Born iterative reconstruction using perturbed-phase field estimates

Jeffrey P. Astheimer

Department of Electrical and Computer Engineering, University of Rochester, Rochester, New York 14627

Robert C. Waag

Department of Electrical and Computer Engineering and Department of Imaging Sciences, University of Rochester, Rochester, New York 14627

(Received 24 January 2008; revised 16 June 2008; accepted 23 June 2008)

A method of image reconstruction from scattering measurements for use in ultrasonic imaging is presented. The method employs distorted-wave Born iteration but does not require using a forward-problem solver or solving large systems of equations. These calculations are avoided by limiting intermediate estimates of medium variations to smooth functions in which the propagated fields can be approximated by phase perturbations derived from variations in a geometric path along rays. The reconstruction itself is formed by a modification of the filtered-backpropagation formula that includes correction terms to account for propagation through an estimated background. Numerical studies that validate the method for parameter ranges of interest in medical applications are presented. The efficiency of this method offers the possibility of real-time imaging from scattering measurements. © 2008 Acoustical Society of America. [DOI: 10.1121/1.2967482]

PACS number(s): 43.80.Ev, 43.60.Pt, 43.35.Wa, 43.80.Qf [TDM]

Pages: 2353–2363

I. INTRODUCTION

In medical ultrasound, reconstruction of sound speed and density variations from scattering measurements promises higher resolution than the usual *b*-scan images that are produced by pulse-echo scanning. In fact, uniqueness results^{1,2} imply that these medium parameters are completely determined by measurements of monochromatic scattering. However, an image resolution that improves on the diffraction limit of the apparatus (sometimes called super-resolution) relies on the nonlinear relationship that exists between medium variations and scattering and is unraveled by inverting a sequence of linearizations. The first linear reconstruction has a resolution determined by the *f*-number for the apparatus. However, once an estimate of the medium is available, attention can focus on deviations from the estimate, and these deviations can be imaged using a revised linear model that incorporates the estimate. In effect, the initial estimate of the medium becomes part of the imaging apparatus, and the *f*-number of this new “apparatus” is smaller than the *f*-number of the original.

The Born approximation is the basis for the linear models used in most iterative reconstruction methods. Examples of such methods are the algebraic technique described in Ref. 3, the sinc-basis moment method discussed in Ref. 4, and the eigenfunction expansion methods detailed in Ref. 5–7. One reason for continuing to pursue new approaches is the high cost in computation time involved in forming nonlinear solutions. Distorted-wave Born estimates described in Ref. 8 are essentially Newton iterations toward the solution of the nonlinear system of Lippman–Schwinger equations⁹ that relate the medium variations to the scattering measurements. Each Newton step requires linearization of the equations in

the vicinity of the current estimate, and this linearization entails propagation of multiple incident fields through the estimated scattering medium.

To make the computation of propagation more efficient, the medium variations can be expanded in terms of a well-chosen finite-dimensional basis of functions. Products of retransmitted eigenfields of the scattering operator were proposed as such a basis in Ref. 5 because these fields concentrate in regions where the scattering is strong and because the resulting solution is characterized by an appealing constrained minimization property [Eqs. (15) and (16) of Ref. 5]. In Refs. 6 and 7, the cost of solving forward problems was eliminated by using estimates of the background medium that were homogeneous cylinders because analytic expressions exist for fields propagated through this object. Although a homogeneous cylinder may be a natural choice for breast imaging by a ring transducer system, other backgrounds (e.g., concentric cylinders) can also be used in similar circumstances.

This study takes a different tack. Instead of requiring the background estimates to have explicit analytic solutions for propagated fields, the estimates are required to be sufficiently smooth so that the propagated fields can be approximated using phase perturbations found by integrating the variations in acoustic path length along rays. This approach is limited to the use of scattering measurements from plane-wave illuminations rather than the eigenfield illuminations used in Refs. 5–7. Because propagated fields are only approximated, modest inaccuracies are introduced in the reconstructions. However, these inaccuracies are offset by significant efficiencies that accrue from the approximations. For example, solving the large systems of linear equations that usually appear in Newton or Born iterations is not necessary. Thus, while the individual iterations are not quite as accurate, the

overall efficiency of the iterations is better and has the improved possibility of being implemented in a way that permits real-time imaging.

Born iterations for large-scale backgrounds can be very specific. For example, if preliminary measurements identify the background as a homogeneous region or as multiple homogeneous regions with prescribed boundaries, then initial iterations can concentrate on determining the constant medium parameters within these regions and can defer a detailed resolution until the final steps. In such cases, the question of convergence is specific to the geometry of the background. This work focuses on an efficient reconstruction of fine details after the background parameters have been firmly established.

The experimental setting for which the methods of this paper are described is the ring transducer system described in Ref. 10. This system consists of an array of transducer elements that extend around the full circumference of a circle and that can be assigned independent complex amplitudes for illumination as well as independent complex weights for accumulating received measurements. The scattering object is placed in the center of the ring and is imaged in two dimensions. The transducer elements are sufficiently close that transmitted amplitudes can be assigned to produce very nearly monochromatic plane waves incident throughout the vicinity of the scattering object.

The theory that supports the reconstruction method is presented in Sec. II. Section III describes numerical experiments that validate the approach. The performance and accuracy of the method are discussed in Sec. IV. Section V gives conclusions from the study. In the Appendix, the exact field that results from propagating an incident plane wave through a homogeneous cylinder is compared with the approximate field that is obtained by phase perturbation for cylinder parameters matching those of the cylindrical object studied in Ref. 7.

II. THEORY

A. Estimation of medium variations

Let $e(\mathbf{x}, \theta_n) = e^{iku(\theta_n)\cdot\mathbf{x}}$ ($n=1, 2, \dots, M$) be monochromatic plane waves with spatial-frequency magnitude k traveling in directions $\theta_1, \dots, \theta_M$ that are evenly distributed around the unit circle so that each $e_n(\mathbf{x})$ satisfies the homogeneous Helmholtz equation

$$(\Delta + k^2)e(\mathbf{x}, \theta_n) = 0. \quad (1)$$

Then, for each n , let $u_B(\mathbf{x}, \theta_n)$ and $u(\mathbf{x}, \theta_n)$ be the solutions to the equations

$$(\Delta + k^2)u_B(\mathbf{x}, \theta_n) = k^2 u_B(\mathbf{x}, \theta_n) \eta_B(\mathbf{x}) \quad (2)$$

and

$$(\Delta + k^2)u(\mathbf{x}, \theta_n) = k^2 u(\mathbf{x}, \theta_n) [\eta_B(\mathbf{x}) + \Delta \eta_B(\mathbf{x})], \quad (3)$$

where $\eta(\mathbf{x}) = \eta_B(\mathbf{x}) + \Delta \eta_B(\mathbf{x})$ is a decomposition of the unknown medium variations as the sum of an assumed background term and a deviation term and where $u_B(\mathbf{x}, \theta_n)$ and $u(\mathbf{x}, \theta_n)$ are both assumed to have a nonradiating component equal to $e(\mathbf{x}, \theta_n)$. All three medium variation terms are as-

sumed to vanish outside a bounded region Ω . The fields $u_B(\mathbf{x}, \theta_n)$ and $u(\mathbf{x}, \theta_n)$ are total fields that result from $\eta_B(\mathbf{x})$ and $\eta(\mathbf{x})$ by applying the incident field $e(\mathbf{x}, \theta_n)$ to the scattering region Ω , respectively.

Now subtract Eq. (1) from Eq. (2) to obtain

$$(\Delta + k^2)[u_B(\mathbf{x}, \theta_n) - e(\mathbf{x}, \theta_n)] = k^2 u_B(\mathbf{x}, \theta_n) \eta_B(\mathbf{x}) \quad (4)$$

and also subtract Eq. (2) from Eq. (3) to obtain

$$\begin{aligned} (\Delta + k^2[1 - \eta_B(\mathbf{x})])[u(\mathbf{x}, \theta_n) - u_B(\mathbf{x}, \theta_n)] \\ = k^2 u(\mathbf{x}, \theta_n) \Delta \eta_B(\mathbf{x}). \end{aligned} \quad (5)$$

The assumptions made earlier regarding the asymptotic behavior of fields $u_B(\mathbf{x}, \theta_n)$ and $u(\mathbf{x}, \theta_n)$ imply that $u_B(\mathbf{x}, \theta_n) - e(\mathbf{x}, \theta_n)$ and $u(\mathbf{x}, \theta_n) - u_B(\mathbf{x}, \theta_n)$ satisfy the Sommerfeld radiation conditions, so integrating both sides of Eq. (4) against the radiating Green's function $G_0(\mathbf{x}, \mathbf{y})$ for the operator $(\Delta + k^2)$ gives

$$u_B(\mathbf{x}, \theta_n) - e(\mathbf{x}, \theta_n) = k^2 \iint u_B(\mathbf{y}, \theta_n) G_0(\mathbf{x}, \mathbf{y}) \eta_B(\mathbf{y}) d^2 \mathbf{y} \quad (6)$$

and integrating both sides of Eq. (5) against the radiating Green's function $G_B(\mathbf{x}, \mathbf{y})$ for the operator $(\Delta + k^2[1 - \eta_B(\mathbf{x})])$ gives

$$u(\mathbf{x}, \theta_n) - u_B(\mathbf{x}, \theta_n) = k^2 \iint u(\mathbf{y}, \theta_n) G_B(\mathbf{x}, \mathbf{y}) \Delta \eta_B(\mathbf{y}) d^2 \mathbf{y}. \quad (7)$$

The left side of Eq. (6) is the change in the field at $\mathbf{x}$ that is caused by the introduction of the medium variations $\eta_B(\mathbf{x})$ and will be denoted as the scattered field $u_B^s(\mathbf{x})$. Similarly, the left side of Eq. (7) is the further change in the field at $\mathbf{x}$ that is caused by the addition of the medium variations $\Delta \eta_B(\mathbf{x})$ and will be denoted as the scattered field $u_\Delta^s(\mathbf{x})$. Using this notation, Eqs. (6) and (7) become

$$u_B^s(\mathbf{x}, \theta_n) = k^2 \iint u_B(\mathbf{y}, \theta_n) G_0(\mathbf{x}, \mathbf{y}) \eta_B(\mathbf{y}) d^2 \mathbf{y} \quad (8)$$

and

$$u_\Delta^s(\mathbf{x}, \theta_n) = k^2 \iint u(\mathbf{y}, \theta_n) G_B(\mathbf{x}, \mathbf{y}) \Delta \eta_B(\mathbf{y}) d^2 \mathbf{y}, \quad (9)$$

respectively.

Equations (8) and (9) give expressions for the scattered fields at location $\mathbf{x}$, but scattering measurements are formed as weighted accumulations of field values across the array of transducer elements at locations $\mathbf{x}_1, \dots, \mathbf{x}_N$ as follows:

$$\begin{aligned} \sum_{i=1}^N \alpha_i u_B^s(\mathbf{x}_i, \theta_n) &= k^2 \iint u_B(\mathbf{y}, \theta_n) \\ &\times \left[\sum_{i=1}^N \alpha_i G_0(\mathbf{x}_i, \mathbf{y}) \right] \eta_B(\mathbf{y}) d^2 \mathbf{y} \end{aligned} \quad (10)$$

and

$$\sum_{i=1}^N \alpha_i u_{\Delta}^s(\mathbf{x}_i, \theta_n) = k^2 \iint u(\mathbf{y}, \theta_n) \times \left[\sum_{i=1}^N \alpha_i G_B(\mathbf{x}_i, \mathbf{y}) \right] \Delta \eta_B(\mathbf{y}) d^2 \mathbf{y}. \quad (11)$$

The weights $\alpha_1, \dots, \alpha_N$ may be chosen in different ways to shape different receiver sensitivity patterns for the bracketed sum inside the integral in Eq. (10). The weights are chosen here to form the M sensitivity patterns

$$\sum_{i=1}^N \alpha_i^{(n)} G_0(\mathbf{x}_i, \mathbf{y}) = e(\mathbf{y}, \theta_n), \quad n = 1, 2, \dots, M \quad (12)$$

that mirror the M incident fields. Reciprocity then implies that the bracketed term in Eq. (11) produces the sensitivity patterns

$$\sum_{i=1}^N \alpha_i^{(n)} G_B(\mathbf{x}_i, \mathbf{y}) = u_B(\mathbf{y}, \theta_n), \quad n = 1, 2, \dots, M. \quad (13)$$

Pairing each incident field with each receiver sensitivity pattern results in the M^2 set of measurements given by

$$M_{nm}^B = k^2 \iint u_B(\mathbf{y}, \theta_n) e(\mathbf{y}, \theta_m) \eta_B(\mathbf{y}) d^2 \mathbf{y} \quad (14)$$

and also the M^2 measurements given by

$$M_{nm}^{\Delta} = k^2 \iint u(\mathbf{y}, \theta_n) u_B(\mathbf{y}, \theta_m) \Delta \eta_B(\mathbf{y}) d^2 \mathbf{y}. \quad (15)$$

The right sides of Eqs. (14) and (15) express the scattering measurements in terms of the medium variations $\eta_B(\mathbf{y})$ and $\Delta \eta_B(\mathbf{y})$, respectively. However, these relationships are non-linear because of the implicit dependency of the fields $u_B(\mathbf{y}, \theta_n)$ and $u(\mathbf{y}, \theta_n)$ on $\eta_B(\mathbf{y})$ and $\Delta \eta_B(\mathbf{y})$, respectively. Equations (14) and (15) can be linearized by invoking the Born approximation that replaces $u_B(\mathbf{y}, \theta_n)$ in Eq. (14) with the homogeneous field $e(\mathbf{y}, \theta_n)$ and replaces $u(\mathbf{y}, \theta_n)$ in Eq. (15) with the field $u_B(\mathbf{y}, \theta_n)$ that propagates through the background variations. This gives the equations

$$M_{nm}^B = k^2 \iint e(\mathbf{y}, \theta_n) e(\mathbf{y}, \theta_m) \eta_B(\mathbf{y}) d^2 \mathbf{y} \quad (16)$$

and

$$M_{nm}^{\Delta} = k^2 \iint u_B(\mathbf{y}, \theta_n) u_B(\mathbf{y}, \theta_m) \Delta \eta_B(\mathbf{y}) d^2 \mathbf{y}. \quad (17)$$

Equation (17) is a general relationship between scattering measurements and medium variations that will be used for reconstruction. Equation (16) that applies to the reconstruction of medium variations in an empty background is a special case of Eq. (17) because if $\eta_B(\mathbf{x})=0$, then the terms $u_B(\mathbf{y}, \theta_n)$ and $u_B(\mathbf{y}, \theta_m)$ in the integrand of Eq. (17) become $e(\mathbf{y}, \theta_n)$ and $e(\mathbf{y}, \theta_m)$, respectively. Thus, Eq. (16) is used for an initial estimate and Eq. (17) is used for successive refinements.

Equations (16) and (17) can be solved for the medium variations by estimating a pseudoinverse for the integral kernel given by the imaginary part of the Green's function for

the medium.¹¹ A simpler approach used here is to adopt specific finite-dimensional expansions for $\eta_B(\mathbf{y})$ and $\Delta \eta_B(\mathbf{y})$ and the forms

$$\eta_B(\mathbf{y}) = \sum_{i \leq j=1}^M Q_{ij}^B \overline{e(\mathbf{x}, \theta_j)} e(\mathbf{x}, \theta_k) \quad (18)$$

and

$$\Delta \eta_B(\mathbf{y}) = \sum_{i \leq j=1}^M Q_{ij}^{\Delta} \overline{u_B(\mathbf{x}, \theta_j)} u_B(\mathbf{x}, \theta_k) \quad (19)$$

suggested by the inner products in Eqs. (16) and (17) that define the measurements.^a The limitation $i \leq j$ for the summation indices is imposed to avoid redundancy in the field products. Substituting these expansions into Eqs. (16) and (17) gives

$$M_{nm}^B = k^2 \sum_{i \leq j=1}^M \left[\iint e(\mathbf{y}, \theta_n) e(\mathbf{y}, \theta_m) \times \overline{e(\mathbf{x}, \theta_j)} e(\mathbf{x}, \theta_k) d^2 \mathbf{y} \right] Q_{ij}^B \quad (20)$$

and

$$M_{nm}^{\Delta} = k^2 \sum_{i \leq j=1}^M \left[\iint u_B(\mathbf{y}, \theta_n) u_B(\mathbf{y}, \theta_m) \times \overline{u_B(\mathbf{x}, \theta_j)} u_B(\mathbf{x}, \theta_k) d^2 \mathbf{y} \right] Q_{ij}^{\Delta}, \quad (21)$$

where the measurement indices are also limited by the constraint $n \leq m$ to avoid redundancy. This yields a system of $M(M+1)/2$ linear equations in the same number of unknowns. Solving these equations provides values for the Q_{ij}^B and Q_{ij}^{Δ} coefficients that can be used in Eqs. (18) and (19) to estimate the medium variations. The estimate given by Eq. (18) may be described as a Born estimate of the medium variations $\eta_B(\mathbf{x})$ relative to a homogeneous background, while the estimate given by Eq. (19) may be described as a distorted-wave Born estimate of the deviations $\Delta \eta_B(\mathbf{x})$ of the medium variations from the background $\eta_B(\mathbf{x})$.

B. Explicit solutions

The above methods for the estimates of medium variations from scattering measurements require solving the large matrix equations given in Eqs. (20) and (21). However, some modest approximations lead to analytic solutions and eliminate the need for lengthy numerical inversions. Equation (20) is the simpler of the two equations and serves as a model for the calculation.

Substituting exponential expressions for the e fields in the double integral in Eq. (20) gives

$$\begin{aligned} & \int_{\Omega} e^{jk[\mathbf{u}(\theta_n) + \mathbf{u}(\theta_m) - \mathbf{u}(\theta_j) - \mathbf{u}(\theta_k)] \cdot \mathbf{x}} d^2 \mathbf{x} \\ &= \hat{\chi}_{\Omega}(k[\mathbf{u}(\theta_j) + \mathbf{u}(\theta_k) - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)]), \end{aligned} \quad (22)$$

where $\chi_{\Omega}(\mathbf{x})$ is the characteristic function of the scattering volume, i.e., $\chi_{\Omega}(\mathbf{x})=1$ when $\mathbf{x} \in \Omega$ and is 0 otherwise, and

$\hat{\chi}_\Omega(\boldsymbol{\nu})$ denotes the Fourier transform of $\chi_\Omega(\mathbf{x})$. Using the right side of Eq. (22) in place of the integral in Eq. (20) gives

$$M_{nm}^B = k^2 \sum_{i \leq j=1}^M \hat{\chi}_\Omega(k[\mathbf{u}(\theta_i) + \mathbf{u}(\theta_j) - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)]) Q_{ij}^B. \quad (23)$$

Since the size of the scattering volume Ω is large relative to the size of the local variations and since the Q_{ij}^B coefficients in Eq. (18) act as Fourier coefficients for the local variations, the Q_{ij}^B coefficients are nearly constant over the range of i and j indices where $\hat{\chi}_\Omega(k[\mathbf{u}(\theta_i) + \mathbf{u}(\theta_j) - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)])$ is appreciable. Hence, Eq. (23) can be approximated by the diagonal system of equations

$$M_{nm}^B = \left[k^2 \sum_{i \leq j=1}^M \hat{\chi}_\Omega \times (k[\mathbf{u}(\theta_i) + \mathbf{u}(\theta_j) - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)]) \right] Q_{nm}^B \quad (24)$$

with solution

$$Q_{nm}^B = M_{nm}^B / \left[k^2 \sum_{i \leq j=1}^M \hat{\chi}_\Omega \times (k[\mathbf{u}(\theta_i) + \mathbf{u}(\theta_j) - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)]) \right]. \quad (25)$$

As Ω grows larger, this result becomes increasingly accurate. An even more explicit expression for the denominator may be obtained by interpreting the sum as a Riemann approximation for the integral

$$\frac{1}{2} \left(\frac{M}{2\pi} \right)^2 \int_0^{2\pi} \int_0^{2\pi} \hat{\chi}_\Omega \times (k[\mathbf{u}(\theta') + \mathbf{u}(\theta'') - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)]) d\theta' d\theta''. \quad (26)$$

The $1/2$ factor that appears outside the integral is included to compensate for extending the limit of the θ' integral to the full 0 to 2π range rather than setting the upper limit to θ'' , which is indicated by the limit $i \leq j$ of the j index in the sum. Applying the change in variables

$$\begin{aligned} \nu_x &= \cos \theta' + \cos \theta'' \\ \nu_y &= \sin \theta' + \sin \theta'', \end{aligned} \quad (27)$$

with Jacobian

$$\begin{vmatrix} \partial_{\theta'} \nu_x & \partial_{\theta''} \nu_x \\ \partial_{\theta'} \nu_y & \partial_{\theta''} \nu_y \end{vmatrix} = \begin{vmatrix} -\sin \theta' & -\sin \theta'' \\ \cos \theta' & \cos \theta'' \end{vmatrix} = |\sin(\theta' - \theta'')| \quad (28)$$

allows Eq. (26) to be rewritten as

$$\frac{1}{2} \left(\frac{M}{2\pi} \right)^2 \int \hat{\chi}_\Omega(k[\boldsymbol{\nu} - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)]) \frac{d^2 \boldsymbol{\nu}}{|\sin(\theta' - \theta'')|}. \quad (29)$$

However, because $|\sin(\theta' - \theta'')| = |\sin(\theta_n - \theta_m)|$, where $\hat{\chi}_\Omega(k[\boldsymbol{\nu} - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)])$ is appreciable, Eq. (29) simplifies to

$$\begin{aligned} & \frac{(M/2\pi)^2}{2|\sin(\theta_n - \theta_m)|} \int \hat{\chi}_\Omega(k[\boldsymbol{\nu} - \mathbf{u}(\theta_n) - \mathbf{u}(\theta_m)]) d^2 \boldsymbol{\nu} \\ &= \frac{M^2}{2k^2 |\sin(\theta_n - \theta_m)|}, \end{aligned} \quad (30)$$

where the integral is evaluated as $4\pi^2 \chi_\Omega(0)/k^2 = 4\pi^2/k^2$. Substituting the right side of Eq. (30) for the denominator of Eq. (25) then gives

$$Q_{nm}^B = (2/M^2) |\sin(\theta_n - \theta_m)| M_{nm}^B, \quad (31)$$

and substituting the coefficient values given by Eq. (31) into the expansion in Eq. (18) gives

$$\eta_B(\mathbf{x}) = (2/M)^2 \sum_{n \leq m=1}^M |\sin(\theta_n - \theta_m)| M_{nm}^B e^{-jk[\mathbf{u}(\theta_n) + \mathbf{u}(\theta_m)] \cdot \mathbf{x}}. \quad (32)$$

Equation (32) is the well-known filtered-backpropagation reconstruction formula¹² that gives an explicit solution for the medium variations in terms of the scattering measurements. The above derivation is included here to provide a framework for establishing a similar expression for $\Delta \eta_B(\mathbf{x})$ based on the distorted-wave Born iteration to which attention is now given.

To simplify the integral in Eq. (21), the u_B fields must be given a more explicit form. This requires making assumptions about the behavior of $\eta_B(\mathbf{x})$. If $\eta_B(\mathbf{x})$ is slowly varying, then the u_B fields can be approximated inside Ω by plane waves with phase corrections that compensate for changes in the speed of propagation along rays. Thus,

$$u_B(\mathbf{x}, \theta_n) \approx \tilde{u}_B(\mathbf{x}, \theta_n) = e^{jk[\mathbf{u}(\theta_n) \cdot \mathbf{x} + \delta(\mathbf{x}, \theta_n)]}, \quad (33)$$

where the $\sim$ indicates approximation and

$$\delta(\mathbf{x}, \theta_n) = \int_0^\infty \eta_B(\mathbf{x} - \beta \mathbf{u}(\theta_n)) d\beta. \quad (34)$$

These phase corrections are easily calculated from the assumed background $\eta_B(\mathbf{x})$.

Substituting these field expressions into the integral that forms the matrix in Eq. (22) gives

$$\begin{aligned} & \int_\Omega e^{jk[\mathbf{u}(\theta_n) + \mathbf{u}(\theta_m) - \mathbf{u}(\theta_i) - \mathbf{u}(\theta_j)] \cdot \mathbf{x}} \\ & \times e^{jk[\delta(\mathbf{x}, \theta_n) + \delta(\mathbf{x}, \theta_m) - \delta(\mathbf{x}, \theta_i) - \delta(\mathbf{x}, \theta_j)]} d^2 \mathbf{x}, \end{aligned} \quad (35)$$

and oscillations in the left term produce cancellations that null the integral except when the direction angles θ_n and θ_m are close to the angles θ_i and θ_j . This implies that only elements near the diagonal of the matrix are appreciable, and the phase correction terms for those elements will nearly cancel. Thus, the integral in Eq. (35) can be replaced with the right side of Eq. (22). This approximation gives the system of equations in Eq. (21) the same form as Eq. (20). Hence, the same steps can be used to arrive at the solution. This yields

$$Q_{nm}^{\Delta} = (2/M)^2 |\sin(\theta_n - \theta_m)| M_{nm}^{\Delta}, \quad (36)$$

and substituting these coefficients and u_B field approximations into the expansion in Eq. (19) gives

$$\Delta \eta_B(\mathbf{x}) = (2/M^2) \sum_{n \leq m=1}^M |\sin(\theta_n - \theta_m)| M_{nm}^{\Delta} e^{-jk[\mathbf{u}(\theta_n) + \mathbf{u}(\theta_m)] \cdot \mathbf{x}} e^{-jk[\delta(\mathbf{x}, \theta_n) + \delta(\mathbf{x}, \theta_m)]}. \quad (37)$$

This startlingly simple result indicates that distorted-wave Born iterations can also be estimated using the filtered-backpropagation formula for a free space reconstruction provided that the expansion is adjusted with the appropriate phase correction terms.^b

C. Incremental scattering measurements

Equation (37) depends on the measurements M_{nm}^{Δ} that are defined in Eq. (10) as the weighted sums,

$$M_{nm}^{\Delta} = \sum_{i=1}^M \alpha_i^{(m)} u_{\Delta}^s(\mathbf{x}_i, \theta_n), \quad (38)$$

where the $\alpha_i^{(m)}$ weights are chosen in accordance with Eq. (12) and the scattered field $u_{\Delta}^s(\mathbf{x}_i, \theta_n)$ is

$$u_{\Delta}^s(\mathbf{x}_i, \theta_n) = u(\mathbf{x}, \theta_n) - u_B(\mathbf{x}, \theta_n). \quad (39)$$

The difference on the right side of Eq. (39) may be further expanded using Eq. (6) to obtain

$$u_{\Delta}^s(\mathbf{x}_i, \theta_n) = u(\mathbf{x}_i, \theta_n) - e(\mathbf{x}_i, \theta_n) - k^2 \iint u_B(\mathbf{y}, \theta_n) G_0(\mathbf{x}_i, \mathbf{y}) \eta_B(\mathbf{y}) d^2 \mathbf{y}, \quad (40)$$

and substituting the right side of this expansion in place of $u_{\Delta}^s(\mathbf{x}_i, \theta_n)$ in Eq. (38) gives

$$M_{nm}^{\Delta} = \left[\sum_{i=1}^N \alpha_i^{(m)} u(\mathbf{x}_i, \theta_n) \right] - \left[\sum_{i=1}^N \alpha_i^{(m)} e(\mathbf{x}_i, \theta_n) \right] - k^2 \iint u_B(\mathbf{y}, \theta_n) e(\mathbf{y}, \theta_m) \eta_B(\mathbf{y}) d^2 \mathbf{y}. \quad (41)$$

The first term on the right side of Eq. (41) represents an explicit field measurement for a scattering experiment in which the incident field is $e(\mathbf{x}, \theta_n)$, and the second term is the same measurement in an experiment with the same incident field but without the scattering medium. The third term, however, represents scattering due to the assumed background and cannot be obtained experimentally. This term can, however, be evaluated by approximating the field $u_B(\mathbf{y}, \theta_m)$ by $e^{jk[\mathbf{u}(\theta_m) \cdot \mathbf{y} + \delta(\mathbf{y}, \theta_m)]}$ as in Eq. (34) and replacing the $e(\mathbf{y}, \theta_n)$ term with the exponential $e^{jk\mathbf{u}(\theta_n) \cdot \mathbf{y}}$ to give

$$k^2 \iint u_B(\mathbf{y}, \theta_n) e(\mathbf{y}, \theta_m) \eta_B(\mathbf{y}) d^2 \mathbf{y} = k^2 \iint e^{jk[\mathbf{u}(\theta_n) + \mathbf{u}(\theta_m)] \cdot \mathbf{y}} e^{jk\delta(\mathbf{y}, \theta_n)} \eta_B(\mathbf{y}) d^2 \mathbf{y}. \quad (42)$$

The right side of Eq. (42) is a spatial Fourier transform of

$e^{jk\delta(\mathbf{y}, \theta_n)} \eta_B(\mathbf{y})$, and the values for different θ_m ($m = 1, 2, \dots, M$) can be efficiently extracted by interpolating values from a single two-dimensional fast Fourier transform.

III. NUMERICAL STUDIES

The reconstruction procedure described above uses the far-field measurements M_{nm} for the total field, together with an assumed background $\eta_B(\mathbf{x})$, to form a correction $\Delta \eta_B(\mathbf{x})$ to the medium variations. The steps in this process are as follows:

- (1) For each incident direction θ_n ($n = 1, \dots, M$), the incident plane wave $e^{jk_0 \mathbf{u}(\theta_n) \cdot \mathbf{x}}$ is propagated through the background $\eta_B(\mathbf{x})$ to obtain the field $u_B(\mathbf{x}, \theta_n)$.
- (2) The far-field pattern M_{nm}^B for the background $\eta_B(\mathbf{x})$ is computed.
- (3) The correction $\Delta \eta_B(\mathbf{x})$ is estimated using the modified filtered-backpropagation expansion

$$\widetilde{\Delta \eta_B}(\mathbf{x}) = (2/M^2) \sum_{m \leq n=1}^M |\sin(\theta_n - \theta_m)| (M_{nm} - M_{nm}^B) \overline{u_B(\mathbf{x}, \theta_n) u_B(\mathbf{x}, \theta_m)}. \quad (43)$$

Equation (43) represents a significant improvement in efficiency over the more complete reconstructions carried out in Refs. 5–7 because the calculation of the inner products $\langle u_B(\mathbf{x}, \theta_i) u_B(\mathbf{x}, \theta_j), \overline{u_B(\mathbf{x}, \theta_n) u_B(\mathbf{x}, \theta_m)} \rangle$ and the inversion of the resulting matrix are unnecessary, but a wavy overbar has been added to the symbol for the correction potential to acknowledge that Eq. (43) is an approximation.

To improve the efficiency of the calculation, two additional approximations are considered. First, the u_B fields in Eq. (43) are replaced with the approximate fields given by Eq. (33). This results in the perturbed-phase filtered-backpropagation expansion

$$\widetilde{\widetilde{\Delta \eta_B}}(\mathbf{x}) = (2/M^2) \sum_{m \leq n=1}^M |\sin(\theta_n - \theta_m)| \times (M_{nm} - M_{nm}^B) \overline{\widetilde{u_B}(\mathbf{x}, \theta_n) \widetilde{u_B}(\mathbf{x}, \theta_m)}, \quad (44)$$

where an additional wavy overbar has been added to the term on the left side to indicate the additional level of approximation, and the use of approximate fields on the right side is also reflected by wavy overbars. This expression retains exact values for the far-field pattern M_{nm}^B . A second approximation entails substituting approximate values $\widetilde{M}_{nm}^B$ that are obtained by using $\widetilde{u_B}$ fields in place of u_B fields in the right side of Eq. (41). This gives the third expansion

$$\widetilde{\widetilde{\widetilde{\Delta \eta_B}}}(\mathbf{x}) = (2/M^2) \sum_{m \leq n=1}^M |\sin(\theta_n - \theta_m)| \times (M_{nm} - \widetilde{M}_{nm}^B) \overline{\widetilde{u_B}(\mathbf{x}, \theta_n) \widetilde{u_B}(\mathbf{x}, \theta_m)}. \quad (45)$$

To determine the effectiveness of these estimates, studies were conducted using both simulated and measured data. In the first study, Eq. (43) was used to reconstruct medium variations $\Delta \eta_B(\mathbf{x})$ in an assumed background medium $\eta_B(\mathbf{x})$

FIG. 1. Radial profiles of reconstructed unit-amplitude point reflectors relative to an assumed background. (a) Filtered backpropagation in an empty background. (b) Modified filtered backpropagation, $\Delta\tilde{\eta}_B(\mathbf{x})$, for a background cylinder with a 24 mm radius and a sound speed of 1.574 mm/ μ s using Bessel-function expansions for the internal fields. (c) Modified filtered backpropagation, $\Delta\tilde{\eta}_B(\mathbf{x})$, for the same background cylinder except using phase perturbation to approximate the internal fields.

consisting of a homogeneous cylinder with a 48 mm diameter. The variations consisted of five unit-amplitude point reflectors that were positioned along a ray at radii that were spaced by one-fifth of the cylinder radius. The far-field measurements $M_{nm}^\Delta = M_{nm} - M_{nm}^B$ were computed numerically using the weak scattering formula in Eq. (17). Figures 1(a) and 1(b) are plots of the reconstruction $\Delta\tilde{\eta}_B(\mathbf{x})$ along the ray where the points are located. In Fig. 1(a), the sound speed for the background cylinder was chosen to be 1.509 mm/ μ s, which matched the sound speed for the ambient medium, so this plot illustrates the effectiveness of conventional filtered backpropagation in the absence of an assumed background. The sound speed for the background cylinder in Fig. 1(b) was 1.574 mm/ μ s, which is a high-contrast level relative to the ambient medium that is well beyond the range of variations that can be reconstructed using conventional filtered backpropagation. The same cylinder was used as the background for the reconstruction shown in Fig. 1(c) except that the estimate $\Delta\tilde{\eta}_B(\mathbf{x})$ given Eq. (44) was used instead of $\Delta\tilde{\eta}_B(\mathbf{x})$. These calculations were all performed at a frequency of 2.5 MHz for 1024 incident plane waves with directions that were spaced evenly around the unit circle. The reconstructions were expanded on a polar grid with 500 radii ranging from 0 to 24 mm and with an angular refinement of $2\pi/1024$ radians. Since the incremental scattering measurements used in this simulation were generated as weak linear effects and the reconstructions are also linear, the plots in Fig. 1 are linear responses to the unit-amplitude point reflectors.

Further calculations were carried out using far-field measurements that were obtained experimentally. These measurements were also used in Ref. 7 and are described in greater detail in that article. The measurements were made using a ring transducer system with 2048 transducer elements. Scattering was produced by a tissue-mimicking phan-

FIG. 2. Tissue-mimicking phantom used as a large-scale high-contrast scattering object for measurements.

tom with dimensions and parameters shown in Fig. 2. The phantom included two scatter-free cylinders and three 0.1 mm diameter nylon filaments.

The scattering measurements were used to produce four different reconstructions. An exact weak scattering result was found using an expansion of products of retransmitted eigenfields as described in Ref. 5. This result may also be interpreted as an expansion in the form of Eq. (19) where the Q_{nm} coefficients are found by solving the system of equations in Eq. (21). This result may be regarded as optimal in the least-squares sense described in Ref. 5 and, in that respect, may be regarded as the most accurate Born estimate. The other three reconstructions were produced using the approximate expansions given in Eqs. (43)–(45). However, since the background cylinder for these reconstructions was strongly attenuating, reciprocal fields were used instead of conjugate fields on the right hand sides of these equations. In each case, reconstructions were computed for 17 frequencies ranging from 1.4 to 2.66 MHz. These potentials were averaged together to produce multifrequency potentials that do not exhibit the oscillatory artifacts that appear in monochromatic potentials.

Impulse responses for each of these reconstructions were computed for a point reflector located midway between the center and the outer radius of the cylinder. Averaging monochromatic responses over frequency, as described above, resulted in a single frequency response for each reconstruction. The real parts of these impulse responses are very nearly circularly symmetric and are essentially independent of the reflector location. The imaginary parts are roughly antisymmetric along the ray from the center of the cylinder to the reflector; they vary with the position of the reflector and are much smaller than the real parts. Ignoring the effects of the imaginary components of the impulse responses allows the reconstructed potentials to be viewed as convolutions of the medium variations with the real parts of the impulse responses.

Frequency responses corresponding to the real parts of the impulse responses from each of the four different reconstructions are plotted in Fig. 3. The frequency responses for the approximation reconstructions are all essentially the

FIG. 3. Spatial-frequency responses corresponding to the real parts of frequency-averaged point-reflector reconstructions. (a) Complete inversion of the system of equations relating far-field measurements of weak scattering to medium variations. (b) Modified filtered backpropagation using Bessel expansions for the internal fields and the far-field pattern of the cylinder. (c) Modified filtered backpropagation using phase perturbation to approximate the internal fields. (d) Modified filtered backpropagation with the far-field pattern of the cylinder also approximated using the phase-perturbed plane waves.

same but differ from the response for the exact weak reconstruction. The large dc spikes in the frequency responses for the approximate reconstructions indicate that the asymptotic values of the impulse responses are nonzero. The flat region of the frequency responses for the approximate reconstructions only extends to $\pm 2\pi f_{\min}/c_1$, where $f_{\min} = 1.4$ MHz and c_1 is the sound speed inside the cylinder, while the flat region of the response for the exact weak solution extends to $\pm 2\pi f_{\min}/c_0$, where c_0 is the speed of sound in water. However, all the responses taper off to 0 at $\pm 2\pi f_{\max}/c_0$, where $f_{\max} = 2.66$ MHz.

Figure 4(a) is a gray-scale image of the exact weak reconstruction, and Figs. 4(b)–4(d) are gray-scale images, respectively, of the reconstructions $\Delta\tilde{\eta}_B(\mathbf{x})$, $\Delta\tilde{\eta}_B(\mathbf{x})$, and $\Delta\tilde{\eta}_B(\mathbf{x})$ given by Eqs. (43)–(45), respectively. Filters were applied to the three approximate reconstructions to scale their frequency responses (as shown in Fig. 3) to match the frequency response for the weak solution. The filtered images differ from one another and also from the exact weak reconstruction because the reconstructions are not completely translation invariant and cannot be completely characterized as convolutions. The Cartesian plot in Fig. 5 provides a more quantitative comparison of the reconstructions along a cross section of the cylinder that intersects two of the filaments and one of the scatter-free regions.

Measurements for these calculations were obtained from 1024 incident plane waves with directions that were spaced evenly around the unit circle. All the reconstructions were expanded on a polar grid with 500 radii ranging from 0 and 24 mm and with an angular refinement of $2\pi/1024$ radians. Cubic splines were used to interpolate the potentials onto a rectangular grid with 1024 points in both dimensions. The gray scale for all the images show values of $\Delta\eta_B(\mathbf{x})$ between -0.04 and $+0.04$ that correspond to sound speed variations between 1.54 and 1.60 mm/ μ s.

IV. DISCUSSION

The studies described above validate the reconstructions given by Eqs. (43)–(45) under appropriate circumstances. The reconstructed points in the inhomogeneous object reconstructions shown in Figs. 1(b) and 1(c) have sidelobes that decay more quickly than those in the homogeneous reconstruction in Fig. 1(a). This improvement is due to the blurring of the Ewald disk caused by dispersion in the direction of plane waves that propagate across the cylinder boundary. The fourth point in the radial profile shown in Fig. 1(c) is slightly depressed, reflecting the increased error in phase-perturbation field approximations toward the edge of the cylinder. The plots in Fig. 1 may be interpreted as illustrations of the system responses for the $\Delta\tilde{\eta}_B(\mathbf{x})$ and $\Delta\tilde{\eta}_B(\mathbf{x})$ reconstructions at different radii and are all very nearly the same as the $J_1(k\rho)/k\rho$ response for the homogeneous case.

Similar inferences are reached through an examination of the reconstructions of experimental data. The $\Delta\tilde{\eta}_B(\mathbf{x})$, $\Delta\tilde{\eta}_B(\mathbf{x})$, and $\Delta\tilde{\eta}_B(\mathbf{x})$ reconstructions shown in Figs. 4(b)–4(d), respectively, are all similar toward the center of the cylinder but perform differently near the cylinder boundary. In each case, the medium variations in the interior of the cylinder are resolved nearly as well as in the image obtained by a full inversion of the inner product matrix shown in Fig. 4(a). This outcome can be investigated by expressing the expansion of Eq. (43) in the integral form as

$$\widetilde{\Delta\eta_B(\mathbf{x})} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sin(\theta - \theta') M^\Delta(\theta, \theta') \times \overline{u_B(\mathbf{x}, \theta) u_B(\mathbf{x}, \theta')} d\theta' d\theta, \quad (46)$$

where the far-field measurements $M^\Delta(\theta, \theta')$ with continuous angular parameters are

$$M^\Delta(\theta, \theta') = k^2 \int_\Omega u_B(\mathbf{y}, \theta) u_B(\mathbf{y}, \theta') \Delta\eta_B(\mathbf{y}) d^2\mathbf{y}. \quad (47)$$

Substituting Eq. (47) into Eq. (46) and interchanging the order of integration give

$$\widetilde{\Delta\eta_B(\mathbf{x})} = \frac{k^2}{4\pi^2} \left[\int_0^{2\pi} \int_0^{2\pi} |\sin(\theta - \theta')| u_B(\mathbf{y}, \theta) u_B(\mathbf{y}, \theta') \times \overline{u_B(\mathbf{x}, \theta) u_B(\mathbf{x}, \theta')} d\theta' d\theta \right] \times \Delta\eta_B(\mathbf{y}) d^2\mathbf{y}. \quad (48)$$

The bracketed double integral on the right side of Eq. (48) is the system response function for the reconstruction process and takes the form of a blurred delta function $\delta(\mathbf{y} - \mathbf{x})$. When the distance between $\mathbf{x}$ and $\mathbf{y}$ is large, rapid oscillations in the u_B fields cause cancellations that null the double integral, so attention can focus on the values of the system response function for $\mathbf{x}$ and $\mathbf{y}$ values that are close to one another. Substituting the perturbed-phase plane wave field approximation

$$u_B(\mathbf{x}, \theta) = e^{jk[\mathbf{u}(\mathbf{x}, \theta) \cdot \mathbf{x} + \delta(\mathbf{x}, \theta)]} \quad (49)$$

for the u_B fields in Eq. (48) gives the expression

Exact Weak Scattering Solution

Modified Filtered Back Propagation

Perturbed-Phase Filtered Back Propagation Based on Approximate Far-Field Estimates

FIG. 4. Cross sections through the two filaments on the left side of reconstructions. (a) Complete inversion of the system of equations relating far-field measurements of weak scattering to medium variations. (b) Modified filtered backpropagation, $\Delta\tilde{\eta}_B(\mathbf{x})$, using Bessel expansions for the internal fields and the far-field pattern of the cylinder. (c) Modified filtered backpropagation, $\Delta\tilde{\eta}_B(\mathbf{x})$, using phase perturbation to approximate the internal fields. (d) Modified filtered backpropagation, $\Delta\tilde{\eta}_B(\mathbf{x})$, with the far-field pattern of the cylinder also approximated using the phase-perturbed plane waves.

$$\widetilde{\Delta\eta}_B(\mathbf{x}) = \frac{k^2}{4\pi^2} \left[\int_0^{2\pi} \int_0^{2\pi} |\sin(\theta - \theta')| e^{jk\{[\mathbf{u}(\theta) + \mathbf{u}(\theta')] \cdot (\mathbf{y} - \mathbf{x}) + [\delta(\mathbf{y}, \theta) - \delta(\mathbf{x}, \theta)] + [\delta(\mathbf{y}, \theta') - \delta(\mathbf{x}, \theta')]\}} d\theta' d\theta \right] \Delta\eta_B(\mathbf{y}) d^2\mathbf{y}, \quad (50)$$

in which the phase perturbations only occur as differences between values at neighboring points. Because the perturbations are slowly varying, these differences can be neglected, yielding the same system response function as in the homogeneous reconstruction. However, this calculation is only valid locally, i.e., in regions where the propagated fields can be accurately estimated by phase perturbation.

For attenuating backgrounds, the spatial frequency k is complex. In that case, reciprocal rather than conjugate fields

should be used in the expansions that represent the medium variations. Thus, reciprocal rather than conjugate $\mathbf{u}_B(\mathbf{x})$ terms should appear on the right hand sides of Eqs. (46) and (49). Use of reciprocal fields produces cancellation of the magnitude variations in the four field products in the integrand on the right hand side of Eq. (48). Equations (49) and (50) are valid for both attenuating and nonattenuating fields.

Equation (43) was derived earlier by making a diagonal approximation to the matrix of inner products in the linear

FIG. 5. Spatial-frequency responses corresponding to the real parts of frequency-averaged point-reflector reconstructions. (a) Complete inversion of the system of equations relating far-field measurements of weak scattering to medium variations. (b) Modified filtered backpropagation using Bessel expansions for the internal fields and the far-field pattern of the cylinder. (c) Modified filtered backpropagation using phase perturbation to approximate the internal fields. (d) Modified filtered backpropagation with the far-field pattern of the cylinder also approximated using the phase-perturbed plane waves.

system of equations that relate weak far-field scattering to medium variations. Thus, Eq. (43) should give a less accurate reconstruction than inversion of the original system of equations. However, the matrix of inner products in the more exact reconstruction produces small slowly varying oscillations that extend into the interior of the cylinder. This behavior is evidenced in Fig. 4(a) by the radial undulations that are superimposed on the medium variations. This effect is less prominent in the approximate reconstructions. However, reconstruction near the edge of the cylinder becomes progressively worse in Figs. 4(b)–4(d), which is consistent with the increased dependency on field approximations in the estimates $\Delta \tilde{\eta}_B(\mathbf{x})$, $\Delta \tilde{\eta}_B^B(\mathbf{x})$, and $\Delta \tilde{\eta}_B^{\tilde{\eta}}(\mathbf{x})$. The eight-point star pattern in the center of these reconstructions is an artifact caused by crosstalk in the data acquisition apparatus.

Figure 4(d) is particularly noteworthy because this reconstruction relies on phase-perturbation approximations for both the u_B fields and the M_{nm}^B far-field measurements. In the case of a cylindrical background, there is no advantage in using these estimates since the u_B fields and the far-field measurements M_{nm}^B can be readily calculated from Bessel-function expansions. However, this is not the case if the boundary is irregular. Recent studies¹³ have demonstrated that more complicated convex boundaries can be determined, *a priori*, from experimental measurements. The perturbed-phase filtered-backpropagation expansion $\Delta \tilde{\eta}_B^{\tilde{\eta}}(\mathbf{x})$ provides a way to make use of this information that avoids having to solve difficult and time consuming forward problems.

The estimates described above are incremental estimates of small-scale medium variations within a high-contrast background of large-scale objects. However, the monochromatic expressions $\Delta \tilde{\eta}_B(\mathbf{x})$, $\Delta \tilde{\eta}_B^B(\mathbf{x})$, and $\Delta \tilde{\eta}_B^{\tilde{\eta}}(\mathbf{x})$ are not ef-

fective for an iterative refinement of the background because small changes in the medium properties of large-scale objects can cause large errors in Born estimates as a result of phase deviations that accumulate in the incident and scattered waves as they propagate through the background. This difficulty can be overcome by forming estimates at multiple frequencies and obtaining phase corrections from the slopes of the phase at each reconstructed point with respect to frequency as described in Ref. 7. Use of these phase corrections led to a rapid convergence of the background parameters in Ref. 7. Another possibility that does not involve reconstruction is to select the background medium parameters to give the best least-squares fit of Eq. (42) to the measured far-field pattern. However, this procedure may find a local minimum if the initial guess is not sufficiently accurate.

The examples and discussion in this study have focused on an assumed background consisting of a homogeneous body, but the perturbed-phase filtered-backpropagation expansion is not limited to this application. In fact, better performance is expected from backgrounds that do not have abrupt discontinuities. The results presented here could be extended to include the use of assumed backgrounds that consist of homogeneous bodies with tapered edges and also the use of iterations that incorporate large-scale interior structures in the assumed background.

V. CONCLUSIONS

Image reconstruction from scattering measurements has the potential of achieving a better resolution than conventional *b*-scan images, but inverse scattering is an iterative process that can have high computational cost. The studies presented in this paper demonstrate that Born iterations can, under the right circumstances, be accomplished using an efficient variation of the filtered-backpropagation formula that does not require a solution to forward scattering problems or an inversion of large matrices. The conditions to which this approach is applicable are local. High-quality reconstruction can be realized in areas of the scattering volume where the fields of propagated plane waves are accurately approximated by phase perturbations. These requirements are satisfied, for example, when small medium variations are embedded in large-scale high-contrast background objects. The specific application that motivated this study is breast imaging by a ring transducer system, but the technique may also apply to other medical imaging applications that involve a resolution of detail within large anatomical features.

ACKNOWLEDGMENTS

David P. Duncan, Jin Jing, and Jason C. Tillett are thanked for their participation in discussions surrounding this study and also for their support in the development of the processing software. Michele M. Foster is thanked for her help in translating this article to a TeX format. This work was partially supported by NIH Grants Nos. HL 50855, CA 74050, and EB 00280 and by the University of Rochester Diagnostic Ultrasound Research Laboratory Industrial Associates.

APPENDIX

The approximate field given by Eq. (34) for plane-wave propagation through a cylinder with constant parameters is compared below with the exact field determined analytically. Let k_1 be the spatial-frequency magnitude that results from the medium parameters inside the cylinder, let k be the external spatial-frequency magnitude, and let the plane wave $u_i(\mathbf{x}) = e^{jkr \cos \theta}$ be the incident field that is applied to the cylinder. An internal field $u_1(\mathbf{x})$ and a scattered field $u_s(\mathbf{x})$ are sought to satisfy

$$\begin{aligned} \Delta u_1(\mathbf{x}) + k_1^2 u_1(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Omega, \\ \Delta u_s(\mathbf{x}) + k^2 u_s(\mathbf{x}) &= 0, \quad \mathbf{x} \in R^2 - \bar{\Omega} \text{ and } u_s \text{ radiating,} \end{aligned} \quad (\text{A1})$$

and also to satisfy the boundary conditions

$$\left. \begin{aligned} u_1(\mathbf{x}) &= u_s(\mathbf{x}) + u_i(\mathbf{x}) \\ (k_1/k) \partial_n u_1(\mathbf{x}) &= \partial_n u_s(\mathbf{x}) + \partial_n u_i(\mathbf{x}) \end{aligned} \right\} \quad \mathbf{x} \in \partial\Omega. \quad (\text{A2})$$

Expansions that comply with these conditions are

$$\begin{aligned} u_i(r, \theta) &= \sum_{n=-\infty}^{\infty} A_n J_n(kr) e^{jn\theta}, \\ \partial_n u_i(r, \theta) &= k \sum_{n=-\infty}^{\infty} A_n J'_n(kr) e^{jn\theta}, \\ u_s(r, \theta) &= \sum_{n=-\infty}^{\infty} D_n H_n^{(1)}(kr) e^{jn\theta}, \\ \partial_n u_s(r, \theta) &= k \sum_{n=-\infty}^{\infty} D_n H_n^{(1)'}(kr) e^{jn\theta}, \\ u_1(r, \theta) &= \sum_{n=-\infty}^{\infty} B_n J_n(k_1 r) e^{jn\theta}, \\ \partial_n u_1(r, \theta) &= k_1 \sum_{n=-\infty}^{\infty} B_n J'_n(k_1 r) e^{jn\theta}. \end{aligned} \quad (\text{A3})$$

The assignments $A_n = j^n$ follow from the expansion

$$e^{jkr \cos \theta} = \sum_{n=-\infty}^{\infty} j^n e^{jn\theta} J_n(kr), \quad (\text{A4})$$

and then values for the B_n and D_n coefficients follow from the application of the boundary conditions to the expansions in Eq. (A3), as described in Ref. 14.

The plots of the magnitude and phase of the interior field $u_1(\mathbf{x})$ are shown in Figs. 6(a) and 6(b). The radius of the cylinder for these computations is 24 mm, and the temporal frequency is 2.5 MHz. The sound speed inside the cylinder is 1.574 mm/ μ s, while the external sound speed is 1.509 mm/ μ s. The magnitude of the internal field is essentially 1 throughout most areas of the scattering volume and agrees with the magnitude of the phase-perturbed field given

FIG. 6. (a) Magnitude of a unit-amplitude plane wave propagating through a homogeneous cylinder. (b) Phase deviation of a plane wave propagating through a cylinder from a plane wave propagating homogeneously. (c) Error in estimating the phase of a plane wave propagating through a cylinder using an acoustic path length along rays. (d) Comparison of the exact far-field pattern of the cylinder with the estimated pattern given by Eq. (41).

by Eq. (33). However, two small areas of significant fluctuations exist near the edges of the cylinder grazed by the incident field, and two crescent-shaped regions of smaller fluctuations exist near the far edge of the cylinder. These smaller fluctuations are attributed to the interference of the forward propagating field with reflection from the far edge. Both effects would diminish if the edges of the cylinder were smooth rather than abrupt.

The plot in Fig. 6(c) is the residual phase fluctuation that remains after the phase perturbation of Eq. (34) has been deducted from the phase of $u_1(\mathbf{x})$ and demonstrates the close agreement in phase between the exact and approximate fields.

Figure 6(d) compares the far-field pattern of the cylinder obtained from the radial limit of the expansion for $u_s(\mathbf{x})$ given in Eq. (A3) with the approximate far-field pattern given in Eq. (41), which employs perturbed-phase field estimates. The absolute values of the patterns are normalized by the peak value of the pattern from the (A3) expansion and are plotted over a range of angles that extend 20° on either side of the forward-scattering direction.

^aFor strongly attenuating background potentials, the use of reciprocal fields rather than conjugate fields for the expansion in Eq. (19) can yield better results. The subsequent analysis applies to expansions of reciprocal fields without significant alteration.

^bDuring the review process, the authors learned that the reconstruction given by Eq. (37) has been published in a somewhat different form in Ref. 15 where the expansion is a time-domain reconstruction in which phase perturbations appear as time delays. The perspective used in Ref. 15 is easily translated to the monochromatic case that is considered here. From the view of Ref. 15, the sums in Eq. (37) combine plane-wave transmitted fields and receiver sensitivities to form a single compound measurement. The exponential terms act as geometric phase factors that focus the measurement at point $\mathbf{x}$. The phase perturbations that appear in the modified expansion given in Eq. (37) are focusing corrections that

compensate for the aberration caused by propagation through the assumed background.

- ¹A. I. Nachman, "Reconstructions from boundary measurements," *Ann. Math.* **128**, 531–576 (1988).
- ²A. I. Nachman, "Global uniqueness for a two-dimensional inverse boundary problem," *Ann. Math.* **143**, 71–96 (1996).
- ³C. Lu, J. Lin, W. Chew, and G. Otto, "Image reconstruction with acoustic measurement using distorted Born iteration method," *Ultrason. Imaging* **18**, 140–156 (1996).
- ⁴T. J. Cavicchi and W. D. O'Brien, "Numerical study of higher-order diffraction tomography via the sinc basis moment method," *Ultrason. Imaging* **11**, 42–74 (1989).
- ⁵T. D. Mast, A. I. Nachman, and R. C. Waag, "Focusing and imaging using eigenfunctions of the scattering operator," *J. Acoust. Soc. Am.* **102**, 715–725 (1997).
- ⁶F. Lin, A. I. Nachman, and R. C. Waag, "Quantitative imaging using a time-domain eigenfunction method," *J. Acoust. Soc. Am.* **108**, 899–912 (2000).
- ⁷R. C. Waag, F. Lin, T. K. Varslot, and J. P. Astheimer, "An eigenfunction method for reconstruction of large-scale and high-contrast objects," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **54**, 1316–1332 (2007).
- ⁸J. R. Taylor, *Scattering Theory* (Wiley, New York, 1972), p. 270.
- ⁹K. Chadan, D. Colton, L. Päiväranta, and W. Rundell, *An Introduction to Inverse Scattering and Inverse Spectral Problems* (SIAM, Philadelphia, 1997), p. 35.
- ¹⁰R. C. Waag and R. J. Fedewa, "A ring transducer system for medical ultrasound research," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **53**, 1707–1718 (2006).
- ¹¹A. J. Devaney and M. L. Oristaglio, "Inversion procedure for inverse scattering within the distorted-wave Born approximation," *Phys. Rev. Lett.* **51**, 237–240 (1983).
- ¹²A. J. Devaney, "A filtered back-propagation algorithm for diffraction tomography," *Ultrason. Imaging* **4**, 336–350 (1982).
- ¹³J. Jin, J. P. Astheimer, and R. C. Waag, "Estimation of scattering object characteristics for image reconstruction by using a nonzero background," Report No. DURL 08-1, Diagnostic Ultrasound Research Laboratory, University of Rochester, Rochester, NY, January 2008.
- ¹⁴P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968), Chap. 8.
- ¹⁵T. D. Mast, "Aberration correction for time-domain ultrasound diffraction tomography," *J. Acoust. Soc. Am.* **112**, 55–64 (2002).