Causal impulse response for circular sources in viscous media

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(Received 6 March 2007; revised 29 January 2008; accepted 30 January 2008)

The causal impulse response of the velocity potential for the Stokes wave equation is derived for calculations of transient velocity potential fields generated by circular pistons in viscous media. The causal Green's function is numerically verified using the material impulse response function approach. The causal, lossy impulse response for a baffled circular piston is then calculated within the near field and the far field regions using expressions previously derived for the fast near field method. Transient velocity potential fields in viscous media are computed with the causal, lossy impulse response and compared to results obtained with the lossless impulse response. The numerical error in the computed velocity potential field is quantitatively analyzed for a range of viscous relaxation times and piston radii. Results show that the largest errors are generated in locations near the piston face and for large relaxation times, and errors are relatively small otherwise. Unlike previous frequency-domain methods that require numerical inverse Fourier transforms for the evaluation of the lossy impulse response, the present approach calculates the lossy impulse response directly in the time domain. The results indicate that this causal impulse response is ideal for time-domain calculations that simultaneously account for diffraction and quadratic frequency-dependent attenuation in viscous media.

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PACS number(s): 43.58.Bh, 43.20.Px, 43.20.Hq, 43.35.Bf [TDM]

Pages: 2107–2116

I. INTRODUCTION

Several applications, including medical ultrasound, SONAR, and seismology, utilize broadband, three-dimensional (3D) sound beams in dissipative media, where attenuation is often modeled via a power-law relation.^{1–3} Since no closed-form Green's functions for arbitrary power-law media have been reported in the literature, loss is often approximated via viscous relaxation. As discussed elsewhere,^{4,5} the transient solutions in a viscous medium in 1D may be approximated in terms of Gaussian functions, thus avoiding the mathematical complexity encountered in the general power-law case. Since viscous media is a special case of power-law media, insight into general dissipative media is gained by studying the approximations available in viscous media.

The underlying physics of viscous diffusion are well documented,⁶ yet time-domain studies of wave propagation in viscous and other lossy media have been primarily limited to one-dimensional (1D) plane wave propagation. For example, the classic study by Blackstock⁴ approximated the viscous term in the Stokes wave equation by a third derivative, leading to a non-causal solution. The Stokes equation has also been approximated by the telegrapher's equation,⁷ leading to causal solutions with sharp wavefronts similar to electromagnetic propagation in conductive media. Recent studies derived causal Green's function solutions in the form of contour integrals,⁸ infinite series,⁹ and closed-form approximations.⁵ Although the combined effects of diffraction and loss have been thoroughly studied in the frequency

domain both analytically¹⁰ and numerically,¹¹ most time-domain studies have been analytical. For example, the combined effects of diffraction, dissipation, and nonlinearity were studied numerically in Ref. 12 under the parabolic approximation. In Ref. 13, a general approach combining linear with frequency loss¹⁴ and time-domain diffraction was proposed. A general 3D numerical model incorporating power-law loss and finite apertures was developed in Ref. 15.

In order to understand the interplay between diffraction and frequency-dependent attenuation, time-domain impulse response expressions for simple piston geometries are needed. Although impulse response expressions exist for lossless, homogeneous media for circular^{16,17} and rectangular¹⁸ apertures, no analytical expressions for the transient field radiated by finite planar apertures in a viscous medium have been published. An analytical expression for the on-axis velocity potential produced by a focused spherical shell was recently derived in Ref. 19, although field points off axis were not considered. In particular, the effect of viscous loss on the impulse response of a baffled circular piston has not been addressed.

The goal of this work is to derive the causal impulse response of the velocity potential for circular sources in viscous media satisfying the Stokes wave equation. In Sec. II, a method for computing solutions to the Stokes wave equation is developed. By decomposing a causal, approximate Green's function into diffraction and loss components, an analytical relationship between the lossless and lossy impulse response is determined. In Sec. III, this method is applied to the transient fast near field method²⁰ to compute the lossy impulse

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responses for the uniform circular piston. In Secs. IV and V, the lossy impulse responses are numerically evaluated and physically interpreted.

II. THEORY

In this section, an approximate causal Green's function solution to the Stokes wave equation is analyzed. The lossy impulse response in a viscous medium is then related to the corresponding impulse response in a lossless medium. The medium is assumed to be linear, isotropic, and homogeneous and bounded by an infinite, rigid baffle in the $z=0$ plane. Moreover, the transient Green's function $g(R, t)$ is assumed to satisfy the viscous wave equation, or Stokes equation:⁵

$$\nabla^2 g - \frac{1}{c_0^2} \frac{\partial^2 g}{\partial t^2} + \gamma \frac{\partial}{\partial t} \nabla^2 g = -\delta(t) \delta(\mathbf{R}), \quad (1)$$

where c_0 is the thermodynamic speed of sound and γ is the relaxation time of the medium. The relaxation time, which is proportional to the coefficient of shear viscosity μ , measures the time necessary to restore equilibrium to the translational degrees of freedom in the fluid following a small thermodynamic disturbance. If $\gamma=0$, equilibrium is restored immediately and Eq. (1) reduces to the lossless wave equation. The viscous term in Eq. (1) is a singular perturbation which transforms the lossless, second-order hyperbolic wave equation into a third-order parabolic equation. The dispersion relation between wavenumber k and angular frequency ω are computed via a space-time Fourier transform applied to Eq. (1), which are summarized by Eqs. (58) and (59) in Ref. 9:

$$k(\omega) = \frac{\omega}{c_0 \sqrt{1 + (\omega\gamma)^2}} \sqrt{1 - i\omega\gamma} \quad (2)$$

yielding an attenuation coefficient that is approximately proportional to ω^2 for $\omega\gamma \ll 1$.

A. Green's function decomposition

Unfortunately, no closed-form analytical Green's function is known to exist for the Stokes waves equation. However, an asymptotic causal Green's function for the 3D problem has been found.²¹ Alternatively, the 3D Green's function is derived in Appendix A with the methodology developed in Ref. 21, yielding

$$g(R, t) \approx \frac{u(t)}{4\pi R \sqrt{2\pi t \gamma}} \exp\left(-\frac{(t - R/c_0)^2}{2t\gamma}\right), \quad (3)$$

where $R=|\mathbf{r}-\mathbf{r}'|$ is the distance between observer $\mathbf{r}$ and source $\mathbf{r}'$ and $u(t)$ is the Heaviside unit step function. Equation (3), which is valid for small viscosity ($\gamma \rightarrow 0$) and large times ($t \rightarrow \infty$), supports infinite propagation speed, yet is strictly causal due to the Heaviside function $u(t)$. As shown in Sec. IV, Eq. (3) is an excellent, causal approximation except in the extreme near field or highly viscous media.

Equation (3) is recognized as the composition of (a) the Green's function for the 3D wave equation and (b) the Green's function for the (1D) diffusion equation. This observation motivates decomposing Eq. (3) into individual diffraction and loss factors via

$$g(R, t) = \int_{-\infty}^{\infty} \left[\frac{\delta(t - t' - R/c_0)}{4\pi R} \right] \times \left[u(t) \frac{1}{\sqrt{2\pi t \gamma}} \exp\left(-\frac{t'^2}{2t\gamma}\right) \right] dt', \quad (4)$$

where the sifting property of the Dirac delta function applied to Eq. (4) yields Eq. (3). Identifying the first bracketed expression as a diffraction component and the second bracketed expression as a loss component via

$$g_D(R, t) = \frac{\delta(t - R/c_0)}{4\pi R}, \quad (5a)$$

$$g_L(t, t') = u(t) \frac{1}{\sqrt{2\pi t \gamma}} \exp\left(-\frac{t'^2}{2t\gamma}\right), \quad (5b)$$

allows Eq. (4) to be expressed as

$$g(R, t) = g_L(t, t') \otimes g_D(R, t) \equiv \int_{-\infty}^{\infty} g_D(R, t - t') g_L(t, t') dt', \quad (6)$$

where the convolution denoted by $\otimes$ is performed with respect to the t' variable. In Eq. (6), the subscript "D" refers to diffraction, whereas the subscript "L" refers to loss. Equation (5a) is responsible for diffraction and the directivity of the radiating aperture. Equation (5b) is a Gaussian function centered at $t'=0$ having width, or standard deviation, $\sqrt{t\gamma}$. The t' variable is interpreted as "propagation time" (fast), whereas t is interpreted as the "diffusion time" (slow). In fact, Eq. (5b) is the causal Green's function for the 1D diffusion equation²²

$$\left(\frac{\partial}{\partial t} - \frac{\gamma}{2} \frac{\partial^2}{\partial t'^2} \right) g_L(t, t') = \delta(t) \delta(t'), \quad (7)$$

where $c_0 t'$ plays the role of the spatial variable. Thus, Eq. (6) is interpreted as the non-stationary convolution of the 1D Green's function for the diffusion equation with the Green's function with the 3D wave equation. As shown below, the decomposition embodied in Eq. (6) facilitates the generalization of the impulse response for circular apertures to viscous media.

B. Lossy impulse response function

The impulse response function $h_L(\mathbf{r}, t)$ in the half space is obtained by integrating Eq. (3) over the surface of the radiating aperture and multiplying by two to account for the infinite, planar baffle in the $z=0$ plane. Analytically, Eq. (6) allows the lossy impulse response to be evaluated for a circular piston. Integrating the decomposed Green's function over the aperture S and multiplying by two yields

$$h_L(\mathbf{r}, t) = g_L(t, t') \otimes \int_S \frac{\delta(t - R/c_0)}{2\pi R} d^2 \mathbf{r}'. \quad (8)$$

The second factor in Eq. (8) is identical to the standard impulse response for a uniform aperture, which has been calculated previously for a circular source. Therefore, the lossy impulse response function is expressed as a non-stationary convolution of the standard impulse response with a loss

FIG. 1. Piston and coordinate geometry. The piston, centered at the origin and surrounded by an infinite rigid baffle in the $z=0$ plane, has a radius a . The radial and axial observation coordinates are denoted by x and z , respectively. The distance between origin and observer is given by r and the angle between the radial and axial coordinates is θ .

function, where the convolution is taken over t' :

$$h_L(\mathbf{r}, t) = g_L(t, t') \otimes h(\mathbf{r}, t). \quad (9)$$

Equation (9) maps solutions of the hyperbolic wave equation to solutions of the parabolic Stokes equation, similar to the Q transform.²³ Since the loss function $g_L(t, t')$ is independent of R , the integration in Eq. (9) is independent of the geometry associated with the aperture S , allowing the lossy impulse response to be calculated from a known lossless impulse response.

III. CIRCULAR PISTON

Figure 1 displays the geometry and coordinate systems used in the following derivations. The lossy impulse response is determined for a uniform piston at all observation points in both the near and far fields. Equation (9) is evaluated analytically, yielding either closed-form or single-integral expressions which are then evaluated numerically. The transient velocity potential is then recovered via fast Fourier transform (FFT)-based convolutions.

A. Near field

1. Lossless impulse response

The near field solution to the transient, lossless, baffled circular piston problem may be expressed piecewise in terms of the rect function and the inverse cosine function.^{16,18} The non-stationary convolution is difficult to compute with these piecewise expressions; therefore, a single-integral expression that is valid for all observation points is instead utilized. In Ref. 20, a single integral expression for the pulsed pressure was derived. Assuming a Dirac delta excitation $v(t) = \delta(t)$ in Eq. 12 in Ref. 20 and utilizing the expression $h(x, z, t) = \int_{-\infty}^t p(x, z, t') dt' / \rho_0$ yields the time-domain impulse response

$$h(x, z, t) = \frac{c_0 a}{\pi} \int_0^\pi \frac{x \cos \psi - a}{x^2 + a^2 - 2ax \cos \psi} \times [u(t - \tau_1) - u(t - \tau_2)] d\psi, \quad (10)$$

where the delays τ_i are specified by

$$\tau_1 = \sqrt{x^2 + z^2 + a^2 - 2ax \cos \psi} / c_0 \quad \text{and} \quad (11a)$$

$$\tau_2 = z/c_0. \quad (11b)$$

In Eqs. (10) and (11), x is the radial coordinate and z is the axial coordinate. The integration over ψ can be evaluated in closed form, yielding the piecewise-defined lossless impulse response given in Ref. 18. However, the integral representation given by Eq. (10) leads directly to the lossy impulse response that follows.

2. Lossy impulse response

The lossless near field solution given by Eq. (10) solves the transient wave equation assuming a uniform piston in an infinite rigid baffle. The lossy impulse response is computed by inserting Eq. (10) into Eq. (9) and evaluating the unit step functions in Eq. (10). The integration over t' is evaluated in terms of the error function²⁴ $\text{erf}(z)$, yielding

$$h_L(x, z, t) = \frac{c_0 a}{2\pi} u(t) \int_0^\pi \frac{x \cos \psi - a}{x^2 + a^2 - 2ax \cos \psi} \times \left[\text{erf}\left(\frac{t - \tau_1}{\sqrt{2t\gamma}}\right) - \text{erf}\left(\frac{t - \tau_2}{\sqrt{2t\gamma}}\right) \right] d\psi, \quad (12)$$

with τ_1 and τ_2 defined in Eq. (11). On axis ($x=0$), the lossy impulse response reduces to

$$h_L(0, z, t) = \frac{c_0 u(t)}{2} \left[\text{erf}\left(\frac{t - z/c_0}{\sqrt{2t\gamma}}\right) - \text{erf}\left(\frac{t - \sqrt{z^2 + a^2/c_0}}{\sqrt{2t\gamma}}\right) \right]. \quad (13)$$

As in the lossless case, the first and second delays in Eq. (13) correspond to the closest point and the furthest point on the radiating piston, respectively. In Eqs. (12) and (13), The “slow” time scale is embodied in the denominator of the erf functions $\sqrt{2t\gamma}$, whereas the “fast” time scale is embodied in the numerator. The lossy impulse response described by Eqs. (12) and (13) possesses an infinitely long tail that decays like a Gaussian.

B. Far field

1. Lossless impulse response

Although Eq. (12) is valid for all field points, a simpler expression exists in the far field, where the effects of attenuation are more pronounced. Here, the definition of the far field is the same as that utilized by Morse and Ingard,²⁵ where the observation distance is much larger than the radius of the aperture ($r \gg a$). This definition of the far field is distinguished from the traditional far field distance $r > a^2/\lambda$, which is only valid over a finite frequency band. As shown in Appendix B, the lossless impulse response in the far field is represented by the single-integral expression

$$h(r, \theta, t) = \frac{c_0 a}{\pi r \sin \theta} \int_0^\pi \cos \psi u(t - r/c_0 + a \sin \theta \cos \psi / c_0) d\psi, \quad (14)$$

which is valid for $r \gg a$ and $\theta > 0$. Note that Eq. (14) has support on the time interval $[r/c_0 - a \sin \theta / c_0, r/c_0]$

$+a \sin \theta/c_0]$, which follows from the fact that $\int_0^\pi \cos \psi d\psi = 0$. On axis ($\theta=0$), the impulse response reduces to

$$h(z, 0, t) = \frac{a^2}{2z} \delta(t - z/c_0), \quad (15)$$

which is the limiting case of Eq. (10) as $z \rightarrow \infty$.

2. Lossy impulse response

The far field viscous lossy impulse response is now computed from Eqs. (14) and (15). For the off-axis case, Eq. (14) is substituted into Eq. (9). Interchanging the order of integration and evaluating the unit-step function in the integrand yields

$$h_L(r, \theta, t) = \frac{c_0}{\pi r \sin \theta} \frac{u(t)}{\sqrt{2\pi t \gamma}} \int_0^\pi \int_{-\infty}^{\tau'(\psi)} \exp\left(-\frac{t'^2}{2t\gamma}\right) \times \cos \psi dt' d\psi, \quad (16)$$

where

$$\tau'(\psi) = t - r/c_0 + a \sin \theta \cos \psi/c_0. \quad (17)$$

The integration over t' is evaluated using $2/\sqrt{\pi} \int_{-\infty}^x e^{-z^2} dz = 1 + \text{erf}(x)$. Due to the $\cos \psi$ term in Eq. (16), the constant term vanishes, yielding a single integral expression

$$h_L(r, \theta, t) = \frac{c_0 a u(t)}{2\pi r \sin \theta} \int_0^\pi \text{erf}\left(\frac{t - r/c_0 + a \sin \theta \cos \psi/c_0}{\sqrt{2t\gamma}}\right) \times \cos \psi d\psi. \quad (18)$$

The on-axis case is computed by inserting Eq. (15) into Eq. (9) and utilizing the sifting property of the Dirac delta function, yielding

$$h_L(r, 0, t) = u(t) \frac{a^2 \exp(-(t - r/c_0)^2/2t\gamma)}{2r \sqrt{2\pi t \gamma}}, \quad (19)$$

which can also be derived by letting $z \rightarrow \infty$ in Eq. (13). Unlike the lossless case, the lossy impulse response does not have compact support in time. Since both the Gaussian and error functions have infinite support, the lossy impulse responses defined by Eqs. (18) and (19) are also non-zero for all positive time values. In physical terms, this semi-infinite region of support implies wave components traveling with phase speeds varying from zero to infinity, but the infinite phase speeds are infinitely attenuated,⁵ so the result is causal.

IV. NUMERICAL RESULTS

A. Green's function evaluation

A reference solution to the Stokes wave equation is evaluated using the material impulse response function (MIRF) approach outlined in Ref. 3, and the result is compared to Eq. (3). The wavenumber $k(\omega)$ is calculated via the reference dispersion relation given by Eq. (2), allowing the material transfer function to be calculated in the frequency domain. As specified by Eq. (10) in Ref. 3, the reference Green's function, or MIRF, is then recovered by inverse Fourier transforming the material transfer function (MTF), or frequency-domain Green's function, via the inverse fast Fourier transform (FFT), where

rier transforming the material transfer function (MTF), or frequency-domain Green's function, via the inverse fast Fourier transform (FFT), where

$$g(R, t) = \mathcal{F}^{-1}\left(\frac{e^{ik(\omega)R}}{4\pi R}\right) \quad (20)$$

Since $g(R, t)$ is necessarily real, conjugate symmetry of the MTF is enforced in the frequency domain. The relative L^2 error, defined via

$$\text{error} = \frac{\sqrt{\int_{-\infty}^{\infty} |\phi(t) - \phi_{\text{ref}}(t)|^2 dt}}{\sqrt{\int_{-\infty}^{\infty} |\phi_{\text{ref}}(t)|^2 dt}} \quad (21)$$

is calculated with the MIRF approach used as the reference.

Figure 2 summarizes the results of this comparison for various combinations of observation distance R and viscous relaxation time γ . Panel (a), which shows the numerical reference and approximate Green's functions using $R=1$ mm and $\gamma=0.01$ μs , displays a significant disparity between the reference and asymptotic Green's functions when γ is relatively large and R is relatively small. Panel (b), which is evaluated with $R=1$ mm and $\gamma=0.01$ μs , shows a much smaller variation between the asymptotic and reference Green's functions. Panels (c) and (d) show the Green's functions for $R=1$ mm and $\gamma=0.01$ μs and $\gamma=0.001$ μs , respectively; these panels show very close agreement between the reference and asymptotic Green's functions. The asymptotic and reference Green's functions are virtually identical in panel (d) of Fig. 2, implying that Eq. (3) is an excellent approximation for $R > 1$ mm and $\gamma < 0.001$ μs .

The root mean square (RMS) error is a function of the dimensionless parameter $R/(\gamma c_0)$. A quantitative error analysis comparing the asymptotic Green's function to the MIRF result is displayed in Fig. 3 in terms of $R/(\gamma c_0)$ on a log-log plot. From the slope of the curve in Fig. 3, the RMS error is approximately proportional to $\sqrt{\gamma c_0}/R$. As expected, the error increases as R increases and decreases as R increases. For a 1% error threshold, $R/(\gamma c_0) \geq 3.3 \times 10^3$, yielding the relation $\gamma \leq 3 \times 10^{-4} R/c_0$. By fixing γ and letting R vary, the minimum acceptable distance is determined by $R \geq 5 \times 10^3 c_0 \gamma$.

B. Circular piston evaluation

Since the lossy impulse response solutions utilize an approximate transient Green's function to the Stokes wave equation, a numerical comparison is made to a reference frequency-domain solution. The numerical reference is computed by computing the Fourier transform of the velocity potential $\hat{\Phi}(\mathbf{r}, \omega) = \hat{v}(\omega) \hat{h}(\mathbf{r}, k(\omega))$ using the dispersion relations $k(\omega)$ given by Eq. (2). This product is computed over the effective bandwidth of $\hat{v}(\omega)$ and then numerically inverse Fourier transformed using an inverse FFT. For a uniform circular piston of radius a , the on-axis ($r=0$) transfer function $\hat{h}(0, z; \omega)$ exists in closed form, thereby obviating the need for numerical integration.

FIG. 2. Comparison of the asymptotic form of the Green's function for the Stokes wave equation (Eq. (3)) and the reference Green's function computed numerically via the MIRF approach. The Green's function is shown for (a) $R=0.1$ mm and $\gamma=0.01$ μ s, (b) $R=0.1$ mm and $\gamma=0.001$ μ s, (c) $R=1.0$ mm and $\gamma=0.01$ μ s and (d) $R=1.0$ mm and $\gamma=0.001$ μ s.

Figure 4 displays the on-axis velocity potential for a circular piston of radius $a=10$ mm in a viscous medium for the same combinations of z and γ utilized in Fig. 2. A Hanning-weighted toneburst

FIG. 3. The RMS error for the asymptotic Green's function relative to the reference MIRF result. The RMS error is displayed on a log-log plot relative to the dimensionless parameter $R/(\gamma c_0)$. The slope on the log-log plot is approximately 0.5, indicating that RMS error is proportional to $\sqrt{\gamma c_0/R}$.

$$v(t) = \frac{1}{2}(1 - \cos(2\pi t/W))\text{rect}\left(\frac{t}{W}\right)\sin(2\pi f_0 t) \quad (22)$$

with center frequency $f_0=6.0$ MHz and duration $W=0.5$ μ s is applied, where $\text{rect}(t)=u(t)-u(t-1)$. The velocity potential is obtained for (a) $z=0.1$ mm and $\gamma=0.01$ μ s, (b) $z=0.1$ mm and $\gamma=0.001$ μ s, (c) $z=1$ mm and $\gamma=0.01$ μ s, and (d) $z=1$ mm and $\gamma=0.001$ μ s. The error is computed between the lossy impulse response and the MIRF method using Eq. (21), using the MIRF field as reference. The resulting errors for the four cases are (a) 26.7%, (b) 3.26%, (c) 6.89%, and (d) 2.71%, respectively. Thus, larger errors are observed for points closer to the piston and large relaxation times due to the approximation in the Green's function given by Eq. (3). For $z > 10$ mm and $\gamma \leq 0.001$ μ s, the relative L^2 error is less than or equal to 1%. Since the effects of viscous dissipation are negligible in the extreme near field, the lossy impulse response yields an accurate solution that captures the combined effects of diffraction and viscous dissipation. Finally, Eq. (9) accounts for the small differences in the arrival time of each attenuated spherical wave emitted by the radiating aperture via the non-stationary convolution. Hence, no additional error is introduced in the calculation of the lossy impulse response.

Figure 5 plots the RMS error versus axial distance in terms of the normalized distance z/a for the on-axis lossy impulse response relative to the reference MIRF result. Error is shown for four representative viscous relaxation times γ . For $\gamma=0.0001$ μ s, 0.0003 μ s, and 0.001 μ s, the RMS error decreases as a function of normalized distance. For $\gamma=0.003$ μ s, the minimum RMS error occurs at $z/a \approx 2$, however. Unlike the Green's function error shown in Fig. 2, the error here is not controlled by a single parameter. Rather, error decreases slowly as a function of z/a , eventually reaching an error floor that depends on the viscous relaxation time.

C. Evaluation of the lossy impulse response

In this section, the lossy impulse response expressions derived in Secs. II and III are numerically evaluated. The velocity potential $\Psi(\mathbf{r}, t)$ resulting from the temporal convolution of a transient excitation with the lossy impulse response is also computed and discussed. All single-integral

FIG. 4. On-axis ($x=0$ mm) velocity potential for a circular piston of radius $a=10$ mm in a viscous medium. The piston is excited by a Hanning-weighted toneburst with center frequency $f_0=6.0$ MHz and pulse length $W=0.5$ μ s for four combinations of axial distance z and relaxation time γ : (a) $z=0.1$ mm and $\gamma=0.01$ μ s, (b) $z=0.1$ mm and $\gamma=0.001$ μ s, (c) $z=1$ mm and $\gamma=0.01$ μ s, (d) $z=1$ mm and $\gamma=0.001$ μ s. The velocity potential for each combination of z and γ is computed via both the lossy impulse response approach and a frequency-synthesis approach for verification.

FIG. 5. The RMS error for the on-axis lossy impulse response relative to the reference MIRF result for a piston with radius $a=10$ mm. The RMS error is plotted versus the normalized axial distance z/a .

expressions are numerically integrated using Gauss–Legendre quadrature.²⁶ In the following velocity potential computations, the Hanning-weighted toneburst given by Eq. (22) is utilized.

Figures 6 and 7 display the near field impulse response for a circular piston with radius $a=10$ mm for lossless media and viscous media, respectively. Figure 6 shows two snapshots of the lossless impulse response ($\gamma=0$) on a two-dimensional computational grid extending from $x=-40$ mm to $x=40$ mm in the lateral direction and $z=20$ to $z=60$ mm in the axial direction. Snapshots of the impulse response are shown at two instances in time: $t=22$ μ s and $t=34$ μ s. As shown in Fig. 6, the field within the paraxial region ($|x| \leq 10$ mm) in the lossless case is unattenu-

FIG. 6. Snapshots of the lossless impulse response generated by a circular piston with radius $a=10$ mm at $t=22$ μ s and $t=34$ μ s are displayed in panels (a) and (b). The constant amplitude component within the paraxial region $|x| < a$ represents the direct wave. The remaining component represents the edge wave generated by the discontinuity in particle velocity at $x=a$.

FIG. 7. Snapshots of the lossy impulse response generated by a circular piston with radius $a=10$ mm with $\gamma=0.01$ μs for $t=22$ μs and $t=34$ μs are displayed in panels (a) and (b). Unlike the lossless impulse response depicted in Fig. 6, the direct wave is attenuated due to viscous diffusion. The edge wave also experiences additional attenuation relative to Fig. 6.

ated, while the corresponding region in Fig. 7 experiences attenuation and stretching due to viscous diffusion. For increasing relaxation times, the field becomes progressively closer to that generated by an omni-directional source as the directivity of the aperture is weakened far from the source.

The near field velocity potential field generated by a piston with radius $a=10$ mm within a viscous medium with relaxation time $\gamma=0.001$ μs is displayed in Fig. 8 at two successive snapshots in time. At $t=22$ μs , distinct direct and edge waves are evident, with little apparent viscous spreading. As time increases, the edge and direct waves become less distinct, while spreading and attenuation due to viscous loss become more pronounced. Unlike the lossless case, the direct wave in Fig. 8 experiences a significant decrease in amplitude.

Figure 9 displays both the near field and far field lossy impulse responses for a circular piston ($a=5$ mm). Panel (a) evaluates the lossy impulse response at $r=50$ mm both on axis ($\theta=0$) and off axis ($\theta=\pi/12$ and $\phi=0$). In this region, the far field approximation differs significantly from the near field solution. Panel (b) evaluates the impulse response at $r=200$ mm both on axis ($\theta=0$) and off axis ($\theta=\pi/12$ and $\phi=0$). In this region, the far field approximation agrees with the near field solution, indicating that the simplified far field expressions accurately represent the lossy impulse response at distances far from the source.

FIG. 8. Normalized velocity potential field produced by a circular piston of radius $a=10$ mm excited by a Hanning-weighted toneburst in a viscous medium with relaxation time $\gamma=0.001$ μs . Snapshots of the normalized velocity potential for $t=22$ μs and $t=34$ μs are displayed in panels (a) and (b).

FIG. 9. Lossy impulse response for a circular piston ($a=5$ mm) and $\tau_s=0.001$ μs . In panel (a), the near field impulse response and the far field impulse response were evaluated at $r=50$ mm both on axis ($\theta=0$) and off axis ($\theta=\pi/12$ and $\phi=0$). Panel (b) shows the near field and far field impulse responses evaluated at $r=200$ mm on axis ($\theta=0$) and off axis ($\theta=\pi/12$ and $\phi=0$).

V. DISCUSSION

A. Physical interpretation

The physical interpretation of the uniformly excited circular piston discussed in Ref. 27 is also applicable to transient propagation in viscous media. Physically, the first term in Eq. (12) corresponds to the *edge wave* generated by the discontinuity at the edge of the piston, whereas the second term corresponds to the *direct wave* due to the bulk motion of the piston. In the lossless case given by Eq. (10), the direct wave contribution localized within the paraxial region of the radiator maintains a constant amplitude; however, for the lossy wave, the direct wave decays as z increases. As Fig. 6 shows, the direct and edge waves are clearly discernible in the lossless case, while Fig. 7 shows the smooth transition between edge and direct waves in the presence of loss.

Numerical evaluations of the lossy impulse response show distinct behavior within the near field, the transition region, and the far field. For the values of γ evaluated in Fig. 9, the diffraction component dominates in the near field, with only a small amount of smoothing in the vicinity of the head and tail of the response. For observation points in the transition region between the near and far fields, the effects of diffraction and loss are both apparent. In this transition region, the impulse response is both smooth and asymmetric. Finally, in the far field, the effects of viscous loss predominate, yielding increasingly symmetric and broad responses with a reduced directivity.

Large values of γ were chosen to determine the range of values for which the approximation holds. The viscous relaxation time is related to the attenuation coefficient α at a fixed frequency f via the relation²⁸ $\gamma = c_0 \alpha / (2\pi^2 f^2)$. For instance, soft tissue at $f = 3$ MHz with an attenuation coefficient of $\alpha = 0.345 \text{ cm}^{-1}$ and sound speed $c_0 = 1.5 \text{ mm}/\mu\text{s}$ corresponds to $\gamma = 3 \times 10^{-4} \mu\text{s}$. Applying the error analysis developed in the Results section, R must be chosen larger than 1.5 mm to maintain less than 1% error for these parameters.

B. Implementation issues

Unlike the MIRF method and other schemes that synthesize fields in the frequency domain, the lossy impulse response solution presented here is computed directly in the time domain without inverse FFTs. Numerical inverse FFTs pose several difficulties, including (1) additional computational burden and (2) incursion of numerical error due to undersampling in the frequency domain. Since the bandwidth of the MIRF is a function of distance from the source, the Nyquist sampling frequency is a function of distance. Therefore, an efficient MIRF implementation requires multi-rate sampling. The lossy impulse response method eliminates these problems by replacing numerical inverse Fourier transforms that utilize a compact time-domain expression with a constant sampling rate for all observation points. In addition, computing snapshots of the impulse response at one particular point in time is particularly straightforward with the lossy impulse response method presented in Eq. (12).

C. Numerical evaluation and aliasing

As discussed in Ref. 29, evaluation of the lossless impulse response requires artificially high sampling rates in order to accommodate the discontinuities in the derivative of the impulse response, which result in large bandwidths. These discontinuities are magnified on axis and in the far field, where the lossless impulse response is represented by a short-duration rect function in the near field and a scaled delta function in the far field. However, these numerical difficulties are caused, in part, by an inaccurate physical model which assumes zero loss and viscous spreading. In reality, some loss is always present, effectively acting as a low-pass filter and removing the high-frequency components of the impulse response. This filtering effect is reflected in the lossy impulse response expressions, which are infinitely smooth, implying a Fourier transform that decays faster than $1/f^n$ for any $n > 1$ where f represents the frequency.³⁰ In contrast, the Fourier transform of the lossless impulse response, due to discontinuities on axis, decays as $1/f$. Due to this rapid decay in the frequency domain, the lossy impulse response is bandlimited, whereas the lossless response is band unlimited. The numerical difficulties arising from the band unlimited lossless impulse response are thus eliminated by the inclusion of a loss mechanism, which effectively band limits the impulse response. Since both the diffraction and loss components are evaluated simultaneously in the present formulation [cf. Eq. (6)], sampling errors are reduced with the lossless impulse response.

D. Other dispersive media

Since the Stokes wave equation serves as a first approximation for loss in biological tissue, more accurate models, such as power-law media,^{2,15,31,32} also demand consideration. In power-law media, phase velocity is an increasing function of frequency, resulting in an asymmetric loss function function g_L . In a future effort, loss functions for power-law media will be derived, allowing the lossy impulse responses generated by various piston sources to be computed in tissue-like media. The general power-law case will utilize the same machinery as the viscous case with an altered loss function. Therefore, the main benefits of the lossy impulse response approach (no inverse FFTs, lack of aliasing, etc.) should apply in the more general power-law setting. Since all calculations are directly in the time domain, the lossy impulse response approach should provide a more intuitive solution to transient, dispersive problems. Unlike the frequency-squared case, the general loss function for power-law media cannot be approximated via a simple expression such as a Gaussian. Thus, the derivation of the impulse response for finite aperture radiators in power-law media becomes more complicated, necessitating more advanced mathematics. Nevertheless, these analytical models will facilitate validation of numerical scattering studies in linear dissipative media.³³

VI. CONCLUSION

The lossy impulse response for a circular piston is derived by decomposing an asymptotic Green's function into diffraction and loss operators. This decomposed Green's

function relates the solutions to the Stokes wave solution and the lossless wave equation via Eq. (9), facilitating the derivation of the lossy impulse response in Eq. (12). The resulting expressions are strongly causal and straightforward to implement numerically. Equation (12) also eliminates the aliasing problems associated with the lossless impulse response. The error incurred via the approximate Green's function given by Eq. (3) is analyzed as function of relaxation time γ and distance from the source R , yielding the simple relation between γ and R for a 1% error threshold. Velocity potential computations display a significant attenuation of the direct wave within the paraxial region of the piston due to viscous diffusion. The validity of the far field expression given by Eq. (18) is also studied as a function of distance. Both the physical and numerical properties of the resulting lossy impulse responses are analyzed, revealing the critical role of dissipation in the far field.

ACKNOWLEDGMENTS

Insightful discussions with Liyong Wu, GE Medical Systems, and Thomas L. Szabo, University of Boston, are acknowledged with pleasure. This work was funded in part by NIH Grant Nos. 1R01 CA093669 and 1R21 CA121235.

APPENDIX A: DERIVATION OF THE ASYMPTOTIC GREEN'S FUNCTION

Fourier transforming Eq. (1) from the space-time domain $(\mathbf{R}, t)$ to the spectral-frequency domain $(\mathbf{p}, \omega)$ via the convention utilized in Ref. 5 produces

$$\left(-p^2 + \frac{\omega^2}{c_0^2} - \gamma i p^2 \omega\right) g_{\omega p} = -1, \quad (\text{A1})$$

where $p = |\mathbf{p}|$ is the magnitude of the spatial frequency vector $\mathbf{p}$. Solving for $g_{\omega p}$ and performing an inverse Fourier transform over ω gives

$$g_p = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{\omega p} \exp(i\omega t) d\omega \quad (\text{A2})$$

$$= -\frac{c_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{(\omega - \omega_+)(\omega - \omega_-)} d\omega, \quad (\text{A3})$$

where $\omega_{\pm} = i\gamma c_0^2 p^2 / 2 \pm c_0 p \chi$ and $\chi = \sqrt{1 - \gamma^2 c_0^2 p^2 / 4}$. Evaluating Eq. (A3) via contour integration yields

$$g_p(t) = \frac{c_0 u(t)}{\chi p} \exp\left(-\frac{\gamma c_0^2 p^2 t}{2}\right) \sin(c_0 p \chi t). \quad (\text{A4})$$

The 3D Green's function is recovered via a threefold inverse Fourier transform expressed in spherical coordinates (p, θ_p, ϕ_p) , giving

$$g(R, t) = \frac{1}{8\pi^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} g_p(t) \exp(ipR \cos \theta_p) \times \sin \theta_p p^2 d\phi_p d\theta_p dp. \quad (\text{A5})$$

The ϕ_p integral and the θ_p integral are evaluated, yielding

$$g(R, t) = \frac{1}{2\pi^2 R} \int_0^\infty p \sin(pR) g_p(t) dp. \quad (\text{A6})$$

Inserting Eq. (A4) into Eq. (A6) and exploiting the evenness of the integrand produces

$$g(R, t) = \frac{c_0 u(t)}{4\pi^2 R} \int_{-\infty}^{\infty} \frac{\sin(pR) \exp\left(-\frac{\gamma c_0^2 p^2 t}{2}\right) \sin(c_0 p \chi t)}{\chi} dp. \quad (\text{A7})$$

Note that Eq. (A7) is an exact Fourier integral representation of the 3D Green's function. Unfortunately, this integral cannot be evaluated in closed form. However, approximating $\chi = 1$ yields

$$g(R, t) = \frac{c_0 u(t)}{4\pi^2 R} \int_{-\infty}^{\infty} \sin(pR) \exp\left(-\frac{\gamma c_0^2 p^2 t}{2}\right) \sin(c_0 p t) dp. \quad (\text{A8})$$

By exploiting the cosine addition formula, the following identity is obtained:

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} \exp(-ap^2) \sin(Rp) \sin(bp) dp \\ &= \frac{1}{\sqrt{4\pi a}} [\exp(-(R-b)^2/4a) - \exp(-(R+b)^2/4a)]. \end{aligned} \quad (\text{A9})$$

Evaluating the inverse transform of Eq. (A8) by taking $a = \gamma c_0^2 t / 2$ and $b = c_0 t$ yields

$$g(R, t) \approx \frac{u(t)}{4\pi R \sqrt{2\pi \gamma t}} \left[\exp\left(-\frac{(t - R/c_0)^2}{2\gamma t}\right) - \exp\left(-\frac{(t + R/c_0)^2}{2\gamma t}\right) \right]. \quad (\text{A10})$$

Physically, the first term in Eq. (A10) represents an outgoing wave, while the second term represents an incoming wave. For large times $\gamma t \gg 1$, the incoming wave is negligible, yielding Eq. (3).

APPENDIX B: DERIVATION OF LOSSLESS FAR FIELD EXPRESSION

The single-integral representation of the far field lossless impulse response function utilized in Sec. III is derived from their well-known frequency-domain counterparts. In the far field region, the impulse response is expressed via an inverse Fourier transform

$$h(r, \theta, t) = \mathcal{F}^{-1}[\hat{h}(r, \theta, \omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikr}}{2\pi r} D(\theta; \omega) e^{-i\omega t} d\omega, \quad (\text{B1})$$

where $D(\theta; \omega)$ is the far-field pattern and $k = \omega / c_0$ is the (lossless) wavenumber. Equation (14) is computed via inverse Fourier transforming the classical frequency-domain result. The far field pattern is given by

$$D(\theta; \omega) = \pi a^2 \frac{2J_1(ka \sin \theta)}{ka \sin \theta}, \quad (\text{B2})$$

where $J_1(z)$ is the Bessel function of the first kind of order 1. To invert the Bessel function in Eq. (B1), an integral representation is utilized:

$$J_n(z) = \frac{(i)^n}{\pi} \int_0^\pi \cos n\psi e^{-iz \cos \psi} d\psi. \quad (\text{B3})$$

Inserting Eqs. (B3) with $n=1$ and (B2) into Eq. (B1) yields

$$h(r, \theta, t) = \frac{c_0 a}{\pi r \sin \theta} \mathcal{F}^{-1} \left[\frac{e^{i\omega r/c_0}}{-i\omega} \int_0^\pi \cos \psi e^{-ika \sin \theta \cos \psi} d\psi \right]. \quad (\text{B4})$$

Noting the spectral integration and applying the shifting properties of the Fourier transform yields Eq. (14).

- ¹P. He, "Simulation of ultrasound pulse propagation in lossy media obeying a frequency power law," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **45**, 114–125 (1998).
- ²R. S. C. Cobbold, N. V. Sushilov, and A. C. Weathermon, "Transient propagation in media with classical or power-law loss," *J. Acoust. Soc. Am.* **116**, 3294–3303 (2004).
- ³T. L. Szabo, "The material impulse response for broadband pulses in lossy media," *Proceedings of the IEEE Ultrasonics Symposium*, Honolulu, HI (2003), pp. 748–751.
- ⁴D. T. Blackstock, "Transient solution for sound radiated into a viscous fluid," *J. Acoust. Soc. Am.* **41**, 1312–1319 (1967).
- ⁵M. J. Buckingham, "Causality, Stokes' wave equation, and acoustic pulse propagation in a viscous fluid," *Phys. Rev. E* **72**, 026610 (2005).
- ⁶P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (Princeton University Press, Princeton, NJ, 1968), pp. 270–300.
- ⁷P. M. Jordan, M. R. Meyer, and A. Puri, "Causal implications of viscous damping in compressible fluid flows," *Phys. Rev. E* **62**, 7918 (2000).
- ⁸R. Ludwig and P. L. Levin, "Analytical and numerical treatment of pulsed wave propagation into a viscous fluid," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **42**, 789–792 (1995).
- ⁹G. C. Gaunaurd and G. C. Everstine, "Viscosity effects on the propagation of acoustic transients," *J. Vibr. Acoust.* **124**, 19–25 (2002).
- ¹⁰G.-P. J. Too, "New phenomena on King integral with dissipation," *J. Acoust. Soc. Am.* **101**, 119–124 (1997).
- ¹¹P. T. Christopher and K. J. Parker, "New approaches to linear propagation of acoustic fields," *J. Acoust. Soc. Am.* **90**, 507–521 (1991).
- ¹²Y.-S. Lee and M. F. Hamilton, "Time-domain modeling of pulsed finite-amplitude sound beams," *J. Acoust. Soc. Am.* **97**, 906–917 (1995).
- ¹³J. A. Jensen, D. Gandhi, and W. D. O'Brien, "Ultrasound fields in an attenuating medium," *Proceedings of the IEEE Ultrasonics Symposium*, Baltimore, MD (1993), pp. 943–946.

- ¹⁴K. V. Gurumurthy and R. M. Arthur, "A dispersive model for the propagation of ultrasound in soft-tissue," *Ultrason. Imaging* **4**, 355–377 (1982).
- ¹⁵M. G. Wismer and R. Ludwig, "An explicit numerical time domain formulation to simulate pulsed pressure waves in viscous fluids exhibiting arbitrary frequency power law attenuation," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **42**, 1040–1049 (1995).
- ¹⁶F. Oberhettinger, "On transient solutions of the baffled piston problem," *J. Res. Natl. Bur. Stand., Sect. B* **65B**, 1–6 (1961).
- ¹⁷G. R. Harris, "Review of transient field theory for a baffled planar piston," *J. Acoust. Soc. Am.* **70**, 10–20 (1981).
- ¹⁸J. C. Lockwood and J. G. Willette, "High-speed method for computing the exact solution for the pressure variations in the nearfield of a baffled piston," *J. Acoust. Soc. Am.* **53**, 735–741 (1973).
- ¹⁹J. Djelouah, N. Bouaoua, A. Alia, H. Khelladi, and D. Belgrounde, "Theoretical and experimental study of the diffraction of focused ultrasonic waves in viscous fluids," *2003 World Congress on Ultrasonics*, Paris 1347–1350 (2003).
- ²⁰J. F. Kelly and R. J. McGough, "A time-space decomposition method for calculating the nearfield pressure generated by a pulsed circular piston," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **53**, 1150–1159 (2006).
- ²¹D. G. Crighton, *Modern Methods in Analytical Acoustics: Lecture Notes* (Springer-Verlag, London, 1996).
- ²²I. Stakgold, *Green's Functions and Boundary Value Problems*, 2nd ed. (Wiley, New York, 1998), p. 198.
- ²³K. H. Lee and G. Xie, "A new approach to imaging with low frequency electromagnetic fields," *Geophysics* **58**, 780–796 (1993).
- ²⁴M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1972), pp. 295–309.
- ²⁵P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968), p. 388.
- ²⁶P. J. Davis and P. Rabinowitz, *Numerical Integration* (Academic, New York, 1975), pp. 138–140.
- ²⁷G. E. Topholme, "Generation of acoustic pulses by baffled plane pistons," *Mathematika* **16**, 209–224 (1969).
- ²⁸L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, 4th ed. (Wiley, New York, 2000), p. 213.
- ²⁹D. P. Orofino and P. C. Pedersen, "Multirate digital signal-processing algorithm to calculate complex acoustic pressure fields," *J. Acoust. Soc. Am.* **92**, 563–582 (1992).
- ³⁰R. N. Bracewell, *The Fourier Transform and Its Applications* (McGraw-Hill, New York, 1978), p. 144.
- ³¹T. L. Szabo, "Time-domain wave equations for lossy media obeying a frequency power-law," *J. Acoust. Soc. Am.* **96**, 491–500 (1994).
- ³²N. V. Sushilov and R. S. Cobbold, "Frequency-domain wave equation and its time-domain solutions in attenuating media," *J. Acoust. Soc. Am.* **115**, 1431–1436 (2004).
- ³³M. G. Wismer, "Finite element analysis of broadband acoustic pulses through inhomogeneous media with power law attenuation," *J. Acoust. Soc. Am.* **120**, 3493–3502 (2006).