Published in final edited form as:

J Stat Plan Inference. 2008 October 1; 138(10): 2847–2866. doi:10.1016/j.jspi.2007.10.021.

Testing Lack of Symmetry in Spatial-Temporal Processes

Man Sik Park^{a,*} and Montserrat Fuentes^b

^aDepartment of Biostatistics, Korea University, Seoul 136-701, Korea

^bDepartment of Statistics, North Carolina State University, Raleigh, NC 27695-8203, USA

Abstract

Symmetry and separability of spatial-temporal covariances are the main assumptions that are frequently taken for granted in most applications because of the simplicity of constructing covariance structure. However, many studies in environmental sciences show that real data have complex spatial-temporal dependency structures resulting from lack of symmetry or violation of other standard assumptions of the covariance function. In this study, we propose new formal tests for lack of symmetry by using spectral representations of the spatial-temporal covariance functions. The advantage of the proposed tests is that classical analysis of variance (ANOVA) models can be used for detecting lack of symmetry inherent in spatial-temporal processes. We evaluate the performance of the tests with simulation studies and we apply them to air pollution data.

Keywords

Air pollution; asymmetry; nonseparability; spatial-temporal process; Matérn covariance; spectral density function

1 Introduction

Symmetry and separability are the main assumptions used in spatial statistics about a covariance function. The basic concept of symmetry is the relationship between any two elements, (i, j) and (j, i) in any variance-covariance matrix, $\Sigma = \{\sigma_{ij}\}_{i,j=1,\dots,n}$, that is, $\sigma_{ij} = \sigma_{ji}$. In spatial or spatial-temporal processes, we can think of the following case

$$\sigma_{ij} = \sigma_{kl}, \quad (1)$$

where (k, l) element may not be diagonally located with (i, j) element. Two concepts of symmetry in this study will be founded on (1). Separability inherent in spatial or spatial-temporal processes is the property of a variance-covariance matrix that can be represented as the addition or the multiplication of small variance-covariance matrices. Symmetry and separability of a covariance for spatial or spatial-temporal processes are highly related. Separability provides many advantages such as the simplified representation of the covariance matrix and, consequently, remarkable computational benefits. Symmetry is related to the spatial or the spatial-temporal dependencies that are closely related to the structural change of neighborhood. This characteristic has been assumed because of mathematical convenience, modeling parsimony, or calculational efficiency. The common advantage of symmetry and separability is the simplification obtained for modeling purpose. However, many studies in

*Corresponding author. Email addresses: bayesia@korea.ac.kr (Man Sik Park), fuentes@stat.ncsu.edu (Montserrat Fuentes).

environmental sciences show that real data have complex spatial-temporal dependency structures that are difficult to model and estimate by separability and symmetry.

There has been considerable research on separability while symmetry has not been highly focused yet. Modeling nonseparable covariance functions is one of the keys for more reliable prediction in the environmental research. Cressie and Huang (1999) introduced a new class of nonseparable, stationary spatial-temporal covariance functions with space-time interaction, for which the separable covariance function is a special case. Gneiting (2002) also proposed general classes of nonseparable, stationary spatial-temporal covariance functions which are directly constructed in the space-time domain and are based on Fourier-free implementation. Fuentes et al. (2005) proposed a new class of nonseparable and nonstationary spatial-temporal covariance models, which has a unique parameter indicating spatial-temporal dependency. In addition to the modeling issue, many studies about testing lack of separability have been accomplished. Shitan and Brockwell (1995) used an asymptotic χ^2 test for stationary spatial autoregressive processes. Guo and Billard (1998) proposed the Wald test for testing lack of a doubly-geometric process under the temporal setting. A likelihood ratio test for lack of separability for *i.i.d* multivariate processes was proposed by Mitchell (2002), and Mitchell et al. (2006). Fuentes (2006) developed a formal test for lack of separability and lack of stationarity of spatial-temporal covariance functions by applying a two-factor analysis of variance (ANOVA) procedure, which is applicable to more general spatial-temporal covariance models.

In terms of testing for lack of symmetry, the most relevant work is Scaccia and Martin (2005), and Lu and Zimmerman (2005). Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{D}\}$ denotes a spatial process where $\mathbf{s} = (s_1, \dots, s_d)'$ is a spatial site over a fixed domain $\mathbb{D} \subset \mathbb{R}^d$, and $\mathbb{R}^d$ is a d -dimensional Euclidean space. Then the covariance function is defined as

$$C(\mathbf{h}|\theta) \equiv \text{cov} \{Z(\mathbf{s}_i+\mathbf{h}), Z(\mathbf{s}_i)\},$$

where $\mathbf{s}_i = (s_1^i, \dots, s_d^i)'$, $\mathbf{h} = (h_1, \dots, h_d)'$, and θ is a covariance parameter vector. For two-dimensional rectangular lattice data, Scaccia and Martin (2005) developed new tests of axial symmetry and separability which are, respectively, defined by, for all h_1 and h_2 ,

$$C(h_1, h_2 | \theta) = C(-h_1, h_2 | \theta), \quad (2)$$

and

$$C(h_1, h_2 | \theta) = C_1(h_1 | \theta_1) \cdot C_2(h_2 | \theta_2),$$

where C_1 and C_2 are the positive-definite covariances of the corresponding spatial lags, h_1 and h_2 , and $\theta = (\theta_1', \theta_2')$. Their tests are performed in two stages: testing axial symmetry first and then, if the hypothesis of axial symmetry is not rejected, testing separability. Under an $n_1 \times n_2$ rectangular lattice dataset, their tests are based on the periodogram denoted by

$$I(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \sum_{h_1=-n_1+1}^{n_1-1} \sum_{h_2=-n_2+1}^{n_2-1} C(h_1, h_2 | \theta) \cos(h_1 \omega_1 + h_2 \omega_2).$$

Lu and Zimmerman (2005) also proposed diagnostic tests of axial symmetry shown in (2) and complete symmetry defined by

$$C(h_1, h_2 | \theta) = C(-h_1, h_2 | \theta) = C(-h_2, h_1 | \theta) = C(h_2, h_1 | \theta),$$

for all h_1 and h_2 . Their tests of symmetries are also based on certain ratios of spatial periodograms. However, these noteworthy studies are only applicable for spatial processes, not spatial-temporal ones and, therefore, no formal tests for lack of symmetry in spatial-temporal processes have been developed yet although the modeling of asymmetric spatial-temporal processes has been researched by Stein (2005), and Park and Fuentes (2006).

In this study, we propose new formal tests by using spectral representation of the spatial-temporal covariance functions. The advantage of the tests is that classical analysis of variance (ANOVA) models can be used for detecting lack of symmetry inherent in spatial-temporal processes.

This paper is organized as follows. In Section 2 we introduce spectral representation under the spatial-temporal setting. We define two types of symmetry of a spatial-temporal covariance function in Section 3. New tests for lack of symmetry in spatial-temporal processes based on the spectral representation are proposed in Section 4 and Section 5. The performances of the tests are evaluated with simulation studies in Section 6 and with the real application in Section 7. Finally, we present some conclusions and final remarks in Section 8.

2 The Spectral Representation of Stationary Processes

In this section, we introduce spectral representation of stationary spatial-temporal processes, which is a major key for building new tests for lack of symmetry. Suppose that $\{Z(\mathbf{s}, t) : \mathbf{s} \in \mathbb{D} \subset \mathbb{R}^d, t \in [0, \infty)\}$ denotes a spatial-temporal process where t indicates a measuring time. Then $\{Z(\mathbf{s}, t)\}$ can be expressed in the spectral domain by sinusoidal forms with different frequencies $(\boldsymbol{\omega}, \tau)$, where $\boldsymbol{\omega}$ is a d -dimensional spatial frequency vector, and τ is a temporal frequency. If $Z(\mathbf{s}, t)$ is a stationary process with the corresponding covariance defined by

$$C(\mathbf{h}; u) \equiv \text{cov} \{Z(\mathbf{s}_i + \mathbf{h}, t_k + u), Z(\mathbf{s}_i, t_k)\}, \quad (3)$$

then we can write the process in the following Fourier-Stieltjes integral (Yaglom, 1987):

$$Z(\mathbf{s}, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \exp(i\mathbf{s}'\boldsymbol{\omega} + i\tau t) dY(\boldsymbol{\omega}, \tau),$$

where Y is a random process with complex symmetry except for the constraint, $dY(\boldsymbol{\omega}, \tau) = dY^c(-\boldsymbol{\omega}, -\tau)$, which ensures that $Z(\mathbf{s}, t)$ is real-valued. Here c stands for complex conjugate. Using the spectral representation of Z , the covariance function $C(\mathbf{h}; u)$ can be represented as

$$C(\mathbf{h}; u) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \exp(i\mathbf{h}'\boldsymbol{\omega} + i\tau u) G(d\boldsymbol{\omega}; d\tau), \quad (4)$$

where $(\mathbf{h}; u) = (\mathbf{s}_i - \mathbf{s}_j; t_k - t_l)$ for $\mathbf{s}_i, \mathbf{s}_j \in \mathbb{D}$ and $t_k, t_l \in [0, \infty)$, and the function G is a positive finite measure called the spectral measure or spectral distribution function for Z . The spectral measure G is the expected squared modulus of the process Y denoted by

$$E \{ |Y(\omega, \tau)|^2 \} = G(\omega; \tau).$$

We can easily see that $C(\mathbf{h}; u)$ in (4) is always positive-definite for any finite positive measure G . If G has a density with respect to Lebesgue measure, the spectral density g is the Fourier transform of the spatial-temporal covariance function:

$$g(\omega; \tau) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \exp(-i\mathbf{h}'\omega - i\tau u) C(\mathbf{h}; u) d\mathbf{h} du, \quad (5)$$

and the corresponding covariance function is given by

$$C(\mathbf{h}; u) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \exp(i\mathbf{h}'\omega + i\tau u) g(\omega; \tau) d\omega d\tau. \quad (6)$$

Using this spectral framework, we can easily make a new class of covariance functions from Fourier transformation of the corresponding spectral densities.

3 Symmetry In Spatial-Temporal Processes

We summarized spectral representation of a stationary spatial-temporal process in Section 2. In this section, we define two types of symmetry under the spatial-temporal setting: axial symmetry in time and axial symmetry in space. The covariance shown in (3) is assumed to be stationary, that is,

$$\text{cov} \{Z(\mathbf{s}_i, t_k + u), Z(\mathbf{s}_i, t_k)\} = \text{cov} \{Z(\mathbf{s}_i, t_l + u), Z(\mathbf{s}_i, t_l)\}$$

for arbitrary u . The first type of symmetry is axial symmetry in time.

Definition 3.1

A process is called axially symmetric in time if

$$C(\mathbf{s}_i - \mathbf{s}_j; u) = C(\mathbf{s}_i^* - \mathbf{s}_j^*; -u), \quad (7)$$

for any temporal lag, $u \neq 0$, and arbitrary four different locations (i, j, i^, j^*) satisfying $\mathbf{s}_i - \mathbf{s}_j = \mathbf{s}_i^* - \mathbf{s}_j^*$.*

Under stationarity in space, (7) is reduced to

$$C(\mathbf{h}; u) = C(\mathbf{h}; -u),$$

where $\mathbf{s}_i = \mathbf{s}_j + \mathbf{h}$ and $\mathbf{s}_i^* = \mathbf{s}_j^* + \mathbf{h}$. The distance vectors on the spatial domain are invariant, and the temporal lags have the same magnitudes but different signs.

The second type of symmetry is axial symmetry in space.

Definition 3.2

A process is called (k) axially symmetric in space if

$$C(\mathbf{h};u)=C(\mathbf{h}_k^a;u), \quad (8)$$

where $\mathbf{h}_k^a=(h_1, \dots, h_{k-1}, -h_k, h_{k+1}, \dots, h_d)'$ for k fixed.

As can be seen in (8), for a fixed temporal lag u , all the spatial lags are the same except for one spatial lag, which has different sign. A process that does not satisfy any of the two types of symmetry is called asymmetric in space and time.

In this section, we have defined two types of symmetry of a spatial-temporal covariance function. In real applications, these concepts might not be realistic, especially when analyzing environmental data which, in general, have very complicated spatial-temporal dependency structures, but we can not handle lack of symmetry or asymmetry unless any concepts of symmetry are clearly defined.

4 Test for Lack of Axial Symmetry in Time

In this section we explain a new formal testing method for lack of axial symmetry in time (Definition 3.1) in spatial-temporal process. In Gneiting (2002), axial symmetry in time is referred as full symmetry defined as

$$C(\mathbf{h};u)=C(\mathbf{h};-u)=C(-\mathbf{h};u)=C(-\mathbf{h};-u).$$

4.1 Spatial-temporal processes with axial symmetry in time

We now introduce spectral representation of a spatial-temporal covariance function with axial symmetry in time. Consider $\{Z(\mathbf{s}, t) : \mathbf{s} \in \mathbb{D} \subset \mathbb{R}^d, t \in [0, \infty)\}$ a spatial-temporal zero-mean process observed at $N \times T$ space-time coordinates. By Bochner's theorem, we can always write the positive-definite spatial-temporal covariance in (6) in terms of the corresponding valid spectral density function, g in (5) by assuming stationarity in space and time for the covariance function. If C is integrable, then (5) can be expressed as, for $\mathbf{h}$ fixed and $\tau \in [0, \infty)$,

$$\begin{aligned} g(\omega; \tau) &= (2\pi)^{-(d+1)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \exp \{-i\mathbf{h}'\omega - iu\tau\} C(\mathbf{h};u) d\mathbf{h} du \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \exp \{-i\mathbf{h}'\omega\} f(\mathbf{h};\tau) d\mathbf{h}, \end{aligned}$$

where $f(\mathbf{h}; \tau)$ is the cross-spectral density function of $Z(\mathbf{a}, \cdot)$ and $Z(\mathbf{a} + \mathbf{h}, \cdot)$, and is defined as

$$f(\mathbf{h};\tau)=(2\pi)^{-1} \int_{\mathbb{R}} \exp \{-iu\tau\} C(\mathbf{h};u) du = f^c(-\mathbf{h};\tau). \quad (9)$$

Without stationarity in space, we can write the cross-spectral density function in (9) as

$$f_{ab}(\tau)=(2\pi)^{-1} \int_{\mathbb{R}} \exp \{-iu\tau\} \text{cov} \{Z(\mathbf{a},t), Z(\mathbf{b},t+u)\} du = f_{ba}^c(\tau), \quad (10)$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{D}$. In general, f could be a complex function. However, under the axial symmetry in time, $f_{ba}(\tau)$ is represented as

$$f_{ba}(\tau)=f_{ab}(\tau). \quad (11)$$

So, $f_{ab}(\tau)$ is real-valued from (10) and (11).

Now we define the phase between $\mathbf{a}$ and $\mathbf{b}$ as

$$\phi_{ab}(\tau) = \tan^{-1} \left\{ \frac{\text{Im} \cdot f_{ab}(\tau)}{\text{Re} \cdot f_{ab}(\tau)} \right\},$$

where $\text{Im} \cdot f$ and $\text{Re} \cdot f$ are, respectively, the imaginary part and the real part of f . The phase can be interpreted as the angle of the vector which consists of cosine magnitude (real part) and sine magnitude (imaginary part). So, if there is any difference (delay) in spectral densities between the two time processes, $Z(\mathbf{a}; \cdot)$ and $Z(\mathbf{b}; \cdot)$, then we have a nonzero phase. If a covariance function is axially symmetric in time, that is, if the difference is not statistically significant, then we have

$$\phi_{ab}(\tau) = \phi_{ba}(\tau) = 0.$$

4.2 Estimating the cross-spectral density f and the phase ϕ

We introduce a nonparametric estimate of the cross-spectral density proposed by Fuentes (2006) and the asymptotic properties of the phase based on the cross-spectral density. For an arbitrary site $\mathbf{a}$, we can define the tapered Fourier transform, $J_{\mathbf{a}}(\tau)$ as

$$J_{\mathbf{a}}(\tau) = \sum_{t=1}^{T-1} K\left(\frac{t}{T}\right) Z(\mathbf{a}; t) \exp\{-i\tau t\},$$

where K is a tapering function and, in this study, is considered constant, i.e. $K(x) = 1$ for all x . The spectral window, $W_T(\mu)$ can be estimated by

$$\widehat{W}_T(\mu) = \frac{1}{B_T} \sum_{t=-\infty}^{\infty} W_T\left(\frac{\mu + 2\pi t}{B_T}\right), \quad (12)$$

where B_T is a temporal bandwidth parameter. In the real application, the following weight function is considered,

$$\widehat{W}_T\left(\frac{2\pi s}{T}\right) = \frac{T}{2\pi} (2M+1)^{-1},$$

where $M = B_T T$ and $s \leq M$. We can finally estimate the cross-spectral density function between $Z(\mathbf{a}; \cdot)$ and $Z(\mathbf{b}; \cdot)$ by

$$\widehat{f}_{ab}(\tau) = \frac{2\pi}{T} \sum_{t=1}^{T-1} \widehat{W}_T\left(\tau - \frac{2\pi t}{T}\right) \widehat{I}_{ab}^T\left(\frac{2\pi t}{T}\right), \quad (13)$$

where the sample cross-periodogram $\widehat{I}_{ab}^T(\tau)$ is defined by

$$\widehat{I}_{ab}^T(\tau) = \left[2\pi \sum_{t=1}^{T-1} K^2 \left(\frac{t}{T} \right) \right]^{-1} J_{\mathbf{a}}(\tau) J_{\mathbf{b}}^c(\tau).$$

Under the assumptions A.1 through A.3 (see Appendix), the expected value of the estimated cross-spectral density function, $\widehat{f}_{ab}(\tau)$ can be obtained as

$$\begin{aligned} E \left\{ \widehat{f}_{ab}(\tau) \right\} &= \frac{2\pi}{T} \sum_{t=1}^{T-1} \widehat{W}_t \left(T - \frac{2\pi t}{T} \right) \tilde{f}_{ab} \left(\frac{2\pi t}{T} \right) + O \left(\frac{1}{T} \right) \\ &= \int_{\mathbb{R}} W_t(\mu) \tilde{f}_{ab}(\tau - B_t \mu) d\mu + O \left(\frac{1}{B_T T} \right), \end{aligned}$$

where the error term is uniform in τ , and

$$\tilde{f}_{ab}(\tau) = \int_{\mathbb{R}^d} q_{\rho}(\mathbf{a} - \mathbf{x}) q_{\rho}(\mathbf{b} - \mathbf{x}) \tilde{f}_{\mathbf{a}+\mathbf{x}, \mathbf{b}+\mathbf{x}}(\tau) d\mathbf{x},$$

for $-\infty < \tau < \infty$. In this study, we regard $q_{\rho}(\mathbf{x})$ as a tensor product of d one-dimensional filters, $q_{\rho}(\mathbf{x}) = \prod_{i=1}^d q(x_i)$, where q is of the form

$$q(x) = 1/\rho \cdot \mathbf{1}(|x| \leq \rho/d),$$

where $\mathbf{1}(\cdot)$ is an indicator function. $\tilde{f}_{ab}(\tau)$ is the smoothed cross-spectral density function within a band of frequencies in the region of τ and a region in space in the neighborhood of $\mathbf{a}$ and $\mathbf{b}$, and the covariance between $\widehat{f}_{\mathbf{a}_i \mathbf{b}_i}(\tau)$ and $\widehat{f}_{\mathbf{a}_j \mathbf{b}_j}(\lambda)$ is expressed by

$$\begin{aligned} \lim_{T \rightarrow \infty} B_T T \text{cov} \left\{ \widehat{f}_{\mathbf{a}_i \mathbf{b}_i}(\tau), \widehat{f}_{\mathbf{a}_j \mathbf{b}_j}(\lambda) \right\} &= 2\pi \int_{\mathbb{R}} W_t^2(\mu) d\mu \\ &\times \left(\eta \{ \tau - \lambda \} \left[\tilde{f}_{\mathbf{a}_i \mathbf{a}_j}(\tau) \tilde{f}_{\mathbf{b}_i \mathbf{b}_j}(-\tau) \right] + \eta \{ \tau + \lambda \} \left[\tilde{f}_{\mathbf{a}_i \mathbf{b}_j}(\tau) \tilde{f}_{\mathbf{a}_j \mathbf{b}_i}(-\tau) \right] \right), \end{aligned} \quad (14)$$

where

$$\eta\{\tau\} = \begin{cases} 1 & \text{if } \tau \equiv 0 \pmod{2\pi} \\ 0 & \text{otherwise.} \end{cases}$$

Fuentes (2006) shows that, if we define the distance between pairs $(\mathbf{a}_i, \mathbf{b}_i)$ and $(\mathbf{a}_j, \mathbf{b}_j)$ as the minimum distance between any of the two sites in the first pair and any of the two sites in the second pair, then the estimated cross-spectral density functions, $\widehat{f}_{\mathbf{a}_i \mathbf{b}_i}(\tau)$ and $\widehat{f}_{\mathbf{a}_j \mathbf{b}_j}(\lambda)$, are approximately independent if either

C.1 $\|\tau - \lambda\|$ is sufficiently large so that

$$\int_{\mathbb{R}} |W_t(\mu + \tau)|^2 |W_t(\mu + \lambda)|^2 d\mu = 0,$$

that is, $\|\tau - \lambda\| \gg \text{bandwidth of } |W_t(\mu)|^2$ or

C.2 the distance between pairs $(\mathbf{a}_i, \mathbf{b}_i)$ and $(\mathbf{a}_j, \mathbf{b}_j)$ is greater than the bandwidth of $q_{\rho}(\mathbf{s})$.

In practice, we can make the covariance in (14) almost zero by having the frequencies τ and λ and the pairs $(\mathbf{a}_i, \mathbf{b}_i)$ and $(\mathbf{a}_j, \mathbf{b}_j)$ sufficiently apart. As mentioned above, the phase is zero in case of axial symmetry in time. So we can get the asymptotic normality of the estimated phase, $\hat{\phi}_{\mathbf{ab}}(\tau)$, with mean 0 and covariance defined as

$$\lim_{T \rightarrow \infty} B_T T \text{cov} \{ \hat{\phi}_{\mathbf{ab}}(\tau), \hat{\phi}_{\mathbf{ab}}(\lambda) \} = \pi \int_{\mathbb{R}} W_T^2(\mu) d\mu (\eta \{ \tau - \lambda \} - \eta \{ \tau + \lambda \}) \left[|R_{\mathbf{ab}}(\tau)|^{-2} - 1 \right], \quad (15)$$

where the coherency between $Z(\mathbf{a}; t)$ and $Z(\mathbf{b}; t)$, $R_{\mathbf{ab}}(\tau)$ is defined as

$$R_{\mathbf{ab}}(\tau) = \frac{\tilde{f}_{\mathbf{ab}}(\tau)}{\left[\tilde{f}_{\mathbf{aa}}(\tau) \tilde{f}_{\mathbf{bb}}(\tau) \right]^{1/2}}.$$

From (15), the asymptotic variance is simply denoted as

$$\lim_{T \rightarrow \infty} B_T T \text{Var} \{ \hat{\phi}_{\mathbf{ab}}(\tau) \} = \pi \int_{\mathbb{R}} W_T^2(\mu) d\mu (1 - \eta \{ 2\tau \}) \left[|R_{\mathbf{ab}}(\tau)|^{-2} - 1 \right]. \quad (16)$$

Unfortunately, we can not use the asymptotic result of $\hat{\phi}_{\mathbf{ab}}(\tau)$ for the development of a testing method because the asymptotic variance in (16) depends on the relative position of $\mathbf{a}$ and $\mathbf{b}$. So an appropriate transformation is needed. To stabilize the asymptotic variance, we transform $\hat{\phi}_{\mathbf{ab}}(\tau)$ to $\tilde{\phi}_{\mathbf{ab}}(\tau)$ given by

$$\tilde{\phi}_{\mathbf{ab}}(\tau) = \frac{\hat{\phi}_{\mathbf{ab}}(\tau)}{\left[|R_{\mathbf{ab}}(\tau)|^{-2} - 1 \right]^{1/2}}. \quad (17)$$

Then, from (16) and (17), we derive the asymptotic normal distribution of $\tilde{\phi}_{\mathbf{ab}}(\tau)$ with mean 0 and variance given by

$$\begin{aligned} \lim_{T \rightarrow \infty} B_T T \text{Var} \{ \tilde{\phi}_{\mathbf{ab}}(\tau) \} &= \left[|R_{\mathbf{ab}}(\tau)|^{-2} - 1 \right]^{-1} \lim_{T \rightarrow \infty} B_T T \text{Var} \{ \hat{\phi}_{\mathbf{ab}}(\tau) \} \\ &= \pi \int_{\mathbb{R}} W_T^2(\mu) d\mu (1 - \eta \{ 2\tau \}). \end{aligned}$$

However, $R_{\mathbf{ab}}(\tau)$ is unknown in practice, so we newly define $\hat{\phi}_{\mathbf{ab}}^*(\tau)$ as

$$\hat{\phi}_{\mathbf{ab}}^*(\tau) = \frac{\hat{\phi}_{\mathbf{ab}}(\tau)}{\left[\left| \hat{R}_{\mathbf{ab}}(\tau) \right|^{-2} - 1 \right]^{1/2}}. \quad (18)$$

By the Slutsky's theorem, we can obtain the same asymptotic normal distribution of $\tilde{\phi}_{\mathbf{ab}}(\tau)$ in (17) as the one of $\hat{\phi}_{\mathbf{ab}}^*(\tau)$ in (18). Based on the assumptions, C.1 and C.2, we implicitly know that, under axial symmetry in time, ϕ^* evaluated at different pairs and different frequencies can be approximately treated jointly normally distributed and independent (see Appendix).

4.3 Testing lack of axial symmetry in time

With the information of asymptotic distribution of the adjusted phase, $\widehat{\phi}_{ab}^*(\tau)$ in (18), we propose a formal test for lack of axial symmetry in time by employing analysis of variance (ANOVA) procedure. First we compute $\widehat{\phi}_{\mathbf{a},\mathbf{b}_i}^*(\tau_j)$ at arbitrary two sites, $\{(\mathbf{a}_i, \mathbf{b}_i)\}_{i=1}^m$ and a set of temporal frequencies, $\{\tau_j\}_{j=1}^n$ that cover the space-time domain. Arbitrary two sites should be selected based on the condition that, for $\mathbf{h}$ fixed,

$$\mathbf{a}_i - \mathbf{b}_i = \mathbf{h} = (h_1, h_2, \dots, h_d)'.$$

In order to apply to a two-factor ANOVA model, we rewrite $\widehat{\phi}_{\mathbf{a},\mathbf{b}_i}^*(\tau_j)$ as follows:

$$\widehat{\phi}_{\mathbf{a},\mathbf{b}_i}^*(\tau_j) = \phi_{\mathbf{a},\mathbf{b}_i}^*(\tau_j) + \epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j), \quad (19)$$

where $\phi_{\mathbf{a},\mathbf{b}_i}^*(\tau_j) = \phi_{\mathbf{a},\mathbf{b}_i}(\tau_j) / [|\mathbf{R}_{\mathbf{a},\mathbf{b}_i}(\tau_j)|^{-2} - 1]^{1/2}$. Here $\epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j)$ asymptotically has the following assumptions:

1. $E\{\epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j)\} = 0, \forall i, j,$
2. $\text{Var}\{\epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j)\} = \sigma_\epsilon^2, \forall i, j,$ and
3. $\text{cov}\{\epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j), \epsilon_{\mathbf{a},\mathbf{b}_k}(\tau_l)\} = 0, \forall i, j, k, l$ satisfying C.1 and C.2.

We also express (19) as

$$\widehat{\phi}_{\mathbf{a},\mathbf{b}_i}^*(\tau_j) = \alpha_i + \beta_j + \epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j), \quad (20)$$

where the parameters $\{\alpha_i\}$ and $\{\beta_j\}$ are “Location” effect and “Temporal Frequency” effect, respectively. Suppose that a spatial-temporal process is stationary in space, then its covariance does not depend on the relative position of the sites, which implies that, under the stationarity in space, “Location” effect, $\{\alpha_i\}$ is not significant. So, lack of the stationarity in space can be detected by the classical ANOVA technique to test the null hypothesis:

$$\widehat{\phi}_{\mathbf{a},\mathbf{b}_i}^*(\tau_j) = \beta_j + \epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j)$$

against the alternative hypothesis shown in (20). Under axial symmetry in time, the phase is zero. So, “Temporal Frequency” effect, $\{\beta_j\}$ is statistically zero. The classical ANOVA procedure is employed to check lack of axial symmetry in time by testing the null hypothesis:

$$\widehat{\phi}_{\mathbf{a},\mathbf{b}_i}^*(\tau_j) = \alpha_i + \epsilon_{\mathbf{a},\mathbf{b}_i}(\tau_j)$$

against the alternative hypothesis shown in (20). In addition, we can also check both lack of axial symmetry in time and lack of stationarity in space simultaneously by examining whether $\alpha_i = \beta_j = 0$ or not.

5 Test for Lack of Axial Symmetry in Space

In this section we explain a new formal testing method for lack of axial symmetry in space (Definition 3.2) in a spatial-temporal process. For the simplification of developing the test, we only consider $\mathbf{h}_1^a = (-h_1, h_2)'$ for $d=2$.

5.1 Spatial-temporal processes with axial symmetry in space

Now we propose a new version of the cross-spectral density function between $Z(\cdot, a_2, t)$ and $Z(\cdot, a_2 + h_2, t + u)$, $k(\omega_1; h_2, u)$ given by

$$k(\omega_1; h_2, u) = \int_{\mathbb{R}} \int_{\mathbb{R}} \exp \{i h_2 \omega_2 + i u \tau\} g(\omega; \tau) d\omega_2 d\tau, \quad (21)$$

for fixed h_2 and u . If C is integrable, then

$$\begin{aligned} g(\omega; \tau) &= (2\pi)^{-3} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \exp \{-i \mathbf{h}' \omega - i u \tau\} C(\mathbf{h}; u) d\mathbf{h} d\tau \\ &= (2\pi)^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp \{-i h_2 \omega_2 - i u \tau\} k(\omega_1; h_2, u) dh_2 du. \end{aligned}$$

Since the function k in (21) is the Fourier transform of the spatial-temporal covariance function with respect to one of the spatial frequencies, we can also write k in an alternative form denoted by

$$k(\omega_1; h_2, u) = (2\pi)^{-1} \int_{\mathbb{R}} \exp \{-i h_1 \omega_1\} C(\mathbf{h}; u) dh_1 = k^c(-\omega_1, h_2; u).$$

If a process is axially symmetric in space, that is, $C(\mathbf{h}; u) = C(\mathbf{h}_1^a; u)$, then

$$k(\omega_1; h_2, u) = k(-\omega_1; h_2, u).$$

So, k is always real-valued.

We also define a new version of the phase between $Z(\cdot, a_2, t)$ and $Z(\cdot, a_2 + h_2, t + u)$, $\psi(\omega_1; h_2, u)$ as

$$\psi(\omega_1; h_2, u) = \tan^{-1} \left\{ \frac{\text{Im}.k(\omega_1; h_2, u)}{\text{Re}.k(\omega_1; h_2, u)} \right\}.$$

If the process has a covariance with axial symmetry in space, then we obtain the following result:

$$\psi(\omega_1; h_2, u) = \psi(-\omega_1; h_2, u) = 0.$$

Otherwise, we have nonzero value of ψ .

5.2 Estimating the cross-spectral density k and the phase ψ

Now we propose a nonparametric estimate of k and study the asymptotic properties of k and ψ . If Z is observed only at $N (= N_1 \times N_2)$ sites on regular grids and at the measuring times T , then, for a_2 and t fixed, we can define $J_{\Delta_1}(\omega_1; a_2, t)$,

$$J_{\Delta_1}(\omega_1; a_2, t) = \Delta_1 \sum_{n_1=1}^{N_1-1} K\left(\frac{n_1}{N_1}\right) Z(\Delta_1 n_1, a_2; t) \exp\{-i\Delta_1 n_1 \omega_1\},$$

where Δ_1 is the unit distance of the first spatial coordinate. We also define the sample spectral window $\widehat{W}_s(\mu)$ by

$$\widehat{W}_s(\mu) = \frac{1}{B_{N_1}} \sum_{j=-\infty}^{\infty} W_s\left(\frac{\mu + 2\pi}{B_{N_1}}\right), \quad (22)$$

where B_{N_1} is a spatial bandwidth parameter and $-\infty < \mu < \infty$. Then $\widehat{k}_{\Delta_1}(\omega_1; h_2, u)$ is represented by

$$\widehat{k}_{\Delta_1}(\omega_1, h_2; u) = \frac{2\pi}{N_1} \sum_{n_1=1}^{N_1-1} \widehat{W}_s\left(\omega_1 - \frac{2\pi n_1}{N_1}\right) \widehat{I}_{\Delta_1}^s\left(\frac{2\pi n_1}{N_1}; h_2, u\right), \quad (23)$$

where

$$\begin{aligned} \widehat{I}_{\Delta_1}^s(\omega_1; h_2, u) &= \left[2\pi \sum_{n_1=1}^{N_1-1} \left\{ K^2\left(\frac{n_1}{N_1}\right) \Delta_1 \right\} \right]^{-1} \\ &\quad \times J_{\Delta_1}(\omega_1; a_2, t) J_{\Delta_1}^c(\omega_1; a_2 + h_2, t + u). \end{aligned}$$

Under the assumptions A.1, A.4 and A.5 (see Appendix), we can obtain the asymptotic properties of the estimated cross-spectral density function, $\widehat{k}_{\Delta_1}(\omega_1; h_2, u)$ with mean $k_{\Delta_1}(\omega_1; h_2, u)$ defined as

$$\begin{aligned} E\left\{\widehat{k}_{\Delta_1}(\omega_1; h_2, u)\right\} &= \frac{2\pi}{N_1} \sum_{n_1=1}^{N_1-1} \widehat{W}_s\left(\omega_1 - \frac{2\pi n_1}{N_1}\right) \tilde{k}_{\Delta_1}\left(\frac{2\pi n_1}{N_1}; h_2, u\right) + O\left(\frac{1}{N_1}\right) \\ &= \int_{\mathbb{R}} W_s(\mu) \tilde{k}_{\Delta_1}(\omega_1 - B_{N_1} \mu; h_2, u) d\mu + O\left(\frac{1}{B_{N_1} N_1}\right), \end{aligned}$$

where the error term is uniform in ω_1 . The asymptotic variance is also defined as

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} B_{N_1} N_1 \text{cov} \left\{ \widehat{k}_{(b_2^j, t^{\mathbf{b}}; a_2^j, t^{\mathbf{a}})}(\omega_1), \widehat{k}_{(b_2^j, t^{\mathbf{b}}; a_2^j, t^{\mathbf{a}})}(\lambda) \right\} \\ = 2\pi \int_{\mathbb{R}} W_s^2(\mu) d\mu \left(\eta\{\omega_1 - \lambda\} \left[\widehat{k}_{(a_2^j, t^{\mathbf{a}}; a_2^j, t^{\mathbf{a}})}(\omega_1) \widehat{k}_{(b_2^j, t^{\mathbf{b}}; b_2^j, t^{\mathbf{b}})}(-\omega_1) \right] \right. \\ \left. + \eta\{\omega_1 + \lambda\} \left[\widehat{k}_{(b_2^j, t^{\mathbf{b}}; a_2^j, t^{\mathbf{a}})}(\omega_1) \widehat{k}_{(b_2^j, t^{\mathbf{b}}; a_2^j, t^{\mathbf{a}})}(-\omega_1) \right] \right), \quad (24) \end{aligned}$$

where

$$\begin{aligned} \tilde{k}_{\Delta_1}(\omega_1; h_2, u) &= \int_{\mathbb{R}} \int_{\mathbb{R}} q_{\rho}(a_2 - x_1, t^{\mathbf{a}} - x_2) q_{\rho}(b_2 - x_1, t^{\mathbf{b}} - x_2) \\ &\quad \times k_{\Delta_1}(\omega_1; b_2 - a_2 + x_1, t^{\mathbf{b}} - t^{\mathbf{a}} + x_2) dx_1 dx_2 \end{aligned}$$

for $-\infty < \omega_1 < \infty$ and

$$\widehat{k}_{(a_2^i, t^{a_i}; b_2^i, t^{b_i})}(\omega_1) \equiv \widehat{k}_{\Delta_1}(\omega_1; b_2^i - a_2^i, t^{b_i} - t^{a_i})$$

with $b_2 - a_2 = b_2^i - a_2^i = b_2^j - a_2^j = h_2$ and $t^b - t^a = t^{b_i} - t^{a_i} = t^{b_j} - t^{a_j} = u$. To show the asymptotic independence of $\widehat{k}_{(a_2^i, t^{a_i}; b_2^i, t^{b_i})}(\omega_1)$ and $\widehat{k}_{(a_2^j, t^{a_j}; b_2^j, t^{b_j})}(\lambda)$ is analogous to the previous approach. The two estimated cross-spectral densities above are approximately independent if either

C.3 $\|\omega_1 - \lambda\|$ is sufficiently large so that

$$\int_{\mathbb{R}} |W_s(\mu + \omega_1)|^2 |W_s(\mu + \lambda)|^2 d\mu = 0,$$

C.4 the distance between pairs $(a_2^i, t^{a_i}; b_2^i, t^{b_i})$ and $(a_2^j, t^{a_j}; b_2^j, t^{b_j})$ is greater than the bandwidth of $q_p(\mathbf{x})$.

We can practically make the asymptotic covariance in (24) almost zero by taking the frequencies ω_1 and λ , and the pairs $(a_2^i, t^{a_i}; b_2^i, t^{b_i})$ and $(a_2^j, t^{a_j}; b_2^j, t^{b_j})$ sufficiently apart. As mentioned above, the phase ψ is zero under axial symmetry in space. So, we can get asymptotic normality of the estimated phase, $\widehat{\psi}_{\Delta_1}(\omega_1; h_2, u)$ with mean 0 and covariance defined as, for $b_2 - a_2 = h_2$ and $t^b - t^a = u$,

$$\begin{aligned} & \lim_{N_1 \rightarrow \infty} B_{N_1} N_1 \text{cov} \left\{ \widehat{\psi}_{\Delta_1}(\omega_1; b_2 - a_2, t^b - t^a), \widehat{\psi}_{\Delta_1}(\lambda; b_2 - a_2, t^b - t^a) \right\} \\ &= \pi \int_{\mathbb{R}} W_s^2(\mu) d\mu \left(\eta \{ \omega_1 - \lambda \} - \eta \{ \omega_1 + \lambda \} \right) \left[|Q_{\Delta_1}(\omega_1; h_2, u)|^{-2} - 1 \right], \end{aligned} \quad (25)$$

where a new version of the coherency, $Q(\omega_1; h_2, u)$ between two arbitrary points in two-dimensional space, (a_2, t^a) and (b_2, t^b) , is defined by

$$Q(\omega_1; h_2, u) = \frac{k_{(a_2, t^a; b_2, t^b)}(\omega_1)}{\left[k_{(a_2, t^a; a_2, t^a)}(\omega_1) k_{(b_2, t^b; b_2, t^b)}(\omega_1) \right]^{1/2}}.$$

In general, we can not directly use the asymptotic result of $\widehat{\psi}_{\Delta_1}(\omega_1; h_2, u)$ in order to make a new test for lack axial symmetry in space because the asymptotic variance in (25) depends on h_2 and u . In order to make the asymptotic variance independent of h_2 and u , we transform $\widehat{\psi}_{\Delta_1}(\omega_1; h_2, u)$ to $\tilde{\psi}_{\Delta_1}(\omega_1; h_2, u)$ defined by

$$\tilde{\psi}_{\Delta_1}(\omega_1; h_2, u) = \frac{\widehat{\psi}_{\Delta_1}(\omega_1; h_2, u)}{\left[|Q_{\Delta_1}(\omega_1; h_2, u)|^{-2} - 1 \right]^{1/2}}. \quad (26)$$

In practice, $Q_{\Delta_1}(\omega_1; h_2, u)$ is a unknown parameter. So, by using the estimated coherency, $\widehat{Q}_{\Delta_1}(\omega_1; h_2, u)$, we newly define $\widehat{\psi}_{\Delta_1}^*(\omega_1; h_2, u)$ as

$$\widehat{\psi}_{\Delta_1}^*(\omega_1; h_2, u) = \frac{\widehat{\psi}_{\Delta_1}(\omega_1; h_2, u)}{\left[\left| \widehat{Q}_{\Delta_1}(\omega_1; h_2, u) \right|^{-2} - 1 \right]^{1/2}}. \quad (27)$$

If we use an appropriate $\widehat{Q}_{\Delta_1}(\omega_1; h_2, u)$ as an estimate of $Q_{\Delta_1}(\omega_1; h_2, u)$, then we can get the same asymptotic distribution of $\widehat{\psi}_{\Delta_1}^*(\omega_1; h_2, u)$ in (27) as the one of $\widehat{\psi}_{\Delta_1}(\omega_1; h_2, u)$ in (26). In the appendix, we will study the asymptotic normality and the independence of $\widehat{\psi}_{\Delta_1}^*$ evaluated at different pairs and different frequencies under axial symmetry in space.

5.3 Testing for axial symmetry in space

Now we propose a formal test for axial symmetry in space for spatial-temporal processes. For the m pairs, $\{(a_2^i, t^{a_i}; b_2^i, t^{b_i})\}_{i=1}^m$ in two-dimensional space consisting of the second spatial and the temporal coordinates, and a set of first spatial frequencies, $\{\omega_j\}_{j=1}^n$, we can get $\widehat{\psi}_i^*(\omega_j) \equiv \widehat{\psi}_{\Delta_1}^*(\omega_j; b_2^i - a_2^i, t^{b_i} - t^{a_i})$. Arbitrarily, the pairs are selected based on the condition that, for h_2 and u fixed,

$$b_2^i - a_2^i = h_2, \text{ and } t^{b_i} - t^{a_i} = u.$$

In order to apply to traditional two-factor ANOVA procedure, we rewrite $\widehat{\psi}_i^*(\omega_j)$ as follows:

$$\widehat{\psi}_i^*(\omega_j) = \psi_i^*(\omega_j) + e_i(\omega_j), \quad (28)$$

where

$$\psi_i^*(\omega_j) = \frac{\psi_{\Delta_1}(\omega_j; b_2^i - a_2^i, t^{b_i} - t^{a_i})}{\left[\left| Q_{\Delta_1}(\omega_j; b_2^i - a_2^i, t^{b_i} - t^{a_i}) \right|^{-2} - 1 \right]^{1/2}}.$$

Here $e_i(\omega_j)$ asymptotically has the following assumptions:

1. $E\{e_i(\omega_j)\} = 0, \forall i, j,$
2. $\text{Var}\{e_i(\omega_j)\} = \sigma_e^2, \forall i, j,$ and
3. $\text{cov}\{e_i(\omega_j), e_k(\omega_l)\} = 0, \forall i, j, k, l$ approximately.

We also express (28) as

$$\widehat{\psi}_i^*(\omega_j) = \gamma_i + \delta_j + e_i(\omega_j),$$

where the parameters $\{\gamma_i\}$ and $\{\delta_j\}$ are “Space-Time Interaction” effect and “Spatial Frequency” effect, respectively. Since, under the stationarity in space-time, the covariance does not rely on their relative position, it is quite reasonable that “Space-Time Interaction” effect, $\{\gamma_i\}$ is not significant. If axial symmetry in space is in a spatial-temporal process, the phase in (27) is zero, which means that “Spatial Frequency” effect is also zero. Therefore, we can detect

lack of stationarity in space-time as well as lack of axial symmetry in space from the two main effects in the classical two-factor ANOVA model.

In Section 4 and Section 5, we have proposed new formal tests for lack of axial symmetry in time and for lack of axial symmetry in space inherent in spatial-temporal processes. One of the advantages of our methods is that classical ANOVA models are easily employed and, therefore, the interpretation can be more persuadable.

6 Simulation Studies

In this section, we evaluate the performance of the proposed testing methods with simulation studies. The instruction for testing lack of axial symmetry in time is as following:

1. Choose m pairs of sites which are far from each other by the given spatial lags, $\mathbf{h} = (h_1, h_2)'$. Keep the within-pair distance ($\|\mathbf{h}\|$) much smaller than $g_\rho(\mathbf{s})$ in C.2 or the effective range, but the between-pair distance greater than or equal to them in order to make the cross-spectral densities of pairs asymptotically independent.
2. Compute the test statistic shown in (18) for each pair.
3. Apply this statistic to two-factor ANOVA model by considering “Temporal Frequency” effect and “Location” effect.
4. Repeat 1) through 3) under different ways of selecting pairs (Figure 1) in order to search for the specific directions causing lack of axial symmetry in time.

The steps for testing lack of axial symmetry in space are quite similar with the steps above.

1. Find m pairs of points which are far from each other as specified by the second spatial lag (latitudinal lag) and the temporal lag, $(h_2, u)'$. Keep the between-pair distance larger than the effective ranges for space and for time.
2. Compute the test statistic proposed in (27) for each pair.
3. Apply this statistic to two-factor ANOVA model by considering “Spatial (Longitudinal) Frequency” effect and “Space-Time Interaction” effect.
4. Repeat 1) through 3) under different ways of selecting pairs (Figure 3) to search for the specific directions causing lack of axial symmetry in space.

Since the underlying covariance is set to a asymmetric, stationary spatial-temporal one, we briefly explain the asymmetric exponential covariance defined by

$$C(\mathbf{h}; u) = \sigma_1 \exp \left\{ -\sqrt{\beta^2(u - \mathbf{h} \cdot \mathbf{v})^2 + \alpha^2 \|\mathbf{h}\|^2} \right\} + \sigma_0 \mathbf{1}(\|\mathbf{h}\| = u = 0), \quad (29)$$

where σ_0 is the nugget, σ_1 is the partial sill, and α and β are the decaying rates of spatial correlation and of temporal correlation. Here, the asymmetry parameter vector, $\mathbf{v} = (v_1, v_2)' \in \mathbb{R}^2$ controls (lack of) symmetry realized in spatial-temporal processes. For example, $\mathbf{v} = \mathbf{0}$ yields the covariance satisfying axial symmetry in time. If only one element in $\mathbf{v}$ is zero, then axial symmetry in space is satisfied. We call asymmetry in space and time, otherwise (see Park and Fuentes, 2006).

Now we explain the fundamental simulation setup for realizing the tests. The number of Monte-Carlo simulations is set to 100 and, at each iteration, the observations are generated from the multivariate normal distribution with the mean 0 and the variance-covariance matrix in (29).

The covariance parameters are preassigned as follows; $\sigma_0 = 0.01$, $\sigma_1 = 1$, $\alpha = 0.02$, and $\beta = 0.75$. Here the spacing unit for space is 10 and the unit for time is 1.

6.1 Testing lack of axial symmetry in time

For the test for lack of axial symmetry in time, we consider the spatial domain with 16 pairs of two sites as shown in Figure 1 and we generate 51 observations over time at each selected site. The within-pair distance is set to 20 for North-South and East-West directions and $(10 \times \sqrt{5})$ for the other directions. We also set the between-pair distance greater than or equal to the spatial effective range, $3/\alpha = 150$ (15 spacing units). The temporal frequencies, $\{\tau_j\}$ are selected as follows; $\tau_j = \pi(5(j-1)+3)/25$ with $j = 1, \dots, 5$. Then we construct the test statistic for lack of axial symmetry in time, $\hat{\phi}_{a,b_i}^*(\tau_j)$ in (20) at the following temporal frequencies; $3\pi/25$, $8\pi/25$, $13\pi/25$, $\dots$, $23\pi/25$. As can be seen in Figure 1, we consider the six different ways (directions) determined by the pairs on the spatial domain.

Figure 2 displays the contour plots of empirical powers for “Temporal Frequency” effect under asymmetry in space and time ($\mathbf{v} \neq 0$) for each direction shown in Figure 1. Figure 2 shows that the direction of pair is directly related to the detectability of this test for lack of axial symmetry in time, for example, in case of ESE-WNW direction, the empirical power increases as v_1 and v_2 move away from $(0, 0)$ in the ESE-WNW direction (see Figure 1(a), Figure 2(a)) and lack of axial symmetry is not detected well when v_1 and v_2 are along the direction which is exactly perpendicular to the direction of pair. So, it is necessary to test lack of axial symmetry in time for different directions. Every other direction considered has similar pattern of empirical powers with ESE-WNW direction. The empirical powers of “Location” effect are inside the range from 0.02 to 0.11 (Figure 3(a)). This result is quite reasonable in that the covariance in (29) is stationary in space as well as in time. So we can conclude that there does not exist any apparent evidence against stationarity in space.

Pearson’s χ^2 test was employed to check the normality condition for the residuals. The empirical probabilities of rejecting the normality assumption are inside the range from 0.05 to 0.22 (Figure 3(b)), which is quite bigger than expected, but we don’t think that this affects the validity of this testing method seriously.

6.2 Testing lack of axial symmetry in space

For testing lack of axial symmetry in space, we consider 16 pairs of two points where the position of each point is represented by a spatial index and a time index (Figure 4). The two points are apart from each other by 2 spacing units in latitude and by 1 unit in time. The spatial frequencies, ω_j are selected as follows; $\omega_j = \pi(9(j-1)+2)/30$ with $j = 1, \dots, 4$. The test statistic, $\hat{\psi}_i^*(\omega_j)$ in (28) is computed at the following spatial frequencies; $2\pi/30$, $11\pi/30$, $20\pi/30$, $29\pi/30$. Then two-factor ANOVA model is employed for each direction shown in Figure 4.

Now we explain the results obtained from the ANOVA approach including the two main effects; “Spatial Frequency” and “Space-Time Interaction”. Figure 5 displays the empirical power of “Spatial (Longitudinal) Frequency” effect under asymmetry in space and time. The null hypothesis is located on the point which is nothing but the solution of $u - \mathbf{h}'\mathbf{v} = 0$ for $\mathbf{h}$ and u given, for instance, $\mathbf{v} = (0, -0.05)'$ for **D.1** and $\mathbf{v} = (0, 0.05)'$ for **D.2**. The empirical power increases as $\mathbf{v}$ moves away from the corresponding null hypothesis, especially along the dashed lines, which can be regarded as the actual alternative hypothesis.

Figure 6 illustrates the empirical power of “Space-Time Interaction” effect under asymmetry in space and time, and empirical probability of rejecting the normality assumption by Pearson’s χ^2 Normality Test of the residuals. From Figure 6(a), stationarity in space and in time can be

somehow guaranteed. As can be seen in Figure 6(b), the normality condition is too much rejected, but is not so inappropriate as to affect the application to the simple ANOVA model.

In this section, we evaluated the performance of new tests for lack of axial symmetry in time and for lack of axial symmetry in space. By simulation studies, we see that lacks of symmetry are well detected under general asymmetry in space and time, and stationarity in space, or in space and time is properly maintained under the asymmetric stationary covariance model. In addition to the two main effects, we may also consider the direction effect in the ANOVA model, but it is probably easier to run the model for each of the directions for the comfortable interpretation.

7 Real Application

In Section 6, we evaluated the performances of the two tests for lack of symmetry proposed in Section 4 and Section 5 with simulation studies. From the results, the proposed testing methods detect the corresponding lack of symmetry very well under general asymmetry in space and time. In this section, we apply the new testing methods to the real air-pollution dataset. Here we consider the daily $PM_{2.5}$ concentrations which are the averages of hourly values, which are obtained from the Models-3/Community Multiscale Air Quality (CMAQ) modeling system with the spatial resolution of $36km \times 36km$. These data were provided by the U.S. Environmental Protection Agency (EPA). The spatial domain of our interest is the eastern U.S. and the southern Canada (Figure 7), and the time domain is January 1st through December 29th, 2001. The main reason why we are interested in $PM_{2.5}$ concentrations is that this air-pollutant is one of the important factors in the public health problem and, according to many environmental studies, has complex spatial or spatial-temporal dependency structure (see Zidek, 1997; Golam Kribria et al., 2002).

Before applying the test for lack of axial symmetry in time, we remove the spatial and the temporal trends. For a $PM_{2.5}$ concentration at site s and time t , $Z(s, t)$, we remove the average over time at each site and the average over space at each time. Then we employ our tests for lack of symmetry to the $PM_{2.5}$ anomaly concentrations subtracted by the spatial and temporal trends. Here the spatial bandwidth $g_p(\mathbf{0})$ is considered for the simplicity for analyzing the data.

7.1 Testing Lack of Axial Symmetry in Time

We obtain the estimates of phase and coherency, $\hat{\phi}_{ab}(\tau)$ and $R^{\wedge}_{ab}(\tau)$ by calculating the estimated cross-spectrum in (13) where the spectral window, $\hat{W}_T(\alpha)$ in (12) has a bandwidth of $2\pi B_T$ with $B_T = 1/28$. In order to make the estimates uncorrelated approximately, we choose the temporal frequencies τ_j for $j = 1, \dots, n = \lfloor B_T T \rfloor$ satisfying that the spacing between the τ_j 's is at least $\pi/14$, the between-pair distance, $(\mathbf{a}_i, \mathbf{b}_i)$ and $(\mathbf{a}_j, \mathbf{b}_j)$ for $i \neq j$ is set large enough and the within-pair distance is also set much small enough. Here $\lfloor a \rfloor$ means the integer nearest to a . The temporal frequencies, τ_j are selected as follows; $\tau_j = \pi(13(j-1)+6)/181$ with $j = 1, \dots, 13$, where the uniform spacing of $13\pi/181$ is slightly longer than $\pi/14$. We then construct the test statistic for lack of axial symmetry in time, $\hat{\phi}_{\mathbf{a}, \mathbf{b}_i}^*(\tau_j)$ in (20) at the following temporal frequencies; $6\pi/181, 19\pi/181, \dots, 175\pi/181$. We consider the 16 pairs, $\{\mathbf{a}_i, \mathbf{b}_i\}$, $i = 1, \dots, 16$ shown in Figure 1. It can be seen that between-pair distance is at least 15 spacing units (unit= $36km$) and the within-pair distance is set to 2 units for East-West direction and North-South direction, and $\sqrt{5}$ units for the other directions. We also take into account the effect of the direction of pair.

Now we explain the result of the test for lack of axial symmetry in time. Table 1 displays the output from the ANOVA procedure for checking lack of axial symmetry as well as lack of stationarity in space for each direction. "Location" effect is significant under the 0.05 level of

significance for every direction, which implies that this spatial-temporal process is nonstationary in space. However, “Temporal Frequency” effect is not significant for North-South direction and for NNW-SSE direction. This means that $C(\mathbf{h}; u) \neq C(\mathbf{h}; -u)$ for the other directions and, therefore, the covariance (or correlation) between arbitrary two sites with the fixed spatial difference changes as time changes, especially in the Northeastern-Southwestern direction. Pearson's χ^2 test presents that the residuals are satisfied with the normality assumption for most of the directions.

7.2 Testing Lack of Axial Symmetry in Space

The estimates of phase and coherency, $\hat{\psi}_{\Delta_1}$ and $\hat{Q}_{\Delta_1}$ in (27) can be obtained from cross-spectrum in (23) where the spectral window, $\hat{W}_s(\alpha)$ in (22) has a bandwidth of $2\pi B_{N_I}$ with $B_{N_I} = 1/12$. The spatial (longitudinal) frequencies, $\{\omega_j\}$ for $j = 1, \dots, n = \lfloor B_{N_I} N_I \rfloor$ are chosen by the way that the spacings between the ω_j 's are at least $\pi/6$, and the distance between any pairs, $(a_2^j, t_i^a; b_2^j, t_i^b)$ and $(a_2^j, t_i^a; b_2^j, t_i^b)$ for $i \neq j$ is set large enough, and the within-distance of each pair is set small enough. Here, we consider the following spatial frequencies, $\{\omega_j\}$; $\omega_j = \pi(2(j-1)+1)/10$ with $j = 1, \dots, 5$, where the uniform spacing of $\pi/5$ is slightly longer than $\pi/6$. The test statistic for lack of axial symmetry in space, $\hat{\psi}_i^*(\omega_j)$ in (28) is constructed at the following temporal frequencies; $\pi/10, 3\pi/10, \dots, 9\pi/10$. We consider the 16 pairs, $\{(a_2^j, t_i^a), (b_2^j, t_i^b)\}$, $i = 1, \dots, 16$ shown in Figure 8. The temporal between-pair distance is set to at least 100 units (unit=1 day) and the spatial between-pair distance is set to more than 15 units (unit=36km). We also take the two different directions into account.

Now we explain the output from the ANOVA model for testing lack of axial symmetry in space. Table 2 shows that (lack of) axial symmetry in time and, even, “Space-Time Interaction” effect are deeply dependent on how we make the pairs. In case of direction **D.3**, only “Spatial Frequency” effect is significant under the 0.05 level of significance, that is, $C(\mathbf{h}; u) \neq C(\mathbf{h}_1^a; u)$ for $h_2 = 72(\text{km})$ and $u = 1$ fixed. This implies that covariance (or correlation) between arbitrary two sites with one spatial lag and time lag fixed as the other spatial lag changes, especially in direction **D.3**, and lack of axial symmetry in space is inherent in this spatial-temporal process. However, when direction **D.4** is considered, only “Space-Time Interaction” effect is significant. Since the covariance between any two measurements depends on their relative position under nonstationarity in space, or in space and time, this nonzero effect can be one evidence against stationarity in space and time. For both directions, the normality assumption for the residuals is satisfied.

In this section, we applied the formal tests to the real air-pollution dataset. Based on the results from the tests for lack of axial symmetry in time (Table 1) and lack of axial symmetry in space (Table 2), we finally reach the conclusion that the spatial-temporal process of $\text{PM}_{2.5}$ anomaly concentration has apparent evidences for lack of axial symmetry in time for the Northeastern-Southwestern direction and for lack of axial symmetry in space. Some of the main factors causing these lacks of symmetry can be external meteorological conditions, for instance, air pressure, temperature, wind direction, and so on. These factors tend to cause spatial-temporal processes to have complicated dependency structures including lack of symmetry.

8 Discussion

In this study, we introduced new concepts of symmetry in spatial-temporal processes and proposed new formal tests for lack of axial symmetry in time and for lack of axial symmetry in space. We evaluated the performances of the tests with simulation studies and with the real application. The main advantage of the tests is that we can easily check not only the existence

of lack of symmetry but also the potential direction causing asymmetry besides the existence of nonstationarity.

In reality, spatial or spatial-temporal processes may have strong evidence for lack of the symmetries introduced in this study. That may be a good motivation for taking asymmetric spatial-temporal covariance functions into account for explaining the given data better (see Park and Fuentes, 2006).

As part of our further research, we will be developing a formal test for lack of diagonal symmetry in space defined by, under stationarity in space,

$$C(\mathbf{h};u) \neq C(\mathbf{h}_{kl}^c;u)$$

where $\mathbf{h}_{kl}^c = (h_1, \dots, h_{k-1}, h_l, h_{k+1}, \dots, h_{l-1}, h_k, h_{l+1}, \dots, h_d)'$ for $k \neq l$. This test could also be approached by the spectral representation that we have used in this study.

Appendix

Assumptions for new testing methods

A.1 $W(\mu)$ is real-valued, even and of bounded variation such that, for $-\infty < \mu < \infty$,

$$\int_{\mathbb{R}} W(\mu) d\mu = 1$$

and

$$\int_{\mathbb{R}} |W(\mu)| d\mu < \infty.$$

.

A.2 For each $\mathbf{h}$,

$$\sum_{u=-\infty}^{\infty} |u| |C(\mathbf{h};u)| < \infty,$$

which implies that the temporal covariance is summable, that is,

$$\sum_{u=-\infty}^{\infty} |C(\mathbf{h};u)| < \infty.$$

A.3 $B_T T \rightarrow \infty$ and $B_T \rightarrow 0$ as $T \rightarrow \infty$.

A.4 for fixed h_2 and u ,

$$\int_{\mathbb{R}} |h_1| |C(h_1, h_2; u)| dh_1 < \infty,$$

which also implies that the spatial covariance is summable, that is,

$$\int_{\mathbb{R}} |C(h_1, h_2; u)| dh_1 < \infty.$$

A.5 $B_{N_I} N_I \rightarrow \infty$ and $B_{N_I} \rightarrow 0$ as $N_I \rightarrow \infty$.

Asymptotic normality of the test statistic for lack of axial symmetry in time

We show the asymptotic normality of $\widehat{\phi}_{ab}^*(\tau)$ in (18). Fuentes (2006) provides the asymptotic normality and the approximate independence of the cross-spectral density function, $\hat{f}_{ab}(\tau)$ evaluated at different frequencies and at different sites. The approximate independence between $\hat{f}_{a_i b_i}(\tau)$ and $\hat{f}_{a_j b_j}(\lambda)$ is also obtained under either of the conditions, C.1 and C.2. Based on the information from Fuentes (2006), we try to find the asymptotic distribution of $\widehat{\phi}_{ab}^*(\tau)$ in (18).

Suppose that $\widehat{\phi}_{ik}^* \equiv \widehat{\phi}_{a_i b_i}^*(\tau_k)$, $\widehat{\Theta} \equiv (\widehat{\phi}_{ik}^*, \widehat{\phi}_{jl}^*)'$, and $\theta^\wedge \equiv (\hat{\phi}_{ik}, \hat{R}_{ik}, \hat{\phi}_{jl}, \hat{R}_{jl})'$, where $\hat{\phi}_{ik}$ and $\hat{R}_{ik}$ are the phase and the coherency at the i_{th} pair at the temporal frequency τ_k . Then, by Taylor-series expansion,

$$\widehat{\Theta} \approx \Theta + \frac{\partial \widehat{\Theta}}{\partial \theta} (\widehat{\theta} - \theta),$$

where $\Theta = (\phi_{ik}^*, \phi_{jl}^*)'$, $\theta = (\phi_{ik}, R_{ik}, \phi_{jl}, R_{jl})'$. Under the null hypothesis that $\phi_{ik} = \phi_{jl} = 0$, we can re-express the previous equation as

$$\widehat{\Theta} = \Theta_0 + \frac{\partial \widehat{\Theta}}{\partial \theta_0} (\widehat{\theta} - \theta_0) + \mathbf{o} \left(\frac{1}{B_T T} \right),$$

where $\Theta_0 = (0, 0)'$, $\theta_0 = (0, R_{ik}, 0, R_{jl})'$,

$$\frac{\partial \widehat{\Theta}}{\partial \theta_0} = \begin{pmatrix} [|R_{ik}|^{-2} - 1]^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & [|R_{jl}|^{-2} - 1]^{-1/2} & 0 \end{pmatrix},$$

and the error term is uniform in τ_k and τ_l . Here $\mathbf{o}(a)$ is a vector with the elements having the convergence order, $\mathbf{o}(a)$. Under the assumptions A.1 through A.3, we asymptotically obtain the mean and the variance Σ_{Θ} , which is denoted by

$$\begin{aligned} \Sigma_{\Theta} &= B_T T \left(\frac{\partial \widehat{\Theta}}{\partial \theta_0} \right) \Sigma_{\theta} \left(\frac{\partial \widehat{\Theta}}{\partial \theta_0} \right)' \\ &= \begin{pmatrix} \frac{B_T T \text{Var}\{\widehat{\phi}_{ik}\}}{|R_{ik}|^{-2} - 1} & \frac{B_T T \text{cov}\{\widehat{\phi}_{ik}, \widehat{\phi}_{jl}\}}{\sqrt{(|R_{ik}|^{-2} - 1)(|R_{jl}|^{-2} - 1)}} \\ \frac{B_T T \text{cov}\{\widehat{\phi}_{jl}, \widehat{\phi}_{ik}\}}{\sqrt{(|R_{ik}|^{-2} - 1)(|R_{jl}|^{-2} - 1)}} & \frac{B_T T \text{Var}\{\widehat{\phi}_{jl}\}}{|R_{jl}|^{-2} - 1} \end{pmatrix}, \end{aligned}$$

where $\Sigma_{\Theta} = E[(\hat{\Theta} - \Theta_0)(\hat{\Theta} - \Theta_0)']$. If either of the conditions, C.1 and C.2 is satisfied, then $\hat{\Phi}_{\mathbf{a}_i \mathbf{b}_i}(\tau_k)$ is approximately independent of $\hat{\Phi}_{\mathbf{a}_j \mathbf{b}_j}(\tau_l)$. Therefore, we finally compute the following asymptotic variance:

$$\Sigma_{\Theta} = \begin{pmatrix} \pi \int_{\mathbb{R}} W_T^2(\alpha) d\alpha (1 - \eta\{2\tau_k\}) & 0 \\ 0 & \pi \int_{\mathbb{R}} W_T^2(\alpha) d\alpha (1 - \eta\{2\tau_l\}) \end{pmatrix}.$$

Asymptotic normality of the test statistic for lack of axial symmetry in space

We provide the asymptotic normality of $\hat{\psi}_i^*(\omega_j)$ and the approximate independence of the cross-spectral density unction, $\hat{\psi}_i^*(\omega_j)$ evaluated at different frequencies and at different pairs. Since $\hat{\psi}^*(\Omega)$ is basically a function of $\hat{k}$ we should take into account the asymptotic properties of this cross-spectral density function. In the main part of this study, we already studied the asymptotic mean and the asymptotic (co)variance of $\hat{k}$. The last thing to do is to prove the asymptotic joint normality by showing that all standardized joint cumulants of order greater than 2 tend to 0 as $N_I \rightarrow \infty$ under the indicated conditions (see Brillinger, 1981, page 437). In a similar way with the approach in Brillinger (1981) we can obtain that the cumulant

$$\text{cum} \left\{ (B_{N_1} N_1)^{1/2} \widehat{k}_{(b_2^1, l^1, a_2^1, a_1^1)}(\pm\omega_1), \dots, (B_{N_1} N_1)^{1/2} \widehat{k}_{(b_2^J, l^J, a_2^J, a_1^J)}(\pm\omega_J) \right\}$$

is of order $B_{N_1}^{-J/2+1} N_1^{-J/2+1}$, which makes it go to 0 as $N_I \rightarrow \infty$ for $J > 2$ due to A.5. Thus, the asymptotic normality follows from Fuentes (2006, Lemma 2). Also, the approximate independence can be obtained under either of the conditions, C.3 and C.4.

Suppose that $\hat{\psi}_{ik}^* \equiv \hat{\psi}_i^*(\omega_k)$, $\widehat{\Xi} \equiv (\hat{\psi}_{ik}^*, \hat{\psi}_{jl}^*)'$, and $\widehat{\xi} \equiv (\hat{\psi}_{ik}, \hat{Q}_{ik}, \hat{\psi}_{jl}, \hat{Q}_{jl})'$, where $\hat{\psi}_{ik}$ and $\hat{Q}_{ik}$ are the phase and the coherency at the i_{th} pair at the frequency ω_k . Then, by Taylor-series expansion,

$$\widehat{\Xi} \approx \Xi + \frac{\partial \widehat{\Xi}}{\partial \xi} (\widehat{\xi} - \xi),$$

where $\Xi = (\hat{\psi}_{ik}^*, \hat{\psi}_{jl}^*)'$, and $\xi = (\psi_{ik}, Q_{ik}, \psi_{jl}, Q_{jl})'$. Under the null hypothesis that $\psi_{ik} = 0$ and $\psi_{jl} = 0$, we can rewrite the previous equation as

$$\widehat{\Xi} = \Xi_0 + \frac{\partial \widehat{\Xi}}{\partial \xi_0} (\widehat{\xi} - \xi_0) + \mathbf{o} \left(\frac{1}{B_{N_1} N_1} \right),$$

where $\Xi_0 = (0, 0)'$, $\xi_0 = (0, Q_{ik}, 0, Q_{jl})'$,

$$\frac{\partial \widehat{\Xi}}{\partial \xi_0} = \begin{pmatrix} [|\hat{Q}_{ik}|^{-2} - 1]^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & [|\hat{Q}_{jl}|^{-2} - 1]^{-1/2} & 0 \end{pmatrix},$$

and the error term is uniform in ω_k and ω_l . Under the assumptions A.1, A.4, and A.5, we asymptotically obtain the mean and the variance Σ_{Θ} , which is denoted by

$$\begin{aligned}\Sigma_{\Xi} &= B_{N_1} N_1 \left(\frac{\partial \Xi}{\partial \xi_0} \right) \Sigma_{\Xi} \left(\frac{\partial \Xi}{\partial \xi_0} \right)' \\ &= \begin{pmatrix} \frac{B_{N_1} N_1 \text{Var}\{\widehat{\psi}_{ik}\}}{|Q_{ik}|^{-2}-1} & \frac{B_{N_1} N_1 \text{cov}\{\widehat{\psi}_{ik}, \widehat{\psi}_{jl}\}}{\sqrt{(|Q_{ik}|^{-2}-1)(|Q_{jl}|^{-2}-1)}} \\ \frac{B_{N_1} N_1 \text{cov}\{\widehat{\psi}_{jl}, \widehat{\psi}_{ik}\}}{\sqrt{(|Q_{ik}|^{-2}-1)(|Q_{jl}|^{-2}-1)}} & \frac{B_{N_1} N_1 \text{Var}\{\widehat{\psi}_{jl}\}}{|Q_{jl}|^{-2}-1} \end{pmatrix},\end{aligned}$$

where $\Sigma_{\Xi} = E[(\widehat{\xi} - \xi_0)(\widehat{\xi} - \xi_0)']$. Under either of the conditions, C.3 and C.4 $\widehat{\psi}_{ik}$ is approximately independent of $\widehat{\psi}_{jl}$. Therefore, we finally compute the following asymptotic variance:

$$\Sigma_{\Xi} \begin{pmatrix} \pi \int_{\mathbb{R}} W_s^2(\alpha) d\alpha (1 - \eta\{2\omega_k\}) & 0 \\ 0 & \pi \int_{\mathbb{R}} W_s^2(\alpha) d\alpha (1 - \eta\{2\omega_l\}) \end{pmatrix}.$$

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Fig. 1.

The six different ways of selecting pairs of two sites for testing lack of axial symmetry in time; Note that the x-axis is the easting and the y-axis is the northing.

Fig. 2. The contour plots of empirical power of “temporal frequency” effect under asymmetry in space and time; Note that the null hypothesis is located on the origin ($\mathbf{v} = \mathbf{0}$).

Fig. 3. The Histograms of Empirical Power of “Location” Effect and empirical probability of rejecting the normality assumption by Pearson’s χ^2 Normality Test of the residuals. Note that these histograms are based on all the directions combined.

(a) D.1

(b) D.2

Fig. 4.

The two different ways of choosing pairs of two points for testing lack of axial symmetry in space; Note that the x-axis is the northing and the y-axis is the time.

Fig. 5.
The contour plots of empirical power of “Spatial Frequency” effect under asymmetry in space and time.

Fig. 6.

The histograms of empirical power of "Space-Time Interaction" effect and empirical probability of rejecting the normality assumption by Pearson's χ^2 Normality Test of the residuals; Note that these histograms are based on all the directions combined.

Fig. 7.

The map of the sites of 3721 ($= 61 \times 61$) centroids of grid cells where each cell is size of $36\text{km} \times 36\text{km}$; Note that the numbers on the right of grid cells are row indice and the ones on the top are column indice.

(a) **D.3** (b) **D.4****Fig. 8.**

The two different ways of choosing pairs of two points for testing lack of axial symmetry in space; Note that the x-axis is the northing and the y-axis is the time.

Table 1

Analysis of variance

Direction	Item	DF	SS	F value	Pr(F)	Pr(χ^2)
ESE - WNW	Frequencies	13	5.31	4.11	< .01	0.73
	sites	15	6.07	4.07	< .01	
	Residuals	180	17.91			
East - West	Frequencies	13	6.14	5.46	< .01	0.29
	sites	15	3.90	3.00	< .01	
	Residuals	180	15.57			
ENE - WSW	Frequencies	13	6.10	6.28	< .01	0.40
	sites	15	2.99	2.67	< .01	
	Residuals	180	13.46			
NNE - SSW	Frequencies	13	1.96	2.13	0.01	0.33
	sites	15	1.95	1.83	0.03	
	Residuals	180	12.79			
North - South	Frequencies	13	1.42	1.45	0.14	< .01
	sites	15	3.69	3.26	< .01	
	Residuals	180	13.58			
NNW - SSE	Frequencies	13	2.06	1.63	0.08	0.17
	sites	15	5.29	3.62	< .01	
	Residuals	180	17.56			

Table 2

Analysis of variance

Direction	Item	Df	SS	F value	Pr(F)	Pr(χ^2)
D.3	Frequencies	5	38.83	2.51	0.04	0.30
	Interactions	15	59.00	1.27	0.25	
	Residuals	180	185.63			
D.4	Frequencies	5	13.75	0.94	0.46	0.89
	Interactions	15	83.25	1.89	0.04	
	Residuals	180	176.00			