Cubic nonlinearity in shear wave beams with different polarizations

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A coupled pair of nonlinear parabolic equations is derived for the two components of the particle motion perpendicular to the axis of a shear wave beam in an isotropic elastic medium. The equations account for both quadratic and cubic nonlinearity. The present paper investigates, analytically and numerically, effects of cubic nonlinearity in shear wave beams for several polarizations: linear, elliptical, circular, and azimuthal. Comparisons are made with effects of quadratic nonlinearity in compressional wave beams. © 2008 Acoustical Society of America. [DOI: 10.1121/1.2890739]

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I. INTRODUCTION

The distinguishing feature of nonlinearity in shear waves is that it is primarily cubic rather than quadratic. A derivation of the nonlinear parabolic wave equation for shear wave beams in isotropic solids appeared in 1986.¹ Explicit expressions in terms of elastic moduli were presented for the coefficients of both the quadratic and cubic terms. The equation was solved by perturbation to demonstrate that while wavefront curvature introduces quadratic nonlinearity, it does not permit second harmonic generation directly by the fundamental. Rather, it was shown that the second harmonic is produced by interaction of the fundamental with the third harmonic generated by cubic nonlinearity.

A number of papers followed on wave propagation with cubic nonlinearity in nondispersive media. The 1987 paper by Lee-Bapty and Crighton² on plane waves develops a weak shock theory describing the discontinuities that form at both positive and negative gradients in the waveform. In the 1990s began a series of papers by Rudenko and co-workers on diffraction, self-focusing, and other effects in beams with cubic nonlinearity. For discussion of this body of work the reader is referred to two papers by Rudenko and Sapozhnikov^{3,4} and the papers cited therein. The forms of the model equations used in these previous investigations restrict their application to nonlinear shear waves with linear polarization. A main purpose of the present investigation is to consider other types of polarization.

Our interest in nonlinear shear waves was motivated by experiments reported by Catheline *et al.*⁵ that demonstrate clearly the harmonic generation and shock formation manifested by plane nonlinear shear waves propagating in soft tissue phantoms. The propagation speed of shear waves in soft tissue is extremely low, on the order of a few meters per second, making it relatively easy to generate disturbances with large acoustic Mach numbers. Catheline *et al.* reported a peak acoustic Mach number of 0.38, which is well in excess of acoustic Mach numbers ordinarily encountered with compressional waves. The close agreement they showed be-

tween measured waveforms containing shocks and numerical simulations based on a generalized Burgers equation confirmed that the nonlinearity was predominantly cubic.

The same group also reported measurements of elastic constants for soft tissue phantoms.⁶ The enormous differences in the values of these constants for a given medium motivated development of a new expansion of the strain energy density in which the effects of shear deformation, which are associated with the small elastic constants, are separated from the effects of compressibility, which are associated with the large constants. The result is a much simplified expression for the strain energy density in nonlinear shear waves.⁷ This new expansion was used to derive evolution equations for plane nonlinear shear waves in soft tissue-like media.⁸

In the present paper, the plane wave model⁸ is extended to account for diffraction in nonlinear shear wave beams in arbitrary isotropic elastic media. Both components of the particle motion perpendicular to the beam axis are included to account for different polarizations of the beam, and not exclusively the linear polarization that is usually considered. A coupled pair of nonlinear parabolic wave equations is obtained that includes both quadratic and cubic nonlinearity. The present paper investigates, analytically and numerically, the effects of cubic nonlinearity in shear wave beams. Investigation of the quadratic nonlinearity is the subject of a future paper.

Selected results in the present paper have appeared previously in a conference proceedings.⁹

II. BASIC EQUATIONS

The general equation of motion in an elastic medium is¹⁰

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k}, \quad (1)$$

where ρ is the density of the material in its undeformed state, u_i are components of the particle displacement vector, and σ_{ik} is the stress tensor defined by

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$$\sigma_{ik} = \frac{\partial \mathcal{E}}{\partial (\partial u_i / \partial x_k)}. \quad (2)$$

The strain energy density $\mathcal{E}$ for an isotropic elastic medium may be expanded as follows:^{1,7,10}

$$\mathcal{E} = \mu I_2 + \left(\frac{1}{2}K - \frac{1}{3}\mu\right) I_1^2 + \frac{1}{3}A I_3 + B I_1 I_2 + \frac{1}{3}C I_1^3 + E I_1 I_3 + F I_1^2 I_2 + G I_2^2 + H I_1^4, \quad (3)$$

where the invariants

$$I_1 = \text{tr } \mathbf{u} = u_{ii}, \quad I_2 = \text{tr } \mathbf{u}^2 = u_{ik} u_{ki}, \quad I_3 = \text{tr } \mathbf{u}^3 = u_{ik} u_{kl} u_{li}, \quad (4)$$

are defined in terms of the Lagrangian strain tensor

$$u_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_k} \right). \quad (5)$$

Here μ and K are the shear and bulk moduli, respectively; A , B , and C are the third-order elastic constants of Landau and Lifshitz;¹⁰ E , F , G , and H are the fourth-order elastic constants introduced by Zabolotskaya¹ (but without the coefficient D in Zabolotskaya's expansion which, as explained elsewhere,⁷ is not required).

To develop equations that describe the propagation of shear waves, the first step is to express the right-hand side of Eq. (1) explicitly in terms of particle displacement, retaining all terms through cubic order. The quadratic terms are given by Gol'dberg.¹¹ The expression for the cubic terms is rather long and therefore not presented here.

Next, we consider a shear wave beam that propagates in the x_3 direction and introduce the retarded time $\tau = t - x_3/c$, where $c = (\mu/\rho)^{1/2}$ is the small-signal wave speed. As usual, a reduced wave equation is sought by introducing a slow scale corresponding to the moving coordinate system. The following ordering scheme is employed, in which ϵ characterizes the amplitude of the particle displacement transverse to the beam axis:

$$u_1 = \epsilon U_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tau), \quad u_2 = \epsilon U_2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tau), \quad (6)$$

$$u_3 = \epsilon^2 U_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tau),$$

where $\tilde{x}_1 = \epsilon x_1$, $\tilde{x}_2 = \epsilon x_2$, and $\tilde{x}_3 = \epsilon^2 x_3$ are the slow-scale coordinates.

After substitution in Eq. (1) one obtains at leading (cubic) order in ϵ , for $i=1$,

$$\begin{aligned} \frac{2\mu}{c} \frac{\partial^2 U_1}{\partial \tilde{x}_3 \partial \tau} &= \mu \left(\frac{\partial^2 U_1}{\partial \tilde{x}_1^2} + \frac{\partial^2 U_1}{\partial \tilde{x}_2^2} \right) + \left(K + \frac{\mu}{3} \right) \frac{\partial S}{\partial \tilde{x}_1} \\ &+ \frac{1}{c^2} \left(\mu + \frac{A}{4} \right) \left(\frac{\partial^2 U_m}{\partial \tau^2} \frac{\partial U_m}{\partial \tilde{x}_1} - \frac{1}{c} \frac{\partial^2 U_3}{\partial \tau^2} \frac{\partial U_1}{\partial \tau} \right. \\ &+ \frac{\partial^2 U_m}{\partial \tau^2} \frac{\partial U_1}{\partial \tilde{x}_m} - \frac{2}{c} \frac{\partial^2 U_1}{\partial \tau^2} \frac{\partial U_3}{\partial \tau} \\ &+ 2 \frac{\partial^2 U_1}{\partial \tilde{x}_m \partial \tau} \frac{\partial U_m}{\partial \tau} \left. \right) \\ &+ \frac{1}{c^2} \left(K + \frac{\mu}{3} + \frac{A}{4} + B \right) \left(\frac{\partial^2 U_m}{\partial \tilde{x}_1 \partial \tau} \frac{\partial U_m}{\partial \tau} \right. \end{aligned}$$

$$\begin{aligned} &+ \frac{\partial S}{\partial \tau} \frac{\partial U_1}{\partial \tau} \left. \right) + \frac{1}{c^2} \left(K - \frac{2\mu}{3} + B \right) S \frac{\partial^2 U_1}{\partial \tau^2} \\ &+ \frac{1}{c^4} \left(\frac{2\mu}{3} + \frac{K}{2} + \frac{A}{2} + B + G \right) \\ &\times \frac{\partial}{\partial \tau} \left\{ \frac{\partial U_1}{\partial \tau} \left[\left(\frac{\partial U_1}{\partial \tau} \right)^2 + \left(\frac{\partial U_2}{\partial \tau} \right)^2 \right] \right\}, \quad (7) \end{aligned}$$

where $m=1, 2$ is a summation index over components in the plane perpendicular to the x_3 axis, and

$$S = \frac{\partial U_1}{\partial \tilde{x}_1} + \frac{\partial U_2}{\partial \tilde{x}_2} - \frac{1}{c} \frac{\partial U_3}{\partial \tau}. \quad (8)$$

At leading order one also obtains for the divergence $\nabla \cdot \mathbf{u} = \partial u_i / \partial x_i = \epsilon^2 S$. The result for $i=2$ is obtained by reversing the subscripts 1 and 2 in Eq. (7). For $i=3$, the leading order is quadratic in ϵ and one obtains

$$\begin{aligned} S &= -\frac{1}{2c^2} \left(K + \frac{\mu}{3} \right)^{-1} \left(K + \frac{4\mu}{3} + \frac{A}{2} + B \right) \\ &\times \left[\left(\frac{\partial U_1}{\partial \tau} \right)^2 + \left(\frac{\partial U_2}{\partial \tau} \right)^2 \right]. \quad (9) \end{aligned}$$

Substitution of Eq. (9) in Eq. (7) and in the corresponding equation for U_2 yields, after reinstating the original coordinates and particle displacements, and using (x, y, z) in place of (x_1, x_2, x_3) ,

$$\begin{aligned} \frac{\partial^2 u_x}{\partial z \partial \tau} &= \frac{c}{2} \nabla_{\perp}^2 u_x + \frac{1}{2c} \left(1 + \frac{A}{4\mu} \right) Q_{xy} \\ &+ \frac{\beta}{3c^3} \frac{\partial}{\partial \tau} \left\{ \frac{\partial u_x}{\partial \tau} \left[\left(\frac{\partial u_x}{\partial \tau} \right)^2 + \left(\frac{\partial u_y}{\partial \tau} \right)^2 \right] \right\}, \quad (10) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u_y}{\partial z \partial \tau} &= \frac{c}{2} \nabla_{\perp}^2 u_y + \frac{1}{2c} \left(1 + \frac{A}{4\mu} \right) Q_{yx} \\ &+ \frac{\beta}{3c^3} \frac{\partial}{\partial \tau} \left\{ \frac{\partial u_y}{\partial \tau} \left[\left(\frac{\partial u_x}{\partial \tau} \right)^2 + \left(\frac{\partial u_y}{\partial \tau} \right)^2 \right] \right\}, \quad (11) \end{aligned}$$

where $\nabla_{\perp}^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the Laplacian in the plane perpendicular to the beam axis, all quadratic terms are contained in

$$\begin{aligned} Q_{xy} &= \frac{\partial^2 u_y}{\partial \tau^2} \frac{\partial u_y}{\partial x} + \frac{\partial^2 u_y}{\partial \tau^2} \frac{\partial u_x}{\partial y} + 2 \frac{\partial^2 u_x}{\partial y \partial \tau} \frac{\partial u_y}{\partial \tau} - \frac{\partial^2 u_y}{\partial y \partial \tau} \frac{\partial u_x}{\partial \tau} \\ &- 2 \frac{\partial^2 u_x}{\partial \tau^2} \frac{\partial u_y}{\partial y} - \frac{\partial^2 u_y}{\partial x \partial \tau} \frac{\partial u_y}{\partial \tau}, \quad (12) \end{aligned}$$

and the coefficient of nonlinearity for the cubic terms is

$$\begin{aligned} \beta &= \frac{3}{4\mu} \left(K + \frac{4}{3}\mu + A + 2B + 2G \right) \\ &- \frac{3}{4\mu} \frac{\left(K + \frac{4}{3}\mu + \frac{1}{2}A + B \right)^2}{K + \frac{1}{3}\mu}. \quad (13) \end{aligned}$$

Equations (10) and (11) differ only by the reversal of x and y .

In the absence of the second term in Eq. (13), Eqs. (10) and (11) agree with Eq. (14) of Zabolotskaya.¹ The second

term in Eq. (13) is associated with cubic nonlinearity arising from quadratic nonlinearity due to the fact that the right-hand side of Eq. (8), by virtue of Eq. (9), is quadratic at leading order when expressed in terms of only transverse particle displacements. The absence of this contribution to the cubic nonlinearity in Eq. (14) of Ref. 1 was noted by Enflo *et al.*¹² but their expression corresponding to the second term in Eq. (13) differs from ours.

The origin of the second term in Eq. (13) is not connected with diffraction and can be understood from Gol'dberg's¹¹ Eqs. (6) and (7) for plane waves. Gol'dberg defines a quadratic nonlinearity coefficient $\gamma = K + \frac{4}{3}\mu + \frac{1}{2}A + B$, the square of which appears in Eq. (13). The longitudinal wave generated by the quadratic nonlinearity of a shear wave is proportional to γ , and the shear wave generated by interaction of the longitudinal wave with the original shear wave introduces another factor of γ , resulting in the coefficient γ^2 associated with this third-order process. The corresponding term derived by Enflo *et al.*¹² contains only one factor of γ .

For soft elastic media characterized by $\mu \ll K$, the second term in Eq. (13) is negligible relative to the first. This can be shown by making use of the following relations between the elastic constants for such media:^{7,13} $A = O(\mu)$, $B = -K + O(\mu)$, and $G = \frac{1}{2}K + O(\mu)$. Substitution in Eq. (13) reveals that the first term is $O(1)$ and second term is $O(\mu/K)$. For soft tissue, with μ/K of order 10^{-5} and possibly smaller, the second term in Eq. (13) is insignificant. In this case, the elastic energy density can be expressed in a reduced form that depends on only three elastic constants,⁷

$$\mathcal{E} = \mu I_2 + \frac{1}{3}A I_3 + D I_2^2, \quad (14)$$

rather than the nine in Eq. (3), and all three constants are of the same order. This reduced equation for the energy density leads to⁸

$$\beta = \frac{3}{2} \left(1 + \frac{\frac{1}{2}A + D}{\mu} \right) \quad (15)$$

in place of Eq. (13).

Enflo *et al.*¹² present a result similar to Eq. (10) for a shear wave beam that is linearly polarized in the x direction, but without the quadratic terms, and without the corresponding equation for u_y . While Q_{xy} and Q_{yx} vanish for plane waves, they do not for beams. For a source that is linearly polarized in the x direction, u_y and therefore Q_{xy} are in fact zero in the source plane. However, in Eq. (11) the first and last terms in Q_{yx} depend only on u_x , giving rise to a nonzero value of u_y due to quadratic nonlinearity. Simply stated, quadratic nonlinearity prevents the existence of a pure linearly polarized shear wave beam.

Cubic nonlinearity can be more pronounced than quadratic nonlinearity due to the quasiplanar nature of wavefronts in directional sound beams. In the present paper we consider only the effects of cubic nonlinearity.

In the absence of quadratic nonlinearity, one may introduce the particle velocities $v_x = \partial u_x / \partial \tau$ and $v_y = \partial u_y / \partial \tau$ in Eqs. (10) and (11) to obtain, following integration with respect to τ , the coupled equations

$$\begin{aligned} \frac{\partial v_x}{\partial z} = & \frac{\beta}{3c^3} \frac{\partial}{\partial \tau} [v_x(v_x^2 + v_y^2)] + \frac{\eta}{2\rho c^3} \frac{\partial^2 v_x}{\partial \tau^2} \\ & + \frac{c}{2} \int_{-\infty}^{\tau} (\nabla_{\perp}^2 v_x) d\tau', \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\partial v_y}{\partial z} = & \frac{\beta}{3c^3} \frac{\partial}{\partial \tau} [v_y(v_x^2 + v_y^2)] + \frac{\eta}{2\rho c^3} \frac{\partial^2 v_y}{\partial \tau^2} \\ & + \frac{c}{2} \int_{-\infty}^{\tau} (\nabla_{\perp}^2 v_y) d\tau', \end{aligned} \quad (17)$$

where the terms accounting for the shear viscosity η are introduced as for plane waves.⁸ Without the diffraction and absorption terms, Eqs. (16) and (17) are the same as Eqs. (53) and (54) in Ref. 8. When the source functions are axisymmetric and can be expressed as $v_{x0}(r, t)$ and $v_{y0}(r, t)$, where $r = (x^2 + y^2)^{1/2}$, it is evident that the radiated sound beam is axisymmetric, in which case $\nabla_{\perp}^2 = \partial^2 / \partial r^2 + r^{-1}(\partial / \partial r)$.

Comparisons will be made with nonlinear compressional wave beams, which are described by the KZK equation for the particle velocity component in the z direction:

$$\frac{\partial v_z}{\partial z} = \frac{\beta_l}{c_l^2} v_z \frac{\partial v_z}{\partial \tau} + \frac{\delta}{2c_l^3} \frac{\partial^2 v_z}{\partial \tau^2} + \frac{c_l}{2} \int_{-\infty}^{\tau} (\nabla_{\perp}^2 v_z) d\tau'. \quad (18)$$

The subscript l is used to identify parameters that are specific to compressional waves. Here, $\tau = t - z/c_l$ is the retarded time, $c_l = [(K + \frac{4}{3}\mu)/\rho]^{1/2}$ the small-signal compressional wave speed, δ is a diffusivity that accounts for sound absorption due to both shear and bulk viscosity, and also heat condition, and

$$\beta_l = -\frac{3}{2} - \frac{A + 3B + C}{K + \frac{4}{3}\mu} \quad (19)$$

is the coefficient of nonlinearity. The coefficient of nonlinearity for a liquid is $\beta_l = 1 + B_l/2A_l$, where B_l/A_l is the parameter of nonlinearity for liquids that is obtained from the equation of state relating the pressure and density.

III. LINEAR POLARIZATION

For beams that are linearly polarized along the x axis, $v_y = 0$ and Eq. (16) reduces to

$$\frac{\partial v_x}{\partial z} = \frac{\beta}{c^3} v_x^2 \frac{\partial v_x}{\partial \tau} + \frac{\eta}{2\rho c^3} \frac{\partial^2 v_x}{\partial \tau^2} + \frac{c}{2} \int_{-\infty}^{\tau} (\nabla_{\perp}^2 v_x) d\tau'. \quad (20)$$

Without the diffraction term, Eq. (20) reduces to Eq. (37) in Ref. 8. Equation (20) is formally equivalent to Eq. (18) for compressional wave beams apart from the cubic rather than quadratic nonlinearity. The linearized forms of the two equations thus possess the same Green's function, which may be used to compare the corresponding solutions for harmonic generation.

We begin by assuming a source condition with time dependence $e^{i\omega t}$ and write $v_x = v_{x1} + v_{x3}$, where $|v_{x3}| \ll |v_{x1}|$ and

$$v_{xn}(r, z, \tau) = \text{Re}[w_{xn}(r, z)e^{in\omega\tau}] \\ = \frac{1}{2}w_{xn}(r, z)e^{in\omega\tau} + \text{c.c.}, \quad n = 1, 3. \quad (21)$$

The integral solutions of Eq. (20) for the first and third harmonic are thus (see, e.g., Ref. 14, Chap. 8)

$$w_{x1}(r, z) = 2\pi \int_0^\infty w_{x1}(r', 0)g_1(r, z|r', 0)r'dr', \quad (22)$$

$$w_{x3}(r, z) = \frac{i\pi\beta k}{2c^2} \int_0^z \int_0^\infty w_{x1}^3(r', z')g_3(r, z|r', z')r'dr'dz', \quad (23)$$

where $k = \omega/c$ and the Green's function is

$$g_n(r, z|r', z') = \frac{ink}{2\pi(z - z')} J_0\left(\frac{nkr r'}{z - z'}\right) \\ \times \exp\left[-n^2\alpha(z - z') - \frac{ink(r^2 + r'^2)}{2(z - z')}\right], \quad (24)$$

and where $\alpha = \eta\omega^2/2\rho c^3$ is the absorption coefficient at the source frequency. For a source with Gaussian amplitude shading described by $w_{x1}(r, 0) = v_0 \exp(-r^2/a^2)$ the integrals yield

$$w_{x1}(r, z) = \frac{v_0}{1 - iz/z_0} \exp\left(-\alpha z - \frac{r^2/a^2}{1 - iz/z_0}\right), \quad (25)$$

$$w_{x3}(r, z) = \frac{\beta v_0^3 k z_0}{4c^2(1 - iz/z_0)^2} \exp\left(-9\alpha z - \frac{3r^2/a^2}{1 - iz/z_0}\right. \\ \left. - i6\alpha z_0\right) \{E_2[-i6\alpha z_0(1 - iz/z_0)] \\ - (1 - iz/z_0)E_2[-i6\alpha z_0]\}, \quad (26)$$

where $z_0 = ka^2/2$ is the Rayleigh distance, and $E_2(\xi)$ is the exponential integral defined by $E_m(\xi) = \int_1^\infty e^{-\xi t} t^{-m} dt$ with $m = 2$. Equation (25) is the same as the solution for a compressional wave.

To facilitate comparison of Eq. (26) with the corresponding solution for second-harmonic generation in a compressional wave beam¹⁴ we set $\alpha = 0$ to obtain¹

$$w_{x3}(r, z) = \frac{i\beta v_0^3 k z}{4c^2(1 - iz/z_0)^2} \exp\left(-\frac{3r^2/a^2}{1 - iz/z_0}\right). \quad (27)$$

The amplitude along the axis of the beam in the near field ($z \ll z_0$) increases in proportion to z , just like the second harmonic in a compressional wave. In the far field ($z \gg z_0$), however, Eq. (27) describes a z^{-1} decay rate, as for a spherical wave in the linear approximation. In contrast, the second harmonic in a compressional wave decays as $z^{-1} \ln z$ in the far field,¹⁴ which is slower than spherical spreading due to the unending transfer of energy from the fundamental in a lossless medium. When absorption is negligible, all harmonics in the shear wave beam decay as z^{-1} in the far field, whereas all harmonics in the compressional wave beam decay more slowly than z^{-1} .

The decay rate of nonlinear shear waves in the far field can be explained by replacing the absorption and diffraction terms in Eq. (20) with a term that accounts only for spherical spreading:

$$\frac{\partial v_x}{\partial z} + \frac{v_x}{z} = \frac{\beta}{c^3} v_x^2 \frac{\partial v_x}{\partial \tau}. \quad (28)$$

Given a waveform $v_x = f(\tau)$ prescribed at some distance $z = z_0$ in the spherical wave field, the implicit solution describing evolution of the waveform prior to shock formation may be written in the form

$$v_x(z, \tau) = \frac{z_0}{z} v(z, \tau), \quad \text{where } v = f\left[\tau + \frac{\beta}{c^3} v^2 z_0 \left(1 - \frac{z_0}{z}\right)\right]. \quad (29)$$

At distances $z \gg z_0$ the waveform ceases to depend on z , which is to say that no further nonlinear distortion of the waveform takes place. The particle velocity waveform $v_x(z, \tau)$ thus ultimately decays as z^{-1} without further change in shape. This phenomenon is referred to as waveform freezing in compressional waves.

When absorption is taken into account the behavior of nonlinearly generated harmonics in the far field is quite different. The asymptotic forms of Eqs. (25) and (26) are

$$w_{x1}(\theta, z) \sim iv_0 z_0 \frac{e^{-\alpha z}}{z} D(\theta) \exp\left(-\frac{1}{2} ikz \tan^2 \theta\right), \quad (30)$$

$$w_{x3}(\theta, z) \sim \frac{\beta v_0^3 k z_0^3}{24\alpha c^2} \frac{e^{-3\alpha z}}{z^3} D^3(\theta) \exp\left(-\frac{3}{2} ikz \tan^2 \theta\right), \quad (31)$$

where

$$D(\theta) = \exp\left[-\frac{1}{4}(ka \tan \theta)^2\right] \quad (32)$$

is the directivity function for a Gaussian beam in the linear approximation, and θ is the angle with respect to the beam axis, defined by $\tan \theta = r/z$. The third harmonic is seen to be proportional to the cube of the fundamental.

Because of the quadratic dependence of the absorption coefficient on frequency, all higher harmonics in a linearly polarized beam follow the trend $w_{xn}(\theta, z) \propto w_{x1}^n(\theta, z)$ in the far field.¹⁴ This absorption law also permits far-field solutions to be used in combination with geometrical acoustics to obtain asymptotic solutions for harmonics when an analytic solution for the primary wave radiated by a particular source may be available only in the far field. Such is the case for radiation by a circular disk of radius a , the source condition for which is $w_{x1}(r, 0) = v_0 H(a - r)$, where H is the Heaviside unit step function. The far-field solutions for the fundamental and third harmonic in this case are given again by Eqs. (30) and (31), respectively, but with

$$D(\theta) = \frac{2J_1(ka \tan \theta)}{ka \tan \theta}. \quad (33)$$

IV. ELLIPTICAL POLARIZATION

For elliptical polarization, the x and y components of the particle velocity are in phase quadrature, and one may write the source condition in the form

$$v_x(r, 0, t) = v_0 \operatorname{Re}[f(r)e^{i\omega t}], \quad v_y(r, 0, t) = sv_0 \operatorname{Im}[f(r)e^{i\omega t}], \quad (34)$$

where s varies from zero for linear polarization along the x axis to unity for circular polarization. For a Gaussian amplitude shaded source with $f(r) = \exp(-r^2/a^2)$ one obtains, in place of Eq. (27),

$$w_{x3}(r, z) = \frac{i(1-s^2)\beta v_0^3 k z}{4c^2(1-iz/z_0)^2} \exp\left(-\frac{3r^2/a^2}{1-iz/z_0}\right), \quad (35)$$

$$w_{y3}(r, z) = \frac{s(1-s^2)\beta v_0^3 k z}{4c^2(1-iz/z_0)^2} \exp\left(-\frac{3r^2/a^2}{1-iz/z_0}\right), \quad (36)$$

with $v_{x3}(r, z, \tau) = \operatorname{Re}[w_{x3}(r, z)e^{i3\omega\tau}]$ and $v_{y3}(r, z, \tau) = \operatorname{Im}[w_{y3}(r, z)e^{i3\omega\tau}]$. The two components differ only by the factor $-is$, as do the corresponding components at the source frequency. This same ratio is not maintained for the third and higher harmonics when higher-order nonlinear interactions are taken into account. Numerical solutions reveal that the main differences arise in the relative phases of the harmonics in the frequency spectra of v_x and v_y . For weak or moderate nonlinearity, the relative amplitudes remain nearly the same.

Note that the third harmonic vanishes for circular polarization, $s=1$, which is the case considered next.

V. CIRCULAR POLARIZATION

Given the absence of third-harmonic generation for circular polarization, we seek a monofrequency solution in the form

$$v_x(r, z, \tau) = \operatorname{Re}[w(r, z)e^{i\omega\tau}], \quad v_y(r, z, \tau) = \operatorname{Im}[w(r, z)e^{i\omega\tau}]. \quad (37)$$

Since

$$v_x^2(r, z, \tau) + v_y^2(r, z, \tau) = |w(r, z)|^2, \quad (38)$$

the quantity $v_x^2 + v_y^2$ is independent of time and may be removed from under the time derivatives in Eqs. (16) and (17). For a circularly polarized plane wave in a lossless medium, where $w = \text{const} \equiv v_0$, the wave propagates as a small signal and the only effect of nonlinearity is to change the propagation speed from c to $(1 + \beta v_0^2/3c^2)c$.⁸ The amplitude w is not constant in a diffracting beam, and in particular it varies across the beam. Different parts of the beam thus propagate at different phase speeds according to the square of the local acoustic Mach number and the sign of the nonlinearity coefficient. Self-defocusing occurs for $\beta > 0$ because the beam propagates faster along its axis than off axis. Conversely, self-focusing occurs for $\beta < 0$.

The evolution equation for w is obtained by substituting Eq. (37) into either Eq. (16) or Eq. (17) to obtain

$$i\left(\frac{\partial w}{\partial z} + \alpha w\right) - \frac{1}{2k}\nabla_\perp^2 w + \frac{\beta k}{3c^2}|w|^2 w = 0, \quad (39)$$

which is a damped nonlinear Schrödinger equation. In the absence of damping it is the classical model equation for self-focusing of linearly polarized optical beams, and it has also been analyzed in connection with circularly polarized optical beams.¹⁵ For a circularly polarized shear wave beam that is radiated by a Gaussian source into a lossless medium, wavefront curvature due to self-action in the near field becomes comparable with wavefront curvature due to diffraction for v_0/c of order $|\beta|^{-1/2}(ka)^{-1}$.

The main point to be made is that cubic nonlinearity in circularly polarized shear wave beams produces no harmonic generation and therefore no waveform distortion. Its only effect is to cause self-focusing or self-defocusing depending on the sign of β .

VI. AZIMUTHAL POLARIZATION

Azimuthal polarization, which produces a torsional wave beam, results from a source that exhibits only rotational motion about the z axis. For simplicity the source is assumed to be axisymmetric, such that in the cylindrical coordinates (r, ϕ, z) one obtains for the components of the particle velocity vector in the plane perpendicular to the z axis $v_r = 0$ and $v_\phi = v_\phi(r, z, \tau)$. Then $v_x = -v_\phi \sin \phi$, $v_y = v_\phi \cos \phi$, $v_x^2 + v_y^2 = v_\phi^2$, and both Eqs. (16) and (17) yield

$$\begin{aligned} \frac{\partial v_\phi}{\partial z} &= \frac{\beta}{c^3} v_\phi^2 \frac{\partial v_\phi}{\partial \tau} + \frac{\eta}{2\rho c^3} \frac{\partial^2 v_\phi}{\partial \tau^2} \\ &+ \frac{c}{2} \int_{-\infty}^{\tau} \left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} \right) d\tau'. \end{aligned} \quad (40)$$

Equation (40) differs from Eq. (20) for linear polarization only by the addition of the term $-r^{-2}$ to the transverse Laplacian $\nabla_\perp^2 = \partial^2/\partial r^2 + r^{-1}(\partial/\partial r)$. Torsional waves are often more conveniently expressed in terms of the angular velocity in the medium, $\Omega = \partial\phi/\partial\tau$, for which $v_\phi = r\Omega$ and Eq. (40) becomes

$$\frac{\partial \Omega}{\partial z} = r^2 \frac{\beta}{c^3} \Omega^2 \frac{\partial \Omega}{\partial \tau} + \frac{\eta}{2\rho c^3} \frac{\partial^2 \Omega}{\partial \tau^2} + \frac{c}{2} \int_{-\infty}^{\tau} \left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{3}{r} \frac{\partial \Omega}{\partial r} \right) d\tau'. \quad (41)$$

From Eq. (41) it is evident that nonlinearity vanishes along the axis of the beam.

We again consider third-harmonic generation, here with $v_\phi = v_{\phi 1} + v_{\phi 3}$ and

$$v_{\phi n}(r, z, \tau) = \operatorname{Re}[w_{\phi n}(r, z)e^{in\omega\tau}], \quad n = 1, 3. \quad (42)$$

The integral solutions of Eq. (40) for the fundamental and third harmonic are given by Eqs. (22) and (23) with the subscript x replaced everywhere by ϕ , and with

$$g_n(r, z | r', z') = -\frac{nk}{2\pi(z-z')} J_1\left(\frac{nkrr'}{z-z'}\right) \times \exp\left[-n^2\alpha(z-z') - \frac{ink(r^2+r'^2)}{2(z-z')}\right]. \quad (43)$$

Equation (43) differs from Eq. (24) by a multiplicative factor i and by J_1 in place of J_0 . The corresponding Green's function for Eq. (41) is obtained by multiplying the right-hand side of Eq. (43) by r'/r .

The source condition is expressed in the form

$$\Omega(r, 0, t) = \Omega_0 \operatorname{Re}[f(r)e^{i\omega t}], \quad (44)$$

such that $w_{\phi 1}(r, 0) = \Omega_0 r f(r)$. We first take $f(r) = \exp(-r^2/a^2)$ and obtain for the angular velocity at the source frequency

$$W_1(r, z) = \frac{\Omega_0}{(1 - iz/z_0)^2} \exp\left(-\alpha z - \frac{r^2/a^2}{1 - iz/z_0}\right), \quad (45)$$

with $\Omega_n(r, z, \tau) = \operatorname{Re}[W_n(r, z)e^{in\omega\tau}]$. In the far field the decay rate associated with spherical spreading is z^{-2} rather than z^{-1} . This is a consequence of the dipole-like nature of the source, and therefore the entire field, because $v_{x,y}(r, \phi, z, \tau) = -v_{x,y}(r, \phi + \pi, z, \tau)$. For any pair of points that are symmetric with respect to the z axis, the particle velocity vectors in the (x, y) plane are equal in magnitude and opposite in direction.

The integral for the third harmonic does not appear to admit a closed-form solution for a Gaussian beam. However, as discussed at the end of Sec. III, an analytic solution for the primary wave in the far field is sufficient to determine the corresponding asymptotic solution for the third harmonic. We thus consider radiation from a circular disk of radius a , for which $f(r) = H(a-r)$ in Eq. (44). The solutions for the fundamental and third harmonic in the far field are found to be

$$W_1(\theta, z) \sim -\frac{1}{2}\Omega_0 z_0^2 \frac{e^{-\alpha z}}{z^2} D(\theta) \exp\left(-\frac{1}{2}ikz \tan^2 \theta\right), \quad (46)$$

$$W_3(\theta, z) \sim -\frac{i\beta\Omega_0^3 z_0^5}{384ac^2} \frac{e^{-3\alpha z}}{z^4} (ka \tan \theta)^2 D^3(\theta) \times \exp\left(-\frac{3}{2}ikz \tan^2 \theta\right), \quad (47)$$

where the directivity function for the primary wave,

$$D(\theta) = \frac{8J_2(ka \tan \theta)}{(ka \tan \theta)^2}, \quad (48)$$

is normalized to be unity on axis.

Like the linear solution for a Gaussian torsional beam, Eq. (46) exhibits a $z^{-2}e^{-\alpha z}$ decay rate. Unlike the linear solution for a Gaussian beam, the directivity function is different from that for a linearly polarized beam radiated by a source with the same amplitude distribution [compare Eqs. (25) and (45), and then Eqs. (33) and (48)]. Due to the factor $(ka \tan \theta)^2$ multiplying $D^3(\theta)$, the third harmonic vanishes along the axis of the beam in the far field. The maximum

amplitude of the third harmonic in the far field is located at the angle θ_0 where $ka \tan \theta_0 = 1.92$, for which $(ka \tan \theta_0)^2 D^3(\theta_0) = 1.41$.

Equations (31) and (47) may now be used to compare the maximum amplitudes in a linearly polarized beam and an azimuthally polarized beam. To relate the source amplitudes, the kinetic energies associated with translation and rotation of the disk are matched to obtain $\Omega_0 = 2^{1/2}v_0/a$, for which the far-field formulas yield $|w_{\phi 3}(\theta_0, z)/w_{x3}(0, z)| \sim 0.12$. In terms of harmonic generation, azimuthal polarization is thus significantly less efficient than linear polarization.

VII. SOLUTION ALGORITHMS

Numerical results are presented in Sec. VIII for shear wave beams radiated in the z direction by a circular disk of radius a . The source condition for elliptically polarized beams, including linearly and circularly polarized beams, is

$$v_x(r, 0, t) = v_0 H(a-r) \cos \omega t, \\ v_y(r, 0, t) = sv_0 H(a-r) \sin \omega t. \quad (49)$$

Diffraction is characterized by the Rayleigh distance $z_0 = ka^2/2$, nonlinearity by the plane wave shock formation distance $z_{sh} = c^3/\beta k v_0^2$, and absorption by the length $z_{abs} = 1/\alpha$. In terms of these length scales, the two dimensionless parameters used to characterize solutions of Eqs. (16) and (17) are $A = z_0/z_{abs}$ for absorption and $N = z_0/z_{sh}$ for nonlinearity. For plane waves, the Gol'dberg number $\Gamma = z_{abs}/z_{sh}$ is used.

The source condition for azimuthally polarized beams is

$$\Omega(r, 0, t) = \Omega_0 H(a-r) \cos \omega t. \quad (50)$$

Since in this case the peak particle velocity in the source plane is $a\Omega_0$, the characteristic shock formation distance is taken to be $z_{sh} = c^3/\beta \omega a^2 \Omega_0^2$. The parameters N and A are otherwise defined as for elliptically polarized beams.

Comparisons are made with solutions of Eq. (18) for compressional wave beams, the source condition for which is

$$v_z(r, 0, t) = v_{z0} H(a-r) \cos \omega t. \quad (51)$$

The parameters N , A , and Γ are defined as for shear waves, except with the shock formation distance given instead by $z_{sh} = c_l/\beta k_l v_{z0}$, where $k_l = \omega/c_l$, because the nonlinearity is quadratic.

For linear polarization, Eq. (20) is solved using the time domain algorithm in Ref. 16 after a minor change is made in the nonlinearity subroutine to account for the cubic nonlinearity. Equation (41) is solved for azimuthal polarization, again with the algorithm in Ref. 16 but without the coordinate transformation. Also, in addition to the change in the nonlinearity subroutine, a minor change is required in the diffraction subroutine. For elliptical polarization, Eqs. (16) and (17) must be solved simultaneously. No implicit analytical solution was discovered for the nonlinearity time step, and a weighted essentially nonoscillatory scheme¹⁷ was used in place of the time base transformation in Ref. 16.

FIG. 1. Axial time waveforms in a compressional wave (first column, v_z) and an elliptically polarized shear wave with $s=0.25$ (second and third columns, v_x and v_y , respectively), both without diffraction [first row, (a)–(c)] and with diffraction [second row, (d)–(f)]. The distance for the plane waves is $z=3z_{sh}$ with $\Gamma=100$, and for the beams it is $z=3z_0$ with $N=1$ and $A=0.01$.

VIII. NUMERICAL RESULTS

Axial time waveforms are presented in Fig. 1 for a compressional wave (first column, v_z) and for an elliptically polarized shear wave with $s=0.25$ (second and third columns, v_x and v_y , respectively), both without diffraction [first row, Figs. 1(a)–1(c)] and with diffraction [second row, Figs. 1(d)–1(f)]. The distance for the plane waves is $z=3z_{sh}$ with $\Gamma=100$, and for the beams it is $z=3z_0$ with $N=1$ and $A=0.01$. The plane compressional waveform (a) has the classical sawtooth shape associated with quadratic nonlinearity. A distinctive feature of shear waves is that there are two shocks per period. In the y component, spikes appear at the locations where shocks appear in the x component. Effects of beam diffraction are illustrated in the lower row. Note that the waveform asymmetry in the compressional wave beam is absent from the shear wave beam. This is because the cubic nonlinearity generates only odd harmonics in the nonlinear shear wave, and any waveform that can be expressed as a Fourier series of odd harmonics is necessarily symmetric, regardless of the relative harmonic phase shifts introduced by diffraction. However, if the initial waveform contains even harmonics, diffraction will introduce some degree of waveform asymmetry in a nonlinear shear wave beam. Waveforms calculated for linear polarization are much the same as those shown in Figs. 1(b) and 1(e).

While unrelated to diffraction, the formation of secondary shocks was observed in simulations of elliptically polarized shear waves at high amplitude. Shown in Fig. 2 are waveforms at the three distances (a) $z=3z_{sh}$, (b) $10z_{sh}$, and (c) $15z_{sh}$ for an elliptically polarized plane wave with $s=0.25$ and $\Gamma=500$. In each frame, the velocity component v_x is on the top, v_y on the bottom. Note the small spikes that appear in v_x at $3z_{sh}$ (a). Their locations coincide with the large spikes in v_y . By $10z_{sh}$ (b) the spikes have evolved into small step shocks that have advanced in the waveform (to earlier retarded times, from right to left in the figure). By $15z_{sh}$ (c) the secondary shocks have disappeared. For sufficiently large Γ these secondary shocks may reappear.

Presented in Fig. 3 are harmonic beam patterns at $z=5z_0$ with $N=0.2$ and $A=0.01$ for (a) a compressional wave beam, and for the (b) x and (c) y components in an elliptically polarized shear wave beam with $s=0.25$. The beam

FIG. 2. Waveforms in an elliptically polarized plane wave with $s=0.25$ and $\Gamma=500$ at (a) $z=3z_{sh}$, (b) $10z_{sh}$, and (c) $15z_{sh}$. In each frame, the velocity component v_x is on the top, v_y on the bottom.

patterns for the even harmonics in the compressional wave are deemphasized by use of a lighter line weighting to facilitate comparison of the odd harmonics with those of the shear wave. As with compressional waves, the number of sidelobes in the n th harmonic beam pattern for the shear wave is n times the number in the beam pattern at the source frequency. This is not predicted by Eq. (31) for the third harmonic in the shear wave. The directivity function $D^3(\theta)$ indicates a beam pattern having the same number of sidelobes as at the source frequency, not three times as many. These additional sidelobes, also observed in the beam patterns for the compressional wave, are near-field effects that become less pronounced as either distance or absorption is increased.¹⁴ For weak absorption, the far field for the nonlinearly generated harmonics is not reached until distances far beyond the Rayleigh distance z_0 .

Results for an azimuthally polarized beam are presented in Fig. 4 for $N=3$ and $A=0.01$. Figure 4(a) shows harmonic beam patterns at $z=z_0$ for the source frequency and its third

FIG. 3. Harmonic beam patterns at $z=5z_0$ with $N=0.2$ and $A=0.01$ for (a) a compressional wave beam, and for the (b) x and (c) y components in an elliptically polarized shear wave beam with $s=0.25$.

FIG. 4. (a) Harmonic beam patterns and (b) axial time waveform at $z=z_0$ in an azimuthally polarized beam with $N=3$ and $A=0.01$.

and fifth harmonics. Note that the levels of the third and fifth harmonics on the beam axis are maxima, not minima. The discrepancy with Eq. (47), which predicts zero for the amplitude of the third harmonic along the beam axis, is because the wave is not yet in the far field. Even though the nonlinear term in Eq. (41) is zero everywhere along the beam axis, harmonics generated nonlinearly off axis in the near field diffract into the axial region. Figure 4(b) shows the axial waveform at $z=z_0$. Like the waveforms in Figs. 1(e) and 1(f) for the elliptically polarized beam, the waveform in the azimuthally polarized beam is also symmetric.

IX. CONCLUSION

Effects of cubic nonlinearity in shear wave beams were analyzed for linear, elliptical, circular, and azimuthal polarizations, and comparisons were made with effects of quadratic nonlinearity in compressional wave beams. The coupled nonlinear evolution equations that were derived also include quadratic nonlinearity due to diffraction. Effects of

quadratic nonlinearity, and the generation of longitudinal components in shear wave beams, are under investigation and will be reported in the future.

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