now, if there are more than one pair of such eigenvalues, then it may be reasonable to use the one with the largest imaginary part to estimate the oscillation frequency. This is because such eigenvalue

gives the largest frequency ω (see statement (ii) of Lemma 1) and hence its higher order harmonics are most likely to be attenuated by the filter $f(s)$. In summary, we have the following.

Proposition 3—Consider the CPG in (2) with (3) under Assumption 1. Suppose there exists $q \in \mathbb{C}^n$ satisfying

$$\begin{aligned} (MK - \lambda I)q &= 0, \quad \Re(\lambda) = 1, \\ \lambda &:= \rho(MK), \quad K := \mathcal{K}(|q|). \end{aligned} \quad (16)$$

Then the MHB analysis predicts that an oscillatory trajectory $q(t)$ exists at frequency $\omega \in \mathbb{R}$, with amplitude $\alpha \in \mathbb{R}^n$ and phase $\phi \in \mathbb{R}^n$:

$$q_i(t) \cong \alpha_i \sin(\omega t + \phi_i) \quad (17)$$

where the triple (ω, α, ϕ) is determined from

$$1/f(j\omega) = \lambda, \quad q_i = \alpha_i e^{j\phi_i}, \quad \omega > 0. \quad (18)$$

The analysis problem is to find $q \in \mathbb{C}^n$ satisfying (16), from which (ω, α, ϕ) can be calculated. The search for q is a version of eigenvalue problems. A complication is that the matrix MK depends on the eigenvector q . Additional difficulty is that the associated linear system L_K with $K = \mathcal{K}(|q|)$ must be marginally stable. At the moment, it seems difficult to develop a globally convergent yet efficient algorithm for solving the problem. Here, we propose a heuristic method.

0. Let $\varepsilon > 0$ be a small number and initialize $k = 0$ and $v_0 = [1 \dots 1]^T$.
1. Let $K_k := \mathcal{K}(|v_k|)$ and find the maximal eigenvalue λ_k and eigenvector x_k of MK_k :
 $(MK_k - \lambda_k I)x_k = 0, \lambda_k = \rho(MK_k), \|x_k\| = 1.$
2. Update $v_{k+1} = e^{\sigma_k} y_k, \sigma_k := \Re(\lambda_k) - 1, y_k := x_k x_k^* v_k.$
3. If $\|v_{k+1} - v_k\| \leq \varepsilon$, then stop. $q := v_{k+1}$ is the solution to (16). Otherwise go to Step 1 and iterate.

The idea behind the algorithm is as follows. We start with v_0 , an initial guess for q . In Step 1, we calculate the maximal eigenvalue/eigenvector pair (λ_k, x_k) . If x_k is parallel to v_k and $\Re(\lambda_k) = 1$, then we have a solution $q := v_k$. These conditions are equivalent to $\sigma_k = 0$ and $y_k = v_k$ where y_k is the orthogonal projection of v_k onto the null space of $MK_k - \lambda_k I$, spanned by x_k . If σ_k is positive/negative, the linear system L_{K_k} is unstable/stable, and the oscillation represented by phasor v_k is expected to grow/reduce in magnitude. Thus, in Step 2, the phasor is updated to be the projection y_k , modified with the amplification factor e^{σ_k} .

Finally, some relaxation factors η_1 and η_2 can be introduced to modify Step 2 as follows:

$$u_k = e^{\eta_1 \sigma_k} y_k, \quad v_{k+1} = v_k + \eta_2 (u_k - v_k).$$

The positive parameters η_i are used as tuning knobs to make a trade-off between the speed of convergence and likelihood of success. In general, the smaller the parameters, the slower the convergence and more likely to succeed. The algorithm presented above uses $\eta_1 = \eta_2 = 1$.

Example 2—To assess reliability of Algorithm, connectivity matrices M were randomly generated so that solution q to (16) was guaranteed to exist.¹ For each fixed number n of neurons in the CPG, 100 samples of such M are generated. Algorithm was then applied to each sampled case and the number of convergent cases was counted to find the success rate. In this numerical experiment, $\varepsilon = 10^{-4}$ (and $\eta_1 = \eta_2 = 1$) was used in Algorithm, and the “success” is defined by the event that Algorithm stops within 1000 iterations. The result is summarized in Table 2. The success rate is fairly high (> 90%) regardless of the number of neurons n . The computational time is the average of the 100 samples, and is only several seconds even for $n = 100$ on a Pentium IV 3.6 GHz processor with a 3 GB memory.

Example 3—Consider the CPG for swimming motion of leech that swims by undulating its body like snakes. The CPG is a chain of segmental oscillators, each of which can be modeled as a recurrent cyclic inhibition (RCI) oscillator (Zheng et al., 2007; Chen et al., 2008). In Fig. 3, the first two segments of the CPG chain are shown. The three neurons on the left form an RCI oscillator, the other three form another, and the two RCI oscillators are coupled to each other. The whole CPG is a chain of 17 RCI oscillators, any two of which are coupled in the same way as shown in Fig. 3 if they are within the distance of 5 segments (the distance between two adjacent oscillators is 1 segment).

The CPG is modeled by (2)–(4) with $f(s) = f_a(s)$, and

$$M = \mu \begin{bmatrix} M_o & A & \cdots & A & 0 \\ D & M_o & A & \ddots & \ddots \\ \vdots & D & M_o & A & \ddots & A \\ D & \ddots & D & M_o & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & A \\ 0 & & D & \cdots & D & M_o \end{bmatrix},$$

$$M_o := - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A := - \begin{bmatrix} 0 & \sigma & 0 \\ 0 & 0 & \sigma \\ 0 & 0 & 0 \end{bmatrix}, \quad D := \begin{bmatrix} 2\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where M_o is the connectivity matrix for neurons within each RCI oscillator, and A and D are the coupling matrix between RCI oscillators. The matrix M repeats 17 times on the diagonal, and matrices A and D repeat 5 times in the first row and column, respectively. Let $q_{ij}(t)$ be the variable for the j^{th} neuron in the i^{th} RCI oscillator, where $i = 1, \dots, 17$, and $j = 1, 2, 3$. Note that $q(t) \in \mathbb{R}^{51}$ and the k^{th} entry of q is q_{ij} with $k = 3(i-1) + j$. The nominal parameter values are $\mu = 6$, $\sigma = 0.1$, and $\omega_o = 5$ rad/s.

The eigenvalues of M are plotted in Fig. 4. Since the RCI oscillators are weakly coupled ($\sigma \ll 1$), the eigenvalues are scattered around those of M_o (-6 and $3(1 \pm \sqrt{3}j)$). The maximal eigenvalue $\rho(M) = 6.12 + 6.86j$ is indicated by $\star$. Since $\Re[\rho(M)] > 1$, existence of oscillatory trajectory is guaranteed by Proposition 1, provided there is no other equilibrium than the origin. It was indeed verified by simulation that the variable $q(t)$ oscillates in the steady state as shown

¹This was done using Proposition 4 to be presented later. In particular, positive parameters $a \in \mathbb{R}^n$ and $\phi, \omega \in \mathbb{R}$, and matrices $X \in \mathbb{R}^{2 \times (n-2)}$ and $Y \in \mathbb{R}^{(n-2) \times (n-2)}$ were randomly generated so that $\rho(Y) = 0$, and then M was calculated from (19).

in Fig. 5, where 17 curves are plotted for $q_{11}, \dots, q_{17,1}$ with the peaks occurring in this order. We see that $q_{i+1,1}$ lags from q_{i1} consistently for all i , and this pattern is used for generation of traveling waves along the body during swimming.

We applied Algorithm ($\eta_1 = 2, \eta_2 = 0.1, \varepsilon = 10^{-4}$) to estimate the oscillation profile by solving the harmonic balance equation in (16). The period of oscillation was estimated to be 1.03 s, while the actual period was 0.89 s. The estimated amplitudes and phases for q_{i1} are summarized in Figs. 6 and 7 in comparison with those obtained from simulation. The estimates are reasonably close to the actual behavior.

5 CPG synthesis

We now turn our attention to the design of CPGs to achieve a given oscillation profile. The essential idea is that CPGs can be designed by reversing the analysis steps. In particular, the connectivity matrix M in the CPG should be chosen to satisfy (16) and (18) for a given triple (ω, α, ϕ) . Note that two of the eigenvalues of MK , i.e., $\rho(MK)$ and its complex conjugate, must be on the vertical line $1 + j\mathbb{R}$, and the remaining $n - 2$ eigenvalues be on or to the left of the line. To ensure that the harmonic balance equation provides a good estimate, one may require, in the design, that the real parts of the $n - 2$ eigenvalues be less than a given number γ . The synthesis problem turns out to be much easier than the analysis problem, and the set of all solutions can be characterized explicitly in a numerically tractable manner.

Proposition 4

Let $\omega \in \mathbb{R}, \alpha \in \mathbb{R}^n, \phi \in \mathbb{R}^n$, and $f(s) = f_a(s)$ or $f_b(s)$ be given, where $\omega > 0$ and $\alpha_i > 0$. Suppose there exist i and j such that $\phi_i - \phi_j \neq m\pi$ for all integer m . Let $M \in \mathbb{R}^{n \times n}$ and $\gamma \in \mathbb{R}$ be given and assume $\gamma < 1$. Define

$$\begin{aligned} M_o &:= R\Omega R^\dagger, & K &:= \mathcal{K}(\alpha), \\ R &:= \begin{bmatrix} c & s \end{bmatrix} \in \mathbb{R}^{n \times 2}, & c_i &:= \alpha_i \cos \phi_i, \\ & & s_i &:= \alpha_i \sin \phi_i, \\ \Omega &:= \begin{bmatrix} 1 & \varpi \\ -\varpi & 1 \end{bmatrix}, & \varpi &:= j \left(1 - \frac{1}{f(j\omega)}\right), \end{aligned}$$

and let N be a matrix whose columns form an orthonormal basis for the null space of R^T . Then the following statements are equivalent, and each implies that the CPG in (2) is expected to have oscillations (17).

- i. M satisfies the harmonic balance conditions (16) and (18) for (ω, α, ϕ) , and $n - 2$ eigenvalues of MK have real parts strictly less than γ .
- ii. There exist real matrices X and Y such that

$$MK = M_o + (RX + NY)N^T, \quad \Re[\rho(Y)] < \gamma. \quad (19)$$

- iii. M satisfies the following conditions:

$$MKR = R\Omega, \quad \Re[\rho(N^T MKN)] < \gamma.$$

Proof—Noting that MK is a real matrix, we see that statement (i) holds if and only if MK has an eigenvalue at $\lambda := 1 + j\varpi$ with eigenvector v , another eigenvalue at $\bar{\lambda}$ with eigenvector $\bar{v}$, and the remaining $n - 2$ eigenvalues to the left of the vertical line $\gamma + j\mathbb{R}$. Due to the supposition

on ω and ϕ , eigenvalues λ and $\bar{\lambda}$ are distinct and eigenvectors v and $\bar{v}$ are linearly independent. Hence, a necessary condition for (i) is given by

$$MK \begin{bmatrix} v & \bar{v} \end{bmatrix} = \begin{bmatrix} v & \bar{v} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}.$$

It can readily be shown that this condition is equivalent to $MKR = R\Omega$. Noting that R has the full column rank, and using a standard linear algebra result, the set of all MK satisfying this condition can be parametrized by

$$MK = \begin{bmatrix} R & N \end{bmatrix} \begin{bmatrix} \Omega & X \\ 0 & Y \end{bmatrix} \begin{bmatrix} R^\dagger \\ N^T \end{bmatrix}$$

where X and Y are the free parameters. We now see that the second condition in (i) regarding the $n - 2$ eigenvalues is equivalent to $\Re[\rho(Y)] < \gamma$. Thus we have shown the equivalence (i) $\Leftrightarrow$ (ii). To see that these are also equivalent to (iii), note that the above equation can be written as

$$MK \begin{bmatrix} R & N \end{bmatrix} = \begin{bmatrix} R & N \end{bmatrix} \begin{bmatrix} \Omega & X \\ 0 & Y \end{bmatrix}.$$

The left column of this equation gives $MKR = R\Omega$, and there exist X and Y satisfying the right column and $\Re[\rho(Y)] < \gamma$ if and only if $\Re[\rho(N^T MKN)] < \gamma$ holds. Thus we have (iii) as another equivalent condition.

In Proposition 4, an explicit formula for the connectivity matrix M is given in statement (ii) with free parameters X and Y . The eigenvalues of MK are $1 \pm j\omega$ and those of Y . In particular, the simplest choice $X = 0$ and $Y = 0$ would put the $n - 2$ eigenvalues at the origin. The freedom in X and Y can be utilized to enforce a certain structure on M . For example, the diagonal entries of M may be restricted to zero to avoid self-feedback to the same neuron. It may also be desired to have only local interactions between neurons, in which case, the tri-diagonal structure may be enforced on M . To impose such structural constraint on M , the characterization in statement (iii) may be more suitable for numerical computations. In particular, the eigenvalue constraint $\Re[\rho(N^T MKN)] < \gamma$ may be enforced by

$$N^T MKN + (N^T MKN)^T < 2\gamma I$$

which is a linear matrix inequality (LMI) in M . Thus, the design problem reduces to the search for M , with a specific structure, satisfying the linear equality $MKR = R\Omega$ and the above LMI. This is a finite dimensional convex problem and hence is numerically tractable. The potential conservatism associated with the LMI condition does not seem to be a problem since γ can be used as a tuning knob to set the value of $\Re[\rho(Y)]$ rather than as a hard bound on it.

The supposition $\phi_i - \phi_j \neq m\pi$ in Proposition 4 means that not all the signals q_i are synchronized to be in-phase or out-of-phase, and there is at least one pair of signals q_i and q_j such that their phase difference is not a multiple of π . When this supposition is violated, then q and λ are both real, and different characterizations for M can be derived.

Example 4

We design a CPG with five neurons to meet the specification shown in Table 3. We used $f(s) = f_b(s)$ with $\omega_1 = \omega_2 = 1$ and designed M using statement (iii) of Proposition 4. In particular, we solved the following LMI problem:

$$\min_M \|M\| \quad \text{s.t.} \quad \begin{cases} MKR = R\Omega, \\ N^T MKN + (N^T MKN)^T < 0 \end{cases}$$

with the structural constraint that M is penta-diagonal and the main diagonal entries are zero. The optimizer is found as

$$M = \begin{bmatrix} 0 & -0.3779 & 1.2213 & 0 & 0 \\ 0.3690 & 0 & 0.1255 & 0.8656 & 0 \\ -0.0744 & 1.0599 & 0 & -0.0208 & 1.0757 \\ 0 & 0.0717 & 0.0421 & 0 & 1.5166 \\ 0 & 0 & -4.2003 & 3.6983 & 0 \end{bmatrix}$$

with the eigenvalues of MK at $1.00 \pm 0.75j$, $-0.80 \pm 0.90j$, and -0.41 . The corresponding CPG architecture is shown in Fig. 8. We see that the neuronal interactions are local; The neurons are arranged in a chain, and connections are made between neurons that are within two units of distance. The actual oscillation profile of the CPG calculated from a simulation is given in Table 3 and Fig. 9, where the five curves are plotted for $q_1, \dots, q_5$ with q_i having increasing amplitudes in this order. The design specifications are approximately satisfied.

6 Discussion

We have developed methods for analysis and synthesis of CPGs. The problems are essentially reduced to eigenvalue/eigenvector problems through the method of multivariable harmonic balance. In particular, our MHB approach has revealed that the oscillation profile (frequency, amplitudes, phases) is encoded in the maximal eigenvalue/eigenvector of the neuronal connectivity matrix M when the amplitudes are uniform. A computational algorithm is proposed to solve the MHB equation, providing estimates of the oscillation profile. The connectivity matrix that leads to a given oscillation profile is characterized in terms of linear matrix equality/inequality, allowing for design of CPGs with topological constraints on the neuronal connectivity. Our approach provides insights into the relation between the neuronal connectivity and the resulting oscillation profile, and has lead to a very simple yet practically useful methods for CPG analysis and synthesis, at the expense of exactness.

The CPG analysis/synthesis problems considered here are closely related to the question addressed by Smale (1976); Does there exist a pair of neurons, diffusively coupled to each other, such that each neuron has a unique globally stable equilibrium yet the coupled system has an attractive periodic solution? Pogromsky et al. (1999) have generalized the problem for an arbitrary number of neurons and shown that the coupled system is Y-oscillatory if the maximal eigenvalue of the coupling matrix is sufficiently large. Our result (Proposition 3) is able to predict the previous result for the particular neuronal dynamics (1). In fact, our result further generalizes the previous results in the sense that the coupling matrix is arbitrary and is not confined to diagonally dominant, symmetric matrices arising from diffusive connections.

The harmonic balance results are approximate in nature and do not have a theoretical guarantee although a number of numerical examples demonstrate reliability and applicability of the proposed methods. In general, solvability of a harmonic balance equation does not imply

existence of a periodic solution. However, a classical result (Theorem 5.2.15 in Mees (1981)) provides additional conditions under which a periodic solution is guaranteed to exist. Moreover, this result gives a bound for the error between the predicted sinusoidal oscillation and the periodic trajectory. It would be interesting to see how this general result can be applied to the class of CPGs considered here. In particular, exploiting the particular structure of our CPGs, the additional condition for existence of a periodic solution and/or the error bounds may simplify enough to be useful for rigorous CPG designs with a theoretical guarantee. Such developments may be possible but the associated conservatism could be a nontrivial issue.

When the connectivity matrix of the CPG has a certain symmetry property, the MHB prediction can be shown to be accurate. Specifically, if the amplitudes are uniform and phases are equally spaced over 2π in Proposition 4, the connectivity matrix M in (19) with $X = 0$ and $Y = 0$ is a rank-two circulant matrix. In this case, the CPG is guaranteed to have a periodic orbit with the prescribed phase property, and the orbit approaches the sinusoidal orbit with the prescribed frequency and amplitude as the number of neurons n goes to infinity (Chen and Iwasaki, 2008). Related to this result is the Hopf bifurcation for symmetric systems considered by Golubitsky and Stewart (1985). Their main theorem states that bifurcating branches of periodic solutions with certain symmetries exist when the fixed-point subspace of that subgroup of symmetries is two dimensional. Close examination of the relation between the results by Chen and Iwasaki (2008) and Golubitsky and Stewart (1985) may lead to extension of the circulant CPGs to a more general class of CPGs with spacio-temporal symmetry.

Acknowledgments

The author gratefully acknowledges helpful discussions with Prof. U. Jönsson and Prof. A.L. Fradkov. This work was supported by NSF 0237708 and NIH/NINDS/CRCNS 1 R01 NS46057-01.

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Fig. 1.
CPG architecture

Fig. 2.
Oscillatory trajectories

Fig. 3.
Leech swimming CPG (Two segments are shown)

Fig. 4.
Eigenvalues of M

Fig. 5.
Time courses of $q_{i1}(t)$

Fig. 6.
Amplitudes of q_{i1}

Fig. 7.
Phases of q_{i1}

Fig. 8.
CPG architecture

Fig. 9.
Time courses of $q_i(t)$

Table 1

Estimated and simulated oscillation profiles for CPG in Example 1

	period	amp	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
est	6.61	2.32	0.0	36.0	72.0	108.0	144.0
sim	7.24	2.51	0.0	36.1	72.0	108.0	144.0

Table 2

Performance of Algorithm for randomly generated CPGs

Number of neurons	20	40	60	80	100
Success rate [%]	96	96	93	94	95
CPU time [s]	0.55	1.18	2.84	4.39	6.16

Table 3

Design specifications and achieved oscillation profile.

	period	a_1	a_2	a_3	a_4	a_5
spec	3.14	0.60	0.80	1.00	1.20	1.40
sim	3.16	0.59	0.80	1.01	1.17	1.54
		ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
spec	0	10.00	30.00	60.00	100.00	
sim	0	11.16	29.35	59.68	97.73	