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A bilinear stress-strain relationship for arteries

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Abstract

A comprehensive understanding of the mechanical properties of blood vessels is essential for vascular physiology, pathophysiology and tissue engineering. A well known approach to study the elasticity of blood vessels is to postulate a strain energy function such as the exponential or polynomial forms. It is typically difficult to fit experimental data to derive material parameters for blood vessels, however, due to the highly-nonlinear nature of the stress-strain relation. In this work, we generalize the strain definition to absorb the elastic nonlinearity and then propose a 2D bilinear stress-strain relation between Kirchhoff stress and the new strain measure. The model is found to represent the Fung's exponential model very well. The novel linearized constitutive relation simplifies the determination of material constants by reducing the nonlinearity and provides a clearer physical interpretation of the model parameters. The limitations of the constitutive model and its implications for vascular mechanics are discussed.

Keywords

Nonlinearity; linearization; stress-strain relation; strain measure; blood vessel

1. Introduction

An understanding of elasticity of blood vessels may be important in the selection of arterial grafts and in designing artificial blood vessels [1-3]. In this regard, material property mismatch has been shown to reduce graft patency, see review [4] and references therein. Furthermore, growth and remodeling are intimately related to the local stresses and strains in the vessel wall [5-10]. Therefore, understanding the relation between applied load and the resultant deformation (or the constitutive relation) of blood vessels is one of the key research areas in biomechanics.

Using the continuum mechanics approach [11], the behavior of blood vessels can be predicted by theoretical and numerical models once the needed mechanical properties are obtained from experiments. Unlike many inanimate materials, however, blood vessels typically exhibit stress-strain relations characterized by hysteretic highly-nonlinear load-deformation curves with anisotropic mechanical properties [12-14]. These features make modeling of blood vessels more difficult than engineering materials.

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Material properties have been largely reported for the exponential, polynomial or logarithmic forms of strain energy functions, from which nonlinear stress-strain relations can be obtained by differentiation of the strain potential [12]. These well-known models, especially the Fung's exponential model, have been found to fit the experimental data very well in a large range of loading. However, the nonlinearity of the stress-strain curves makes the determination of material constants difficult. First, curve-fitting for nonlinear models is difficult because of possible multiple minimum locales of the strain energy. Second, the empirical constants often lack clear physical meanings. Finally, the parameters in these nonlinear models have been found to be very sensitive to the data; i.e., slight variation of measured data can yield significant changes of empirical parameters [12]. As a result, some model parameters may be arbitrarily chosen.

The goal of this paper is to present a linearized stress-strain relation for large arteries under broad loading (stretch ratios in circumferential and axial directions are $1.1 < \lambda_\theta < 1.8$ and $1.1 < \lambda_z < 1.6$). By introducing a new strain measure that absorbs the nonlinearity, it is found that a bilinear constitutive model can represent the Fung's exponential model very well. With some modification, a low order nominal "strain energy" polynomial can also be employed to characterize the mechanical behavior of blood vessels.

We shall briefly review Fung's 2D pseudoelastic model and demonstrate that the one-dimensional Fung-type exponential constitutive relation can be written in a linearized form. Furthermore, we introduce a new strain measure and present a novel bilinear stress-strain relation in two-dimensions, along with some material properties extracted from Fung's model. Finally, the advantages and limitations of the proposed model are discussed and a pseudoelastic "strain potential" function is derived by introducing quadratic terms in the bilinear model.

2. Fung's 2D pseudoelastic model

For simplicity, we consider a straight cylindrical artery as a thin-walled elastic tube deformed only in the circumferential and axial directions. In other words, we neglect the stress and strain in radial direction as well as all shear components. Other behaviors including the existence of residual strains, hysteresis and active contraction are also ignored.

After Fung's exponential-type pseudoelastic strain energy function was proposed [15], it has been widely used to model the highly nonlinear and anisotropic behavior of blood vessels, e.g., [16-18]. In this model, the second Piola-Kirchhoff stress is obtained by differentiating the strain energy with respect to the Green-Lagrange strain. The model approximates the experimentally measured stress-strain relationship very well and requires a minimum number of material parameters [12].

Green strains in the circumferential (θ) and axial directions (z) are related to corresponding principal stretch ratios λ_θ and λ_z by:

$$E_{\theta\theta} = \frac{1}{2}(\lambda_\theta^2 - 1), E_{zz} = \frac{1}{2}(\lambda_z^2 - 1). \quad (1)$$

In the simplest two-dimensional Fung's model without shear deformation, the strain energy per unit volume is given by [12]:

$$W = \frac{C}{2} [\exp(Q) - 1], \quad (2)$$

where

$$Q = a_1(E_{\theta\theta}^2 - E_{\theta\theta}^{*2}) + a_2(E_{zz}^2 - E_{zz}^{*2}) + 2a_4(E_{\theta\theta}E_{zz} - E_{\theta\theta}^*E_{zz}^*). \quad (3)$$

Kirchhoff stresses are obtained by differentiating the strain energy with respect to corresponding Green strains as:

$$\begin{cases} S_{\theta\theta} = \frac{\partial W}{\partial E_{\theta\theta}} = C(a_1 E_{\theta\theta} + a_4 E_{zz}) \exp(Q) \\ S_{zz} = \frac{\partial W}{\partial E_{zz}} = C(a_2 E_{zz} + a_4 E_{\theta\theta}) \exp(Q) \end{cases} \quad (4)$$

It is noted that for $E_{zz} = 0$, $S_{\theta\theta}$ is a nonlinear function of $E_{\theta\theta}$ only. In the axial direction, we have a similar situation. In such cases, if we define a new strain measure to absorb the nonlinearity, it is possible to write the stress-strain relation in a linear form.

To linearize the constitutive relation, the major task is to find a strain measure that can best approximate the nonlinear curvature of load vs. stretch ratio plots obtained in experiments. Since Fung's model represents the experimental measurements well, we only need to examine if Fung's model can be linearized. In the 1D strain case where $E_{zz} = 0$ ($E_{\theta\theta}^* = E_{zz}^* = 0$) can be assumed as well [15]), the Kirchhoff stress in circumferential direction is given by:

$$S_{\theta\theta} = \frac{\partial W}{\partial E_{\theta\theta}} = C a_1 E_{\theta\theta} \exp(a_1 E_{\theta\theta}^2) = c_{11} D_{\theta\theta}, \quad (5)$$

where $c_{11} = C a_1$ and $D_{\theta\theta} = E_{\theta\theta} \exp(a_1 E_{\theta\theta}^2)$. Equation (5) shows that Hooke's law can be obtained in the 1D case if we view $D_{\theta\theta}$ as a strain measure. A similar relation exists in the axial direction.

The expression of Eq. (5) in a linearized form does not highlight the advantages in the 1D model, but it certainly requires less material constants than a polynomial form [19]. In two-dimensions the linearization of the stress-strain relationship is preferred since it may be difficult to experimentally determine material constants due to the nonlinearity of the constitutive law [18].

3. A linearized constitutive model

3.1. A generalized strain measure

Although Green strain is the best choice for Fung's model since it directly yields Kirchhoff stress from the differentiation of strain energy function, we can certainly choose another strain measure to establish a stress-strain relationship. In the spirit of Seth-Hill strain family [20], we propose a new strain measure that takes into account the exponential increase of stress with Green strain to absorb the nonlinearity and thus result in a simpler stress-strain relation. In the circumferential (θ) and axial (z) directions, we define the new strains as:

$$D_{\theta\theta} = E_{\theta\theta} \exp(nE_{\theta\theta}^2), \quad D_{zz} = E_{zz} \exp(nE_{zz}^2), \quad (6)$$

where n is a material constant characterizing the strain nonlinearity with respect to Green strain $E_{\theta\theta}$ and E_{zz} (for $n=0$, Eq. (6) reduces to Eq. (1)). These generalized strains (Green-exponential strains) are products of Green strains and their exponential functions.

3.2. A bilinear stress-strain relation

Considering the coupling between circumferential and axial directions in a biaxial deformation, we postulate a simple bilinear constitutive relation between Kirchhoff stress and the new strain measure as:

$$\begin{cases} S_{\theta\theta} = c_{11}D_{\theta\theta} + c_{12}D_{zz} + b_{12}D_{\theta\theta}D_{zz} \\ S_{zz} = c_{21}D_{\theta\theta} + c_{22}D_{zz} + b_{21}D_{\theta\theta}D_{zz} \end{cases}, \quad (7)$$

in which there are six constants having the unit of stress and one constant n that characterizes the nonlinearity of strain. The constant c_{11} is the elastic stiffness in the circumferential direction when the strain in the axial direction is zero, c_{12} reflects the elastic modulus of $S_{\theta\theta}$ versus D_{zz} when there is no deformation in the circumferential direction, and b_{12} is the bilinear coupling constant for circumferential stress. The other three parameters have similar meanings in the axial direction.

In general, the material constants in Eq. (7) do not have symmetric properties; e.g., $c_{12} \neq c_{21}$ since we cannot invoke the hyper-elasticity assumption (the existence of strain energy potential [12]). Such assumption can reduce the number of constants.

We will check if Fung's model can be approximated by Eq. (7) in the deformation range of interest.

3.3. Evaluation of the bilinear model

The nonlinearity parameter n in the strain defined in Eq. (6) can be obtained with experimental data using a least square method. If stretch ratio in the axial direction is kept constant (e.g., $E_{zz} = 0$), the first equation in Eq. (7) will reduce to Eq. (5), where $a_1 = n$ is a constant to be determined. Of course, if n is extracted from data in the axial direction, it may differ from that obtained from the circumferential direction. Fortunately, n is not very sensitive to the direction chosen and thus we can use either circumferential or axial data to determine it, or use an average value. In practice, we can choose one set of "the most reliable" (or the most representative) experimental data to determine n .

Once n is determined, the values of the elastic constants can be easily obtained using least square method for the bilinear model. In the following examples, we estimate the model constants using Fung's model parameters in the literature.

With Fung's model (Eq. (4)) for the thoracic aorta of a mongrel dog: $C = 120.2$ kPa, $a_1 = 0.320$, $a_2 = 0.451$, $a_4 = 0.0681$, $E_{\theta\theta}^* = E_{zz}^* = 0$ [16], we first plot the distribution of Kirchhoff stress vs. Green strain in the circumferential and axial directions in Figs. 1(a) and (b), respectively. The range of interest is chosen to cover typical physiological values in [16] and beyond: $1.1 < \lambda_\theta < 1.8$ and $1.1 < \lambda_z < 1.6$, i.e., $0.105 < E_{\theta\theta} < 1.12$ and $0.105 < E_{zz} < 0.78$. We then consider the sample data points (11×11 grids) in Fig. 1 as the "experimental data" we need to fit with the

bilinear model. By systematically changing the nonlinearity parameter n , we monitor the least squares errors to obtain the best fit.

Since both θ and z directions result in residuals, we will admit the value of n which minimizes the total relative least squares error, RLSE, as

$$RLSE = \frac{\sqrt{\sum_N (S_{\theta\theta}^d - S_{\theta\theta}^f)^2} + \sqrt{\sum_N (S_{zz}^d - S_{zz}^f)^2}}{\sum_N S_{\theta\theta}^f + \sum_N S_{zz}^f}, \quad (8)$$

where $S_{\theta\theta}^d$ and S_{zz}^d are the “experimental data”, $S_{\theta\theta}^f$ and S_{zz}^f are the fitted stresses calculated from the bilinear model, N is the total number of sampled data. The first term (in numerator) on the right hand side of Eq. (8) is the error contribution from θ direction and the second term is from z direction.

It is found that $n = 0.41$ results in a minimum least squares error (RLSE = 0.174%, see Fig. 2). The corresponding “experimental data” and the fitted bilinear surfaces are plotted in Figs. 3 (a) and (b), respectively. The obtained bilinear model parameters are listed for Case 1 in Table 1.

To more clearly show the linearized stress-strain relationship, we plot the profiles of Fig. 1(a), i.e., $S_{\theta\theta}$ vs. $E_{\theta\theta}$ curves, at $E_{zz} = 0.105, 0.51, 0.78$, in Fig. 4(a). The linearized profiles of Fig. 3(a) are shown in Fig. 4(b). It is clear that when the abscissa in Fig. 4(a) is “stretched”, a linear stress-strain relation in Fig. 4(b) is obtained. Note that here we are using Fung’s model as a benchmark. The direct experimental data can also be used to fit the bilinear model when available.

Most of Fung’s models reported in references [16-18] can be linearized, except for those highly-anisotropic results such as femoral artery and carotid artery in [17], where a_1 differs from a_2 significantly (about 5~10 times). It seems that n might fall into the range bounded by a_1 and a_2 when a_4 is small in Fung’s model (e.g., see the first three cases in Table 1). The correlation coefficient R in the bilinear model fitting of Fung’s models is above 0.99 for all the five cases in Table 1. It is noted that in Case 5, c_{12} is negative as will be discussed in Appendix A.

4. Discussion

We adopted a strain measure to absorb the nonlinearity of the constitutive law and hence make the stress-strain relation much simpler. The resulting bilinear model may be more practical for experimental, theoretical, and numerical stress analysis.

The choice of the strain measure in Eq. (6) and the Kirchhoff stress in the bilinear model is because the resulting stress-strain relation can be related to Fung’s model. Note that Kirchhoff stress is conjugate to Green strain (a special case of the generalized Green-exponential strain). Other forms of strain and stress can be used as well, as will be discussed later.

For isotropic materials that fit the bilinear model, $c_{11} = c_{22}$, $c_{21} = c_{12}$, $b_{21} = b_{12}$. The total number of material constants is four (with n being the nonlinearity parameter). When the coupling effect is negligible ($b_{21} = b_{12} = 0$), the bilinear relation is reduced to Hooke’s law with three constants.

4.1. Nominal strain energy

It has been stated earlier that the proposed bilinear model does not have a strain potential that takes into consideration the hyperelastic features. Nevertheless, we may assume that a polynomial function of the strains in Eq. (6) exists and its differentiations with respect to $D_{\theta\theta}$ and D_{zz} result in the corresponding Kirchhoff stress; namely,

$$S_{\theta\theta} = \frac{\partial W_n}{\partial D_{\theta\theta}}, \quad S_{zz} = \frac{\partial W_n}{\partial D_{zz}}, \quad (9)$$

where the “strain energy” W_n is not an energy potential in the normal sense, since the strains $D_{\theta\theta}$ and D_{zz} are not conjugate to Kirchhoff stress unless the nonlinearity parameter n is zero. In other words, the above equation defines a nominal “strain potential” only for the convenience of deriving stresses. The physical meaning of W_n differs from the classical convention of energy. In a special case when $n = 0$, however, W_n is the traditional strain energy since $D_{\theta\theta}$ and D_{zz} are then equivalent to Green strains.

One can easily verify that the bilinear stress-strain relations in Eq. (7) are not a differential result of a function. Instead, we need to introduce at least one more term in Eq. (7) to meet the requirement of the possible existence of “strain energy”. In general, let us consider a polynomial function given by:

$$W_n = \frac{c_{11}}{2} D_{\theta\theta}^2 + c_{12} D_{\theta\theta} D_{zz} + \frac{c_{22}}{2} D_{zz}^2 + \frac{b_{12}}{2} D_{\theta\theta}^2 D_{zz} + \frac{b_{21}}{2} D_{\theta\theta} D_{zz}^2 + \frac{b_{11}}{3} D_{\theta\theta}^3 + \frac{b_{22}}{3} D_{zz}^3. \quad (10)$$

The above expression appears similar to the strain energy density function in [21] except that the polynomial in [21] is a function of Green strain. With the definitions in Eqs. (9) and (10), we can obtain the following expression:

$$\begin{cases} S_{\theta\theta} = c_{11} D_{\theta\theta} + c_{12} D_{zz} + b_{12} D_{\theta\theta} D_{zz} + \frac{b_{21}}{2} D_{zz}^2 + b_{11} D_{\theta\theta}^2 \\ S_{zz} = c_{12} D_{\theta\theta} + c_{22} D_{zz} + b_{21} D_{\theta\theta} D_{zz} + \frac{b_{12}}{2} D_{\theta\theta}^2 + b_{22} D_{zz}^2 \end{cases}. \quad (11)$$

Compared with Eq. (7), it is seen that in the above constitutive relation, stresses are now quadratic functions of strains.

Interestingly, Choi and Vito [22] proposed a strain energy density function for canine pericardium (also see [13]) of the form:

$$W = b_0 \left[\exp(b_1 E_{\theta\theta}^2) + \exp(b_2 E_{zz}^2) + \exp(2b_3 E_{\theta\theta} E_{zz}) - 3 \right], \quad (12)$$

where b 's are material constants. Since the role of parameter b_0 in the above equation (or constant C in Fung's model [12]) is a scale factor, Eq. (12) can be re-written as:

$$W = b'_1 \exp(n' E_{\theta\theta}^2) + b'_2 \exp(n' E_{zz}^2) + b'_3 \exp(2n' E_{\theta\theta} E_{zz}) - \text{const}, \quad (13)$$

where n' can be interpreted as the “nonlinearity parameter”. This function seems to be more similar to Eq. (10) than Fung’s model, as the linear terms in the resulting stress-strain relation are more similar to those in Eq. (11).

Of course, significant difference still exists between Eq. (10) and Eq. (13). Firstly, the former is only a “nominal” strain potential that is a function of strains not conjugate to Kirchhoff stress. Secondly, the latter includes a coupling term that can not be directly linearized with a new strain definition.

It is emphasized that the “energy” per volume defined in Eq. (10) is measured at a scale different from the conventional definition. In fact, conjugate stress measures [20,23] are usually sought for specific strains to keep the energy measure universal. In the present case, a meaningful stress measure is more important than a consistent energy scale. The real elastic energy in a loading sequence can be readily computed by cumulatively integrating Kirchhoff stress (Eq. (7) or Eq. (11)) with respect to Green strain (Eq. (1)).

It is worth noting that in Eq. (10) there are more constants than in the bilinear model (Eq. (7)). This is because in the former we considered quadratic terms in stress-strain relation (Eq. (11)).

If the nonlinear dependence of $S_{\theta\theta}$ on $D_{\theta\theta}^2$ and S_{zz} on D_{zz}^2 can be absorbed in the definition of strains, those terms with $D_{\theta\theta}^3$ and D_{zz}^3 are not absolutely necessary in the nominal strain potential ($b_{11} = b_{22} = 0$). Then the stresses in Eq. (11) become linear functions of the strains in one direction and quadratic in another direction, e.g., $S_{\theta\theta}$ changes linearly with $D_{\theta\theta}$ when $D_{zz} = \text{const}$. However, determination of constants could become problematic since a full third-order polynomial is usually expected.

4.2. Alternative strain measures

Alternative strain measures to Eq. (6) are possible such as:

$$\varepsilon = \ln(\lambda) \exp(\lambda^n - 1). \quad (14)$$

where λ is the stretch ratio, n characterizes the nonlinearity of the strain measure with respect to the Hencky (logarithmic or natural) strain $\ln(\lambda)$. Other strain measures can also be adopted as long as the exponential increase of stress with stretch ratio can be accounted for.

The Hencky strain is usually believed to be the “best” because an infinite load is expected to create an infinite Hencky strain if a linear stress-strain relation is assumed. Other strain measures in the Hill-Seth family inherently call for nonlinear stress-strain relations. For example, the minimum Green strain is $-1/2$ when stretch ratio is zero, which implies that if stress changes linearly with strain then a finite force load will compress the material to vanish ($\lambda = 0$). This is obviously non-physical and therefore a nonlinear stress-strain relation is required for large loading. It should be pointed out that linear stress-strain relations can not be guaranteed even if we select the “best” strain measure. A good strain measure can only lead to better linear approximation of the real constitutive law in a larger range of loading.

Similar to Eq. (7), a bilinear model can be established between Cauchy stress (the true stress that is used in mechanical equilibrium equations) and the strain defined in Eq. (14) that reduces to the natural strain when the constant $n = 0$. Also, a nominal “strain potential” resembling Eq. (10), whose derivatives are then Cauchy stresses, can be written when the strain is replaced with the definition in Eq. (14). Here, we choose Cauchy stress because it correlates more linearly with the strain in Eq. (14) than the Kirchhoff stress (result is not shown).

Most blood vessels apparently exhibit an exponential-type stress-stretch ratio relation [12], which motivates us to introduce the exponential functions in strain definitions of Eqs. (6) and (14).

4.3. Model's advantage and limitation

It is easier to experimentally determine material constants in a linearized model. The nonlinearity parameter n can be first estimated from a single stress-strain curve. Then all the model parameters are determined by the least square method for a value of n that minimizes the least squares error. Once a criterion, e.g., the minimum of RLSE in Eq. (8), is reached, the material constants are obtained.

The curve-fitting process with biaxial test data for the bilinear model or a lower-order polynomial is expected to be more reliable than for the highly nonlinear models. It is demonstrated in Appendix B that reasonably good fit can be obtained when we reduce the number of constants by assuming $c_{12}=c_{21}$ and $b_{12}=b_{21}$. The elastic moduli are found not to be sensitive to the nonlinearity parameter n (see Appendix B).

In addition, parameters in the linearized model have clearer physical meanings, i.e., n is nonlinearity of strain, other constants are elastic moduli (expected to be positive in simple tests) with respect to the strain defined by n . The variation of the moduli with direction, for instance, the difference between c_{11} and c_{22} reflects the anisotropy of the material.

One limitation of the proposed model is that the nonlinearity parameter n changes the strain measure from vessel to vessel. Thus, we can not compare strains from different vessels without knowing their values of n . However, this is a minor issue since stress and strain definitions must be clearly given in large deformation problems. One good feature of parameter n is that it is narrowly bounded (usually in the range of 0~3).

As aforementioned, the value of n in Eq. (6) may change when we fit it with the experimental data from a different direction, albeit the changes are small. In Fung's model, we have the same situation. Suppose we employ a linear relationship derived from Fung's 1D model in Eq. (5). Using experimental data in the circumferential direction, we fit a_1 and c_{11} . Similarly in the axial direction, we have $S_{zz}=c_{22}D_{zz}=Ca_2E_{zz}\exp(a_2E_{zz}^2)$ and can fit a_2 and c_{22} using corresponding experimental data. The question is: will c_{11}/a_1 be the same as c_{22}/a_2 and equal to C ? The answer is possibly "no". In Fung's model, the scale constant C can somewhat be adjusted freely similar to the parameter n in our model. The two models do not have advantage over each other in this regard.

An additional disadvantage of the present model includes that highly anisotropic materials can not be linearized the same way. Hence, we must employ other models when necessary [14].

The beauty of Fung's model is that it has only four parameters (C , a_1 , a_2 , and a_4). In the bilinear model, we have seven material constants. Although there are significantly more parameters in the linearized model, the elastic moduli in the circumferential and axial directions are decoupled and can be determined independently.

It will be straightforward to extend the bilinear model to three-dimensions when the radial stress and strain are also considered in a thick tube. This extension has been made for Fung's model, e.g., in [24,25]. In the 3D bilinear model, the number of material constants will be much more than that in Fung's model. One possible method of reducing the number of constants is discussed in Appendix B.

In summary, we emphasize that the proposed strain measures (Eqs. (6) and (14)) are not independent quantities (they change with materials). As a consequence, the nominal “strain energy” (e.g., Eq. (10)) has no thermodynamic significance. The major advantages are that the formulation significantly simplifies the parameter estimation of nonlinear materials and the resulting parameters can be interpreted in terms of elastic moduli in the various directions over the full range of elasticity.

5. Conclusions

In this work, we proposed linearized constitutive relations for large blood vessels and examined the bilinear model with some reported constants for Fung’s exponential strain energy function. It is found that Fung’s model can be linearized, which indicates that the bilinear model may be used in analysis where Fung’s model applies. The results show that Kirchhoff stress varies linearly with the new strain measure when an appropriate nonlinearity parameter n is determined. This implies that we can characterize the stress-strain relation of large arteries with linearized models to make the parameter determination process much easier than in the highly-nonlinear models. It is expected that the linearized stress-strain relations are relatively more convenient to use in theoretical and numerical analyses as well.

When linearized with the new strain measures (Eq. (6) or Eq. (14)), the materials have their own characteristic nonlinearity measure given by n . Adopting appropriate strain measures to yield linear stress-strain relations [23] could be useful in other nonlinear materials such as rubber and polymers. It is essential to clearly state what stress and strain measures are being used in large deformation problems.

Appendix A

3D effect on 2D model parameters

The experimental data for Case 5 were from a complicated loading state rather than a simple 2D configuration. The blood vessels were stretched and pressurized in the experiments. This is neither a plane stress nor a plane strain state, since the blood vessels are undergoing circumferential, axial, and radial loading simultaneously (Fig. 5). The linearized 3D stress-strain relation is, for example (for convenience, let us drop the bilinear coupling term and assume Hooke’s law)

$$S_{zz} = c'_{21}D_{\theta\theta} + c'_{22}D_{zz} + c'_{23}D_{rr}. \quad (\text{A.1})$$

Suppose that we keep strain D_{zz} constant and increase strain $D_{\theta\theta}$ by increasing the internal pressure, then the radial strain is negative, i.e., strain $D_{rr} < 0$. The change of S_{zz} depends on the sum of $c'_{21}D_{\theta\theta}$ and $c'_{23}D_{rr}$, where the former increases while the latter decreases if all material constants in Eq. (A.1) are positive.

However, in the 2D stress-strain relation ($S_{zz} = c_{21}D_{\theta\theta} + c_{22}D_{zz}$), we only observe the change of S_{zz} with the increase of $D_{\theta\theta}$. If S_{zz} is compressive (when $|c'_{23}D_{rr}| > c'_{21}D_{\theta\theta}$), then we conclude that c_{21} is negative. Therefore, it is not surprising to find that some parameters are negative in the two-dimensional model (e.g., Case 5 in Table 1).

Theoretically, $\partial S_{\theta\theta} / \partial D_{\theta\theta} > 0$ and $\partial S_{zz} / \partial D_{zz} > 0$ are necessary conditions for the material to stay stable (increase strain results in increasing stress). For instance, in the bilinear model, we have

$$\frac{\partial S_{\theta\theta}}{\partial D_{\theta\theta}} = c_{11} + b_{12} D_{zz}, \quad \frac{\partial S_{zz}}{\partial D_{zz}} = c_{22} + b_{21} D_{\theta\theta}. \quad (\text{A.2})$$

In the physiological loading range, the stable condition always holds true even if some fitted parameters are negative (Table 1).

Equation (A.2) shows the equivalent moduli in circumferential and axial directions when we have finite strains in both directions. Thus, the bilinear coupling constants b_{12} and b_{21} can also be viewed as “strain hardening” moduli; e.g., the elastic modulus $\partial S_{\theta\theta} / \partial D_{\theta\theta}$ increases when D_{zz} is positive (note that both $D_{\theta\theta}$ and D_{zz} are positive under *in vivo* condition).

In realistic computations of stress development in blood vessels using 2D models, the material constants obtained in the loading condition as in [18] (also see Fig. 5) could be more reasonable than those obtained from pure biaxial loading experiments. This is because the 3D effect has been incorporated in the parameters in the former case. On the other hand, such a loading scheme increases the difficulty for extracting and interpreting material constants.

Appendix B

Stability of bilinear model parameters

One of the disadvantages of the bilinear model is that it has more parameters than Fung’s model. By observing the values of c_{12} and c_{21} , one can see that they do not differ too much from each other. The same trend seems to exist for b_{12} and b_{21} .

Although lacking theoretical basis (e.g., the existence of strain energy function), we can reduce the total number of constants by assuming $c_{12} = c_{21}$ and $b_{12} = b_{21}$. When taking their averaged values for these parameters, we measure the “goodness of fit” by RLSE. The results are shown in Table 2 for Cases 1-4. It is seen that the fitting error increases from Table 1. This kind of averaging can be used with sacrificing some “goodness of fit”. In 3D models this may reduce the number of parameters significantly.

To demonstrate the stability of the bilinear model parameters, we conducted another analysis: we considered the values n for Cases 1-4 in Table 1 and varied them by ± 0.5 . Then the same least square method was used to determine the elastic moduli for these non-optimal nonlinearity parameters. The modification of the elastic moduli can be viewed as their sensitivity to the selection of n (indirectly, their sensitivity to the change of experimental measurements). The results are shown in Table 3.

It is clearly shown that the value of RLSE does not increase significantly when n is varied in a relatively large range, for example, 0.41 ± 0.5 (see Case 1 in Table 1 and Table 3). The marginal variations of the elastic moduli (especially c_{11} and c_{22}) reveal that the modulus parameters are physically meaningful as material constants. This is because the nonlinearity is taken care of by the parameter n and the rest of parameters are sought in a linearized manner (more stable than nonlinear fitting). Note that in Table 3 the constant c_{21} becomes negative in Case 3a, indicating that the selected n is out of physical range.

In summary, the nonlinearity parameter n can be chosen in a moderate range but the least square method always results in relatively stable elastic moduli. This is one of the advantages of the bilinear constitutive model over the highly nonlinear ones. The elastic moduli in our bilinear model are “global”, which is more convenient to use than the “local” incremental moduli [26].

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References

1. Xue L, Greisler HP. Biomaterials in the development and future of vascular grafts. *J Vasc Surg* 2003;37(2):472–80. [PubMed: 12563226]
2. Kannan RY, Salacinski HJ, Sales K, Butler P, Seifalian AM. The roles of tissue engineering and vascularisation in the development of micro-vascular networks: a review. *Biomaterials* 2005;26:1857–75. [PubMed: 15576160]
3. Yang J, Webb AR, Pickerill SJ, Hageman G, Ameer GA. Synthesis and evaluation of poly(diols citrate) biodegradable elastomers. *Biomaterials* 2006;27(9):1889–98. [PubMed: 16290904]
4. Kassab GS, Navia JA. Biomechanical considerations in the design of graft: the homeostasis hypothesis. *Annu Rev Biomed Eng* 2006;8:499–535. [PubMed: 16834565]
5. Fung YC. What are the residual stresses doing in our blood vessels? *Ann Biomed Eng* 1991;19:237–49. [PubMed: 1928868]
6. Thubrikar MJ, Robicsek F. Pressure-induced arterial wall stress and atherosclerosis. *Ann Thorac Surg* 1995;59:1594–603. [PubMed: 7771858]
7. Liu SQ. Influence of tensile strain on smooth muscle cell orientation in rat blood vessels. *J Biomech Eng* 1998;120:313–20. [PubMed: 10412397]
8. Taber LA. Biomechanics of cardiovascular development. *Annu Rev Biomed Eng* 2001;3:1–25. [PubMed: 11447055]
9. Galis ZS, Khatri JJ. Matrix metalloproteinases in vascular remodeling and atherogenesis: the good, the bad, and the ugly. *Circ Res* 2002;90:251–62. [PubMed: 11861412]
10. Seliktar D, Nerem RM, Galis ZS. Mechanical strain-stimulated remodeling of tissue-engineered blood vessel constructs. *Tissue Eng* 2003;9(4):657–66. [PubMed: 13678444]
11. Humphrey JD. Continuum biomechanics of soft biological tissues. *Proc R Soc London A* 2003;459(2029):3–46.
12. Fung, YC. *Biomechanics: Mechanical Properties of Living Tissues*. Vol. 2. New York: Springer-Verlag; 1993.
13. Sacks MS, Sun W. Multiaxial mechanical behavior of biological materials. *Annu Rev Biomed Eng* 2003;5:251–84. [PubMed: 12730082]
14. Vito RP, Dixon SA. Blood vessel constitutive models-1995-2002. *Annu Rev Biomed Eng* 2003;5:413–39. [PubMed: 12730083]
15. Fung YC, Fronek K, Patitucci P. Pseudoelasticity of arteries and the choice of its mathematical expression. *Am J Physiol Heart Circ Physiol* 1979;237:H620–31.
16. Zhou J, Fung YC. The degree of nonlinearity and anisotropy of blood vessel elasticity. *Proc Natl Acad Sci USA* 1997;94:14255–60. [PubMed: 9405599]
17. Zhao J, Day J, Yuan ZF, Gregersen H. Regional arterial stress-strain distributions referenced to the zero-stress state in the rat. *Am J Physiol Heart Circ Physiol* 2002;282:H622–9. [PubMed: 11788411]
18. Pandit A, Lu X, Wang C, Kassab GS. Biaxial elastic material properties of porcine coronary media and adventitia. *Am J Physiol Heart Circ Physiol* 2005;288:H2581–7. [PubMed: 15792993]
19. Sokolis DP, Boudoulas H, Karayannacos PE. Assessment of the aortic stress-strain relation in uniaxial tension. *J Biomech* 2002;35:1213–23. [PubMed: 12163311]
20. Farahani K, Naghdabadi R. Conjugate stresses of the Seth-Hill strain tensors. *Int J Solids Struct* 2000;37:5247–55.
21. Patel, DJ.; Vaishnav, RN. The rheology of large blood vessels. In: Bergel, DH., editor. *Cardiovascular Fluid Dynamics*. Vol. 2. New York: Academic; 1972. p. 2-65.
22. Choi HS, Vito RP. Two-dimensional stress-strain relationship for canine pericardium. *J Biomech Eng* 1990;112(2):153–59. [PubMed: 2345445]

23. Curnier A, Zysset P. A family of metric strains and conjugate stresses, prolonging usual material laws from small to large transformations. *Int J Solids Struct* 2006;43(10):3057–86.
24. Chuong CJ, Fung YC. Three-dimensional stress distribution in arteries. *J Biomech Eng* 1983;105(3):268–74. [PubMed: 6632830]
25. Wang C, Garcia M, Lu X, Lanir Y, Kassab GS. Three-dimensional mechanical properties of porcine coronary arteries: a validated two-layer model. *Am J Physiol Heart Circ Physiol* 2006;291(3):H1200–9. [PubMed: 16582016]
26. Lu X, Pandit A, Kassab GS. Biaxial incremental homeostatic elastic moduli of coronary artery: two-layer model. *Am J Physiol Heart Circ Physiol* 2004;287(4):H1663–9. [PubMed: 15371266]

(a)

(b)

Fig. 1. Kirchhoff stress versus Green strain (Eq.(4)) in the thoracic aorta of a mongrel dog (Case 1 in Table 1): (a) $S_{\theta\theta}$ and (b) S_{zz} . Stress unit is kPa.

Fig. 2.

Change of least squares errors with n when fitting the bilinear model in Case 1. The minimum total error is realized at $n = 0.41$. Note that in this figure the denominator in Eq. (8) was kept at 9915.19 kPa which was computed from the result of $n = 0.41$.

(a)

(b)

Fig. 3. Kirchhoff stress versus Green-exponential strain ($n = 0.41$, Eq. (6)): (a) $S_{\theta\theta}$ and (b) S_{zz} . The 121 points are a re-plot of Fig. 1 when the abscissas are “stretched”; the wireframe plots are the fitted bilinear surfaces (Eq.(7)). Stress unit is kPa.

(a)

(b)

Fig. 4.

(a) Profiles of Kirchhoff stress $S_{\theta\theta}$ versus Green strain $E_{\theta\theta}$ (Fig. 1(a)) at $E_{zz} = 0.105, 0.51, 0.78$.

(b) Profiles of Kirchhoff stress $S_{\theta\theta}$ versus Green-exponential strain $D_{\theta\theta}$ (Fig. 3(a)) at $D_{zz} = 0.1055, 0.5674, 1.001$ (i.e., $E_{zz} = 0.105, 0.51, 0.78$). Stress unit is kPa.

Fig. 5.

An element of the tested blood vessel shows that it is under tri-axial loading. Cauchy stress directions are shown schematically. Note that the radial true stress is equal to the negative internal pressure at the inner surface.

Table 1

Bilinear model parameters for Fung's models

Case	Fung's model					Bilinear model						
	C (kPa)	a ₁	a ₂	a ₄	c ₁₁ (kPa)	c ₁₂ (kPa)	b ₁₂ (kPa)	c ₂₁ (kPa)	c ₂₂ (kPa)	b ₂₁ (kPa)	n	RLSE
1	120.2	0.320	0.451	0.0681	33.01	8.99	19.80	5.53	52.24	26.03	0.41	0.174%
2	126.0	0.289	0.397	0.0727	31.73	9.47	18.54	6.67	48.03	23.74	0.37	0.156%
3	49.6	0.338	0.441	0.0099	15.17	0.92	5.74	0.13	20.93	8.02	0.38	0.174%
4	445.1	0.323	0.235	0.197	125.4	70.0	126.1	75.0	87.3	104.7	0.42	0.154%
5*	30.7	1.4	2.3	0.15	6.67	1.62	12.22	-0.47	15.33	16.97	1.62	0.331%

Case 1-3: Thoracic aortas of mongrel dogs, Fung's model parameters are taken from Zhou and Fung [16].

Case 4: Thoracic aorta of Wistar rat, Fung's model parameters are taken from Zhao et al [17].

Case 5: Intact left anterior descending (LAD) of porcine, Fung's model parameters are taken from Pandit et al [18], where $E_{\theta\theta}^*=0.82$, $E_{zz}^*=0.47$. Bilinear model is fitted for $0.105 < E_{\theta\theta} < 0.8$ and $0.105 < E_{zz} < 0.5$.

* Negative constant appears because of three-dimensional loading in experiments.

Table 2
Bilinear model with averaged parameters ($c_{12}=c_{21}$, $b_{12}=b_{21}$)

Case	c_{11} (kPa)	c_{12} (kPa)	b_{12} (kPa)	c_{21} (kPa)	c_{22} (kPa)	b_{21} (kPa)	n	RLSE
1	33.01	7.26	22.92	7.26	52.24	22.92	0.41	0.278%
2	31.73	8.07	18.54	8.07	48.03	23.74	0.37	0.240%
3	15.17	0.53	6.88	0.53	20.93	6.88	0.38	0.307%
4	125.4	72.5	115.4	72.5	87.3	115.4	0.42	0.291%

Table 3
Bilinear model elastic moduli when the nonlinearity parameter n is varied

Case	c_{11} (kPa)	c_{12} (kPa)	b_{12} (kPa)	c_{21} (kPa)	c_{22} (kPa)	b_{21} (kPa)	n	RLSE
1a	34.34	7.54	22.92	5.21	52.56	29.39	0.36	0.193%
1b	31.70	10.31	17.04	5.79	51.88	23.02	0.46	0.187%
2a	32.99	8.12	21.56	6.44	48.27	26.90	0.32	0.182%
2b	30.49	10.71	15.89	6.85	47.74	20.91	0.42	0.170%
3a	15.82	0.26	6.82	-0.077	21.19	9.06	0.33	0.195%
3b	14.53	1.52	4.79	0.32	20.67	7.08	0.43	0.194%
4a	129.8	64.3	143.3	77.1	84.1	118.4	0.37	0.189%
4b	121.0	75.3	110.8	72.9	90.1	92.4	0.47	0.184%