Published in final edited form as:

Proc SPIE Int Soc Opt Eng. 2008 ; 6913: 691321-. doi:10.1117/12.772390.

Exact and approximate cone-beam reconstruction algorithms for C-arm based cone-beam CT using a two-concentric-arc source trajectory

Tingliang Zhuang^a, Joseph Zambelli^a, Brian Nett^a, Shuai Leng^a, and Guang-Hong Chen^{a,b,*}

^aDepartment of Medical Physics, University of Wisconsin, Madison, 53792, USA

^bDepartment of Radiology, University of Wisconsin, Madison, 53792, USA

Abstract

In this paper, we present shift-invariant filtered backprojection (FBP) cone-beam image reconstruction algorithms for a cone-beam CT system based on a clinical C-arm gantry. The source trajectory consists of two concentric arcs which is complete in the sense that the Tuy data sufficiency condition is satisfied. This scanning geometry is referred to here as a CC geometry (each arc is shaped like the letter "C"). The challenge for image reconstruction for the CC geometry is that the image volume is not well populated by the familiar doubly measured (DM) lines. Thus, the well-known DM-line based image reconstruction schemes are not appropriate for the CC geometry. Our starting point is a general reconstruction formula developed by Pack and Noo which is not dependent on the existence of DM-lines. For a specific scanning geometry, the filtering lines must be carefully selected to satisfy the Pack-Noo condition for mathematically exact reconstruction. The new points in this paper are summarized here. (1) A mathematically exact cone-beam reconstruction algorithm was formulated for the CC geometry by utilizing the Pack-Noo image reconstruction scheme. One drawback of the developed exact algorithm is that it does not solve the long-object problem. (2) We developed an approximate image reconstruction algorithm by deforming the filtering lines so that the long object problem is solved while the reconstruction accuracy is maintained. (3) In addition to numerical phantom experiments to validate the developed image reconstruction algorithms, we also validate our algorithms using physical phantom experiments on a clinical C-arm system.

1. INTRODUCTION

Interventional C-arm systems provide image guidance for minimally invasive interventional procedures where high contrast objects such as bony structure and iodinate-filled vasculature are often the primary imaging targets. Besides the intensive investigations in commercial environments, many investigators in academic environments have also investigated C-arm based cone-beam computed tomography (CT) systems¹⁻⁴ using either an x-ray image intensifier (XRII) or a large area flat-panel detector. In these investigations, the image object is often scanned using a single circular(arc) scanning path. From the image reconstruction point of view, it is clear that the single circular(arc) scan does not provide sufficient

* All scientific correspondence should be addressed to G.-H. Chen via email: gchen7@wisc.edu.

projections to achieve an accurate image reconstruction for image points off the scanning plane. The severity of cone-beam artifacts increases when large longitudinal coverage is desired.^{4, 5} Note that the scatter induced artifacts also increase when the longitudinal scanning coverage is increased. These two factors together confound the final cone-beam CT image quality. In this paper, we select to only address the issue of the cone-beam artifacts.

In order to eliminate cone-beam artifacts, the scanning path should fulfil the Tuy data sufficiency condition.⁶ Based on a C-arm gantry, different complete trajectories may be implemented such as two concentric arcs, and arc plus line. In our lab, a flat-panel cone-beam CT data acquisition system was developed^{4, 5} using a GE Innova 4100 (GE Healthcare, Waukesha, Wisconsin) clinical system. Using this system, it is convenient to implement a complete source trajectory using two concentric arcs (each arc has an angular coverage of 210°) as shown in Figure 1. This scanning geometry is referred to as a CC geometry in this paper (each arc is in the shape of the letter "C"). The implementation of the CC trajectory does not require patient bed movement, which is highly desirable in an interventional angio-suite.

The CC scanning trajectory is complete in the sense of the Tuy data sufficiency condition. Therefore, a mathematically exact reconstruction algorithm can be developed to eliminate the cone-beam artifacts that appear when imaging with a single circular(arc) scan. In the past several years, image reconstruction algorithm development has been significantly advanced. Two general image reconstruction frameworks have been developed: the first is filtered backprojection (FBP)-type image reconstruction^{7-11, 14-16} and the second involves filtering the backprojection image of differentiated projection data (FBPD).¹⁷⁻²³ Based upon these general image reconstruction schemes, algorithms have been specifically derived for some circle(arc)-based source trajectories.^{8, 11, 12, 20, 24-29} The algorithms developed in^{11, 20, 25, 26} depend on an important geometrical property of the source trajectory: existence of at least one doubly measured line (DM-line) for each image point. Unfortunately, for the CC geometry, DM-lines do not well populate the image volume. In fact, as shown in Figure 1, for any point inside the pentahedron $O - A - B - C - D$, but not within the two scanning planes (plane AOC and plane BOD), a line passing through any point will not intersect with the source trajectory simultaneously at two points. Based upon this simple observation, DM-line based image reconstruction schemes can not be directly adapted for the CC geometry.

Recently, a shift-invariant FBP image reconstruction formula was developed by Pack and Noo.¹⁶ A very attractive feature of the Pack-Noo formula is that it does not depend on the existence of the DM-lines. Simply speaking, whenever a group of filtering lines can be found to satisfy a condition (in this paper, we refer to this condition as the Pack-Noo condition), then Pack-Noo's formula achieves a mathematically exact reconstruction. A general form of filtering lines which satisfies Pack-Noo condition has been given in the paper by Pack and Noo for an "umbrella" geometry,¹⁶ and has been generalized to other source geometry by Yang et al.³² Most recently, the Pack-Noo reconstruction scheme has been successfully adapted to develop algorithms for several scanning geometries. For a closed geometry such as a saddle trajectory,^{30,31} a circular sinusoid trajectory,^{32,34} or a

two circles trajectory,³² the developed image reconstruction algorithm does solve the long-object problem. While for the twisted helix geometry the Pack-Noo formula does not provide a solution for the long-object problem.³³

In this paper, we develop a shift-invariant cone-beam FBP algorithm for the CC scanning geometry which is based upon the Pack-Noo reconstruction scheme. For a short image object, a mathematically exact image reconstruction algorithm is presented for the CC geometry. However, the exact algorithm does not solve the long object problem. In order to overcome this problem, we design a scheme to deform the filtering lines yielding an approximate image reconstruction algorithm. This approximate algorithm solves the long-object problem while maintaining the accuracy of the reconstruction. We have validated the developed image reconstruction algorithms by using mathematical phantom data and physical phantom data acquired from our experimental system.

2. REVIEW PACK-NOO IMAGE RECONSTRUCTION SCHEME

We briefly review the Pack-Noo image reconstruction scheme in this section. Denote the cone-beam projection of an image function $f(\vec{x})$ from a source point $\vec{y}(t)$ in the direction of $\hat{r}$ as:

$$g(\hat{r}, t) = \int_0^{+\infty} ds f[\vec{y}(t) + s\hat{r}]. \quad (1)$$

Given an image point $\vec{x}$ and projection data acquired from a continuous segment of the source path $\Lambda(t_I, t_F)$ (view angles t_I and t_F parameterize the end points of this path), the filtered backprojection data $\kappa[\vec{x}, \hat{e}, \Lambda(t_I, t_F)]$ is obtained using^{9, 13-15}

$$\kappa[\vec{x}, \hat{e}, \Lambda(t_I, t_F)] = -\frac{1}{2\pi^2} \int_{t_I}^{t_F} dt \frac{1}{|\vec{x} - \vec{y}(t)|} \int_{-\pi}^{\pi} d\gamma \frac{1}{\sin \gamma} \frac{\partial}{\partial q} g[\hat{r}(\hat{\beta}, \hat{e}; \gamma), q] \Big|_{q=t}, \quad (2)$$

where

$$\begin{aligned} \hat{r}(\hat{\beta}, \hat{e}; \gamma) &= \hat{\beta} \cos \gamma + \hat{\beta}^\perp \sin \gamma, \\ \hat{\beta}(\vec{x}, t) &= \frac{\vec{x} - \vec{y}(t)}{|\vec{x} - \vec{y}(t)|}, \quad \hat{\beta}^\perp(\vec{x}, t) = \frac{\hat{e} - \hat{e} \cdot \hat{\beta}(\vec{x}, t) \hat{\beta}(\vec{x}, t)}{|\hat{e} - \hat{e} \cdot \hat{\beta}(\vec{x}, t) \hat{\beta}(\vec{x}, t)|}, \end{aligned} \quad (3)$$

and $\hat{e}$ is an arbitrary unit vector.

As proved by Bontus et al.,¹³ the function $\kappa[\vec{x}, \hat{e}, \Lambda(t_I, t_F)]$ is related to the second order derivative of the Radon transform of the image function ($R'' f(\hat{\omega}, \vec{x} \cdot \vec{\omega})$) by:

$$\kappa[\vec{x}, \hat{e}, \Lambda(t_I, t_F)] = -\frac{1}{8\pi^2} \int d\hat{\omega} R'' f(\hat{\omega}, \vec{x} \cdot \hat{\omega}) \sigma(\vec{x}, \hat{\omega}, \hat{e}, t_I, t_F). \quad (4)$$

Comparing the above formula with the 3D Radon inversion formula, $f(\vec{x}) =$

$-\frac{1}{8\pi^2} \int d\hat{\omega} R'' f(\hat{\omega}, \vec{x} \cdot \hat{\omega})$, one concludes that, if

$$\sigma(\vec{x}, \hat{\omega}, \hat{e}, t_I, t_F) = 1, \quad (5)$$

for any (or almost any) $\hat{\omega} \in S^2$, then $f(\vec{x}) = \kappa[\vec{x}, \hat{e}, \Lambda(t_I, t_F)]$. The same conclusion has also been obtained in¹⁴ using a different approach. Based upon this observation, an image reconstruction algorithm is obtained by selecting a unit vector $\hat{e}$ which satisfies the condition (5).

Pack and Noo further extended the above results. They developed the following image reconstruction formula¹⁶

$$f(\vec{x}) = \sum_l \kappa \left[\vec{x}, \hat{e}_l, \Lambda(t_I^l, t_F^l) \right], \quad (6)$$

given that

$$\sum_l \sigma(\vec{x}, \hat{\omega}, \hat{e}_l, t_I^l, t_F^l) = 1, \quad (7)$$

where, one continuous path of the source trajectory is denoted by $\Lambda(t_I^l, t_F^l)$ with t_I^l and t_F^l denoting the view angles of the starting and ending points of that path. Each continuous path is associated with one unit vector $\hat{e}_l$ which is used to determine the filtering lines. In this paper, condition (7) will be referred to as the Pack-Noo condition. For a unit vector $\hat{e}$ that does not depend on the view angle t , the σ -function in Equation (4) can be calculated using¹⁵

$$\sigma(\vec{x}, \hat{\omega}, \hat{e}, t_I, t_F) = \frac{1}{2} \text{sgn}(\hat{\omega} \cdot \hat{e}) \left\{ \text{sgn} \left[\hat{\omega} \cdot \hat{\beta}(\vec{x}, t_I) \right] - \text{sgn} \left[\hat{\omega} \cdot \hat{\beta}(\vec{x}, t_F) \right] \right\} \quad (8)$$

Utilizing the above formula, a simple selection of unit vectors $\hat{e}_l$ which satisfy the Pack-Noo condition (7) was derived for an "umbrella" trajectory with three legs¹⁶ and a FBP image reconstruction formula was thus obtained. In the next section, we will employ this reconstruction scheme to design an image reconstruction algorithm for the CC-geometry.

3. AN EXACT RECONSTRUCTION ALGORITHM FOR THE SOURCE TRAJECTORY OF TWO CONCENTRIC ARCS

The CC-geometry consisting of two concentric arcs with equal radius R (Figure 2(a)) is parameterized as :

$$\begin{aligned} C_1: R(\cos \mu_1 \cos t_1, \sin \mu_1 \cos t_1), t_1 \in \Lambda_1 = (t_1^I, t_1^F), \\ C_2: R(\cos \mu_2 \cos t_2, \sin \mu_2 \cos t_2), t_2 \in \Lambda_2 = (t_2^I, t_2^F), \end{aligned} \quad (9)$$

where, $\mu_i (i = 1, 2)$ denotes the angle between each scanning plane and the $x - y$ plane. The notations t_i^I and t_i^F denote the view angles that parameterize the end points of the two arcs.

As shown in Figure 2(b), a plane parallel to the xz plane will intersect the source trajectory at four points A, B, C and D. The view angles that parameterize these four points are given by:

$$t_A = t_B = \pi - \arcsin \frac{y}{R}, \quad (10)$$

$$t_C = t_D = \arcsin \frac{y}{R}. \quad (11)$$

It is clear that there exists a continuous source path connecting any two adjacent vertices of the rectangle ABCD (for instance, the path from A to B may be formed using two sub arcs AO and OB). The following formula can be used to reconstruct the density value at any image point $\vec{x}$ which is inside the rectangle ABCD:

$$f(\vec{x}) = \frac{1}{2} \{ \kappa[\vec{x}, \hat{e}_A, \Lambda(t_A, t_B)] + \kappa[\vec{x}, \hat{e}_B, \Lambda(t_B, t_C)] + \kappa[\vec{x}, \hat{e}_C, \Lambda(t_C, t_D)] + \kappa[\vec{x}, \hat{e}_D, \Lambda(t_D, t_A)] \}, \quad (12)$$

where

$$\hat{e}_A = -\hat{e}_C = \hat{z}, \quad \hat{e}_B = -\hat{e}_D = \hat{x}, \quad (13)$$

and the continuous source paths $\Lambda(t_{V1}, t_{V2})$ with $V_1, V_2 = A, B, C, D$ are obtained using sub arcs V_1O and OV_2 . This reconstruction formula written using flat-panel detector coordinates is also given in Appendix A.

3.1. Analysis of the exact reconstruction formula

The key component of the reconstruction formula (Equation (12)) is the κ -function defined in Equation (2). When computing the κ -function using a third generation flat-panel-detector data acquisition geometry, a 1D filtering process is performed along the lines which are the intersection between the detector plane and the plane spanned by the unit vectors $\hat{\beta}$ and $\hat{e}$ as shown in Figure 3(a). For a fixed unit vector $\hat{e}$, all the filtering lines for one view angle will pass through a detector point where the ray starting from the source point in the direction of the unit vector $\hat{e}$ intersects the detector plane (Figure 3(a)). We denote this particular detector point by $E(\hat{e}, t)$.

We now determine whether the long object problem can be solved for the CC-geometry using formula (12). Formula (12) may be rewritten as:

$$f(\vec{x}) = \frac{1}{2} \{ \kappa[\vec{x}, \hat{e}_A, \Lambda(t_A, t_O)] + \kappa[\vec{x}, \hat{e}_A, \Lambda(t_O, t_B)] + \kappa[\vec{x}, \hat{e}_B, \Lambda(t_B, t_O)] + \kappa[\vec{x}, \hat{e}_B, \Lambda(t_O, t_C)] + \kappa[\vec{x}, \hat{e}_C, \Lambda(t_C, t_O)] + \kappa[\vec{x}, \hat{e}_C, \Lambda(t_O, t_D)] + \kappa[\vec{x}, \hat{e}_D, \Lambda(t_D, t_O)] + \kappa[\vec{x}, \hat{e}_D, \Lambda(t_O, t_A)] \}. \quad (14)$$

Thus, we only need to analyze whether longitudinal data truncation is tolerable when calculating each of the component functions $\kappa[\vec{x}, \hat{e}, \Lambda(t_O, t_V)]$, $V = A, B, C, D$, for a specific choice of the unit vector $\hat{e}$, and all image points $\vec{x}$ within a volume of interest (VOI).

Using a CC scanning geometry, the patient bed is usually set up along the z -direction of the coordinate system as shown in Figure 2(b). In this case, the patient can be approximately modelled by a cylindrical object given by

$$x^2 + y^2 \leq r^2, \quad (15)$$

Suppose the VOI is a portion of the cylinder with $|z| < H$. As shown in Figure 3(b), the cone-beam projection of the cylinder at each view angle is denoted by $S(t)$ which lies in between two dashed lines given by Equation (31) (Appendix B). This region is further divided into two sub-regions $S_1(t)$ and $S_2(t)$ by the solid line shown in Figure 3(b). The cone-beam projection of the VOI at each view angle is denoted by $P(t)$. At a given view angle, the filtered data within the region $BP(t)$, which is the overlap region between the regions $P(t)$ and $S_1(t)$, will be backprojected to the image space as shown in Appendix B. It is obvious that if a filtering line that passes through the region $BP(t)$ is completely inside the region $S_1(t)$ for $t \in [t_O, t_V]$, the filtration will utilize the projection data along that line of the whole cylinder when calculating the κ -function. This situation will happen when the fixed point $E(\hat{e}, t)$ belongs to the region $S_1(t)$. In this case, other filtering lines may also require a very large detector to provide necessary projection data as they are intersecting at one point $E(\hat{e}, t)$ within the region $S_1(t)$ (Figure 3(b)). Thus, longitudinal data truncation is not well tolerated when computing the functions $\kappa[\vec{x}, \hat{e}, \Lambda(t_O, t_V)]$.

For the CC scanning geometry given by Equation (9) and a cylindrical image object given by Equation (15), the points $E(\hat{e}_A, t)$ and $E(\hat{e}_C, t)$ are located at the intersection point of the two boundary lines of $S(t)$ and the points $E(\hat{e}_B, t)$ and $E(\hat{e}_D, t)$ are located at one of the boundaries of the region $S_1(t)$ as shown in Appendix B. One filtering line associated with the fixed points $E(\hat{e}_V, t)$, $V = B, D$ belongs to the region $S_1(t)$. Every filtering line associated with the fixed points $E(\hat{e}_V, t)$ for $V = A, C$ is completely inside the region $S_1(t)$. Therefore, filtration using these filtering lines requires the projection data of the entire cylinder to have been measured. Thus, formula (12) can not solve the longitudinal data truncation problem which is inevitable in the clinical setting. An approximate algorithm was developed to overcome this problem and is presented in the next section.

4. AN APPROXIMATE ALGORITHM FOR THE CC GEOMETRY

In this section a practical approximate algorithm for the CC geometry to solve long object problem is developed. The strategy we utilized is to deform the filtering lines so that filtering process will only utilize measured projection data. Several deformation methods may exist. In this paper, the deformation will have the property that the set of filtering lines at each view angle will share a common point on the detector as in the case of exact algorithm.

As shown in Figure 4, the two boundaries of the cone-beam projected cylinder (which defined the region $S(t)$) on the detector plane intersect with the u -axis of the detector coordinate system at two points, $B_{\pm}(t)$. If the fixed point $E(\hat{e}, t)$ is chosen along the u -axis but either to the left of the point $B_+(t)$ or to the right of the point $B_-(t)$ for each view angle,

none of the filtering lines reside completely inside the region $S(t)$ (Figure 4). Thus, longitudinal data truncation is acceptable when computing the functions $\kappa[\vec{x}, \hat{e}, \Lambda(t_O, t_V)]$ using this choice of the point $E(\hat{e}, t)$. Note also, using the exact reconstruction formula as given by Equation (14), each sub arc (AO, BO, CO, DO) will be used twice when forming all the continuous paths along the source trajectory. Since each continuous path is associated with one unit vector $\hat{e}$ which determines the filtering lines, formula (14) shows that the projection data at each view angle will be filtered twice. For developing an approximate algorithm, one may reduce the computational load by using only one set of filtering lines.

Using the flat-panel detector coordinates and notations given in Appendix A, we may formulate the following approximate image reconstruction formula for the CC-geometry which solves the long object problem:

$$f_{approx}(\vec{x}) = -\frac{1}{4\pi^2} \left\{ \int_{t_c}^{t_A} dt_1 \frac{\bar{Q}^1(u_\beta^1, v_\beta^1, t_1)}{L_1(\vec{x}, t_1)} + \int_{t_D}^{t_B} dt_2 \frac{\bar{Q}^2(u_\beta^2, v_\beta^2, t_2)}{L_2(\vec{x}, t_2)} \right\}, \quad (16)$$

where $\bar{Q}^i = \int du h_H(u - u_i) g_d(u, v, t_i)$, $i = 1, 2$ are the filtered data using the filtering lines given by:

$$v = v_i + \frac{v_i - v_E(t_i)}{u_i - u_E(t_i)} (u - u_i). \quad (17)$$

The coordinate of the fixed detector point $(u_E(t_i), v_E(t_i))$ are given by:

$$u_E(t_i) = \lambda u_B(t_i, +), \quad v_E(t_i) = 0, \quad (18)$$

where $\lambda = 1$ is a free parameter. The expressions of $L_i(\vec{x}, t_i)$, $u_B(t_i, +)$ for $i = 1, 2$ are given in the Appendix A and B. The effect of the parameter λ on the reconstruction will be demonstrated in the next section.

5. VALIDATION OF THE ALGORITHMS

Both analytically generated projection data of mathematical phantoms and experimentally measured projection data were utilized to validate our algorithms.

5.1. Mathematical Phantom

A high contrast Defrise phantom and two low contrast Shepp-Logan phantoms were utilized to generate analytical cone-beam projection data. The projection data of one Shepp-Logan phantom is longitudinally truncated (see Appendix C for the parameters for this phantom), while projections of the second Shepp-Logan phantom³⁸ are not truncated. These two phantoms are referred to as the long and short phantoms respectively. The parameter $\lambda = 20$ was chosen here for the approximate algorithm. The reconstruction parameters are given as follows: gantry radius=4, source to detector distance =8, $\mu_1 = \mu_2 = 15^\circ, -15^\circ$, detector size =6

$\times 6$, detector matrix size $=401 \times 401$, number of views for each arc $=420$, scan range for each arc $=[-15^\circ, 195^\circ]$, image matrix $=256 \times 256 \times 256$, reconstruction volume $=2 \times 2 \times 2$.

Figure 5 shows the reconstruction results of a mathematical Defrise phantom using the CC-scan with the proposed algorithm given by Equation (12) and using a single-arc-scan with the FDK algorithm. It is clear that the cone-beam artifacts are only present in the images reconstructed using the single-arc-scan geometry with the FDK algorithm.

Figure 6 shows the reconstruction results of a short and a long low-contrast Shepp-Logan phantom using algorithms (12) and (16). As shown in Figure 6(a), one observes that for a short object, accurate reconstruction is achieved using the exact algorithm (Equation (12)), while reasonable image quality may be obtained using the approximate algorithm (Equation (16)). As shown in Figure 6(b), since longitudinal data truncation is present using a long Shepp-Logan phantom, several cupping artifacts are observed in the reconstructed images using the algorithm given by Equation (12) which is exact only for a short object. However, good reconstruction results are obtained using the approximate algorithm (Equation (16)).

5.2. Experimental Data

Experimental projection data of a physical Defrise phantom and a modified Rando phantom (The Phantom Laboratory, NY) from the CC source trajectory and a single arc source trajectory were acquired using a GE INNOVA 4100 clinical system.^{4, 5} This modified Rando phantom was constructed by inserting two modules of the Catphan (The Phantom Laboratory, NY) into the head portion of the Rando phantom to make a long object. The cone-beam projection data through the Defrise phantom has no data truncation. For the modified Rando phantom, longitudinal data truncation is present in the acquired data. The experimental parameters for each CC scan are: 70kVp, 100mA, 5ms pulse width, 0.3mm Cu added filtration, and a 12:1 anti-scatter grid. For the single arc scan, in order to maintain similar entrance dose levels, the tube current was doubled and other parameters were kept constant. Using experimentally measured projection matrices, calibration information was incorporated into the reconstruction.⁵ The parameters used in the reconstruction are given in the following: gantry radius $=726\text{mm}$, source to detector distance $=1200\text{mm}$, $\mu_1, \mu_2 = 15^\circ, -15^\circ$, detector pixel size $=0.8\text{mm}$, detector matrix size $=426 \times 426$, number of views for each arc $=420$, scan range for each arc $=[-27^\circ, 181^\circ]$, image matrix $=256 \times 256 \times 256$, reconstruction volume $=200 \times 200 \times 200\text{mm}^3$.

Figure 7 shows the reconstructed yz slice of the Defrise phantom using the CC-scan with the proposed algorithms given by Equations (12) and (16), and using a single-arc-scan with the FDK algorithm.³⁵ The reconstructed image using an exact algorithm is free of cone-beam artifacts. While good image quality is also achieved using the approximate algorithm.

Figure 8(a) shows the reconstructed slices of the Rando phantom using the CC-scan with the proposed algorithms given by Equation (12) and Equation (16). Since the projection data is longitudinally truncated, severe artifacts are present in the images reconstructed using formula (12) (indicated by white arrows in the left column of the Figure 8(a)). Figure 8(b) shows images reconstructed using the CC-scan with the approximate algorithm (Equation (16)) and a single-arc-scan with the FDK algorithm. Cone-beam artifacts have been

suppressed as indicated by the white arrows in the Figure 8(b). The air gaps between the adjacent slices of the Rando phantom are clearly defined in the images (left in Figure 8(b)) reconstructed using formula (16), while the slices of the phantom blend together in the reconstruction using an FDK algorithm and a single arc scan. Note that the entrance dose level is maintained to be similar by reduce the tube current for the CC-scan to be half of that for the single arc scan.

6. CONCLUSION

In this paper, shift-invariant FBP type cone-beam image reconstruction algorithms have been developed for the source trajectory of two concentric arcs. This trajectory can be practically implemented using a clinical C-arm system. This trajectory is complete in the sense that the Tuy data sufficiency condition is fulfilled. However, DM-lines do not exist for image points within a large volume. Thus, DM-line based algorithms can not be used in this case.

The algorithms derived here are based upon the Pack-Noo image reconstruction framework and are independent of the existence of DM-lines. Formula (12) can be used to exactly reconstruct a short object where there is no truncation (both longitudinally and transversely) in the projection data. For a long object, an approximate algorithm has also been developed given by Equation (16) based upon formula (12). Numerical simulations and experimental data have been provided to validate the new algorithms. The final selection of the image reconstruction algorithm depends upon the object to be scanned (short object or long-object) and the requirements on the reconstruction accuracy. Note that reconstructions performed on C-arm systems are subject to physical factors which degrade reconstruction accuracy such as scattered radiation, noise and the effects of a polychromatic X-ray spectrum. Thus, for C-arm CT a more pragmatic approach is presented here where we have relaxed the criteria of an exact reconstruction in favor of a more practical algorithm which solves the long object problem. The physical phantom experiments presented here demonstrate the utility of the approximate algorithm for C-arm CT.

The reconstruction formulae developed in this paper utilized a particular choice of filtering lines (Equation (13)). There exists flexibility to choose different filtering lines in the Pack-Noo reconstruction scheme.¹⁶ However, how to use this flexibility to design a mathematically exact algorithm which solves the long object problem is still an open question.

Acknowledgments

The work was partially supported by NIH grants 1R01 EB005712-02, 5T32CA009206-29, and a grant from GE Healthcare. We thank Alonso Gutierrez for his helping us obtain the physical phantoms scanned here. We would also like to thank our network administrator, Dr. Orhan Unal, for his support of our computer network.

APPENDIX A. RECONSTRUCTION FORMULA (EQUATION (12)) IN FLAT-PANEL DETECTOR COORDINATES

Using the third generation flat-panel detector data acquisition geometry, the following local coordinate system is introduced for source points at each arc, $i=1, 2$:

$$\begin{aligned}\hat{w}_i &= (-\cos \mu_i \cos t_i, -\sin t_i, -\sin \mu_i \cos t_i), \\ \hat{u}_i &= (\cos \mu_i \sin t_i, -\cos t_i, \sin \mu_i \sin t_i), \\ \hat{v}_i &= (-\sin \mu_i, 0, \cos \mu_i).\end{aligned}\quad (19)$$

The measured projection data labelled by the detector coordinates can be written as:

$$g_m(u_i, v_i, t_i) = g(\hat{r}, t_i), \quad (20)$$

$$\text{where, } \hat{r} = \frac{1}{\sqrt{D^2 + u_i^2 + v_i^2}} (u_i \hat{u}_i + v_i \hat{v}_i + D \hat{w}_i).$$

Using the detector coordinates, formula (12) may be rewritten as:

$$f(\vec{x}) = \frac{1}{4\pi^2} \left\{ \int_{t_C}^{t_A} dt_1 \frac{Q^1(u_\beta^1, v_\beta^1, t_1)}{L_1(\vec{x}, t_1)} + \int_{t_D}^{t_B} dt_2 \frac{Q^2(u_\beta^2, v_\beta^2, t_2)}{L_2(\vec{x}, t_2)} \right\}, \quad (21)$$

where,

$$Q^1(u_1, v_1, t_1) = Q_x^1(u_1, v_1, t_1) + Q_z^1(u_1, v_1, t_1), \quad Q^2(u_2, v_2, t_2) = Q_x^2(u_2, v_2, t_2) - Q_z^2(u_2, v_2, t_2), \quad (22)$$

$$u_\beta^i = D \frac{\hat{\beta} \cdot \hat{u}_i}{\hat{\beta} \cdot \hat{w}_i}, \quad v_\beta^i = D \frac{\hat{\beta} \cdot \hat{v}_i}{\hat{\beta} \cdot \hat{w}_i}, \quad L_i(\vec{x}, t_i) = |\vec{x} - \vec{y}(t_i)| \cdot \hat{w}_i, \quad (23)$$

where each filtered data

$$Q_n^i(u_i, v_i, t_i) = \int du h_H(u - u_i) g_d(u, v_n, t_i), \quad n=x, z, \quad (24)$$

is obtained by 1D Hilbert filtration of the modified projection data given by

$$g_d(u_i, v_i, t_i) = \frac{D}{\sqrt{D^2 + u_i^2 + v_i^2}} \left(\frac{D^2 + u_i^2}{D} \frac{\partial}{\partial u_i} + \frac{u_i v_i}{D} \frac{\partial}{\partial v_i} - \frac{\partial}{\partial t_i} \right) g_m(u_i, v_i, t_i). \quad (25)$$

along filtering lines given by

$$v_x = v_i + \frac{v_i - D \frac{\cot \mu_i}{\cos t_i}}{u_i + D \tan t_i} (u - u_i), \quad v_z = v_i + \frac{v_i + D \frac{\cot \mu_i}{\cos t_i}}{u_i + D \tan t_i} (u - u_i). \quad (26)$$

The 1D Hilbert filtering kernel is given by:

$$h_H(u) = \int_{-\frac{1}{2\Delta u}}^{\frac{1}{2\Delta u}} d\omega \quad [-i\pi \operatorname{sgn}(\omega)] e^{i2\pi\omega u}, \quad (27)$$

where, u is the sampling interval along the u -axis. The implementation steps of the formula (21) are similar to those given in.36, 37

APPENDIX B. EXPLICIT CALCULATION OF THE COORDINATE OF $E(\hat{e}_V, T)$ AND THE REGION $S(T)$, $S_1(T)$, AND $S_2(T)$

For the CC scanning geometry given by Equation (9) and a cylindrical object given by Equation (15), we will investigate the relative position of the point $E(\hat{e}_V, t_i)$ with $V = A, B, C, D$ for the unit vectors given by Equation (13) and the regions $S(t_i)$, $S_1(t_i)$, and $S_2(t_i)$.

The detector coordinates of the point $E(\hat{e}_V, t_i)$ are given by:

$$u_0^i(\hat{e}_V, t_i) = D \frac{\hat{e}_V \cdot \hat{u}_i}{\hat{e}_V \cdot \hat{w}_i}, \quad v_0^i(\hat{e}_V, t_i) = D \frac{\hat{e}_V \cdot \hat{v}_i}{\hat{e}_V \cdot \hat{w}_i}, \quad (28)$$

where D denotes the distance between the source and the detector. Substituting the explicit expressions of the unit vectors $\hat{e}_V$ in to above formula, we obtain:

$$u_0^i(\hat{e}_V, t_i) = -D \tan t_i, \quad V = A, B, C, D, \quad (29)$$

$$v_0^i(\hat{e}_A, t_i) = v_0^i(\hat{e}_C, t_i) = -D \frac{\cot \mu_i}{\cos t_i}, \quad v_0^i(\hat{e}_B, t_i) = v_0^i(\hat{e}_D, t_i) = D \frac{\tan \mu_i}{\cos t_i}. \quad (30)$$

The cone-beam projection of the cylindrical object denoted by $S(t)$ is bounded by two dashed lines as shown in Figure 9:

$$(\cos \eta \pm \cos \mu_i \sin t_i - \sin \eta \cos t_i)(u_i + D \tan t_i) - \cos \eta \pm \sin \mu_i \left(v_i + D \frac{\cot \mu_i}{\cos t_i} \right) = 0, \quad (31)$$

where

$$\eta \pm = \pm \arccos \frac{r}{R \sqrt{\sin^2 t_i + (\cos \mu_i \cos t_i)^2}} + \arctan \frac{\tan t_i}{\cos \mu_i}. \quad (32)$$

The u -coordinates of the points $B_{\pm}(t_i)$ which are the intersection between these two lines with the u -axis are given by:

$$u_B^i(t_i, \pm) = D \frac{r}{R \cos \eta \pm \cos \mu_i \sin t_i - \sin \eta \cos t_i}. \quad (33)$$

Since the backprojection range at each arc for a given image point $\vec{x} = (x, y, z)$ is given by (Equation (21)):

$$\arcsin \frac{y}{R} < t_i < \pi - \arcsin \frac{y}{R}, \quad (34)$$

the filtered data at view angle t_i will be backprojected into image space for image points with

$$y \leq R \sin t_i. \quad (35)$$

Using the detector coordinates, the above inequality can be rewritten as :

$$u_i \cos t_i + D \sin t_i \geq 0. \quad (36)$$

The backprojection region $BP(t_i)$ for a VOI is thus either to the left or to the right of the line given by

$$u_i = -D \tan t_i \quad (37)$$

depending on the sign of $\cos t_i$. This line also divides the region $S(t_i)$ into two sub-regions. The region which contains $BP(t_i)$ will be denoted by $S_1(t_i)$ and the other region by $S_2(t_i)$. An example of these regions on the detector plane is shown in Figure 9.

For the unit vectors $\hat{e}_V$ given by Equation (13), it is easy to see that all the fixed points $E(\hat{e}_V, t_i)$ are located on the line given by Equation (37). Especially, the points $E(\hat{e}_V, t_i)$ for $V = A, C$ are the intersections of two boundary lines of the cone-beam projection of the cylinder and the line given by Equation (37).

APPENDIX C. PARAMETERS FOR THE LONG SHEPP-LOGAN PHANTOM

The parameters used for the long Shepp-Logan phantom are summarized in the following table.

Table 1

The parameters for the long Shepp-Logan phantom.

Center (x, y, z)	Angle (deg)	Axis Lengths (a,b,c)	Density
(0, 0, 0)	0	(0.69, 0.92, 5.9)	2.0
(0, 0, 0)	0	(0.6624, 0.874, 5.88)	-0.98
(-0.22, 0, -0.5)	108	(0.41, 0.16, 1.21)	-0.02
(0.32, 0, -0.25)	72	(0.31, 0.11, 0.42)	-0.02
(0, 0.35, -0.25)	0	(0.21, 0.25, 0.50)	0.02
(0, 0.1, -0.25)	0	(0.046, 0.046, 0.046)	0.02
(-0.08, -0.65, -0.25)	0	(0.046, 0.023, 0.02)	0.01
(0.06, -0.650, -0.25)	90	(0.046, 0.023, 0.02)	0.01
(0.06, -0.105, 0.625)	90	(0.056, 0.040, 0.10)	0.02
(0, 0.1, 0.625)	0	(0.056, 0.056, 0.1)	-0.02

REFERENCES

1. Fahrig R, Holdsworth DW. Three-dimensional computed tomographic reconstruction using a C-arm mounted XR11: Image based correction of gantry motion nonlinearities. *Med. Phys.* 2000; 27:30–38. [PubMed: 10659735]
2. Siewerdsen JH, Moseley DJ, Burch S, Bisland SK, Bogaards A, Wilson BC, Jaffray DA. Volume CT with a flat-panel detector on a mobile, isocentric C-arm: Pre-clinical investigation in guidance of minimally invasive surgery. *Med. Phys.* 2005; 32:241–254. [PubMed: 15719975]
3. Lauritsch G, Boese J, Wigstrom L, Kemeth H, Fahrig R. Towards cardiac C-arm computed tomography. *IEEE Trans. Med. Imag.* 2006; 25:922–934.
4. Chen G-H, Zambelli J, Nett BE, Supanich M, Riddell C, Belanger B, Mistretta CA. Design and development of C-arm based cone-beam CT for image-guided interventions: Initial Results. *SPIE Proc. Med. Imag.* 2006; 614210(1-12)
5. Zambelli J, Nett BE, Leng S, Riddell C, Belanger B, Chen G-H. Novel C-arm based cone-beam CT using a source trajectory of two concentric arcs. *SPIE Proc. Med. Imag.* 2007; 6510(6510Q):1–10.
6. Tuy HK. An inversion formula for cone-beam reconstruction. *SIAM J. Appl. Math.* 1983; 43:546–552.
7. Defrise M, Clack R. A cone-beam reconstruction algorithm using shift-variant filtering and cone-beam backprojection. *IEEE Trans. Med. Imag.* 1994; 13:186–195.
8. Kudo H, Saito T. Derivation and implementation of a cone-beam reconstruction algorithm for nonplanar orbits. *IEEE Trans. Med. Imag.* 1994; 13:196–211.
9. Katsevich A. A general scheme for constructing inversion algorithms for cone beam CT. *Int. J. Math. Sci.* 2003; 21:1305–1321.
10. Chen G-H. An alternative derivation of Katsevich's cone-beam reconstruction formula. *Med. Phys.* 2003; 30:3217–3226. [PubMed: 14713088]
11. Chen, G-H.; Zhuang, T-L.; Leng, S.; Nett, BE. A showcase of exact cone-beam image reconstruction algorithms for circle-based trajectories. *Proceeding of the 8th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine; Salt Lake City, Utah.* 2005. p. 295-299.
12. Chen G-H, Zhuang T-L, Leng S, Nett BE. Cone-beam filtered backprojection image reconstruction using a factorized weighting function. *Opt. Eng.* 2007; 46:087006.
13. Bontus C, Kohler T, Proksa R. A quasixact reconstruction algorithm for helical CT using a 3-Pi acquisition. *Med. Phys.* 2003; 30:2493–2502. [PubMed: 14528971]
14. Ye YB, Wang G. Filtered backprojection formula for exact image reconstruction from cone-beam data along a general scanning curve. *Med. Phys.* 2005; 32:42–48. [PubMed: 15719953]
15. Pack JD, Noo F. Cone-beam reconstruction using 1D filtering along the projection of M-lines. *Inverse Problems.* 2005; 21:1105–1120.
16. Pack, JD.; Noo, F. Cone-beam reconstruction outside R-lines using the backprojection of 1-D filtered data. *Proceeding of the 8th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine; Salt Lake City, Utah.* 2005. p. 287-290.
17. Zou Y, Pan X. Exact image reconstruction on PI-lines from minimum data in helical cone-beam CT. *Phys. Med. Biol.* 2004; 49:941–959. [PubMed: 15104318]
18. Zou Y, Pan X. An extended data function and its generalized backprojection for image reconstruction in helical cone-beam CT. *Phys. Med. Biol.* 2004; 49:N383–N387. [PubMed: 15609569]
19. Zhuang T-L, Leng S, Nett BE, Chen G-H. Fan-beam and cone-beam image reconstruction via filtering the backprojection image of differentiated projection data. *Phys. Med. Biol.* 2004; 49:5489–5503. [PubMed: 15724538]
20. Pack JD, Noo F, Clackdoyle R. Cone-beam reconstruction using backprojection of locally-filtered projections. *IEEE Trans. Med. Imag.* 2005; 24:70–85.
21. Ye Y, Zhao S, Yu H, Wang G. A general exact reconstruction for cone-beam CT via backprojection-filtration. *IEEE Tran. Med. Imag.* 2005; 24:1190–1198.

22. Zhao S, Yu H, Wang G. A unified framework for exact cone-beam reconstruction formulas. *Med. Phys.* 2005; 32:1712–1721. [PubMed: 16013729]
23. Wang G, Ye Y, Yu H. Approximate and exact cone-beam reconstruction with standard and nonstandard spiral scanning. *Phys. Med. Biol.* 2007; 52:R1–R13. [PubMed: 17327647]
24. Noo F, Clack R, White TA, Roney TJ. The dual-ellipse cross vertex path for exact reconstruction of long objects in cone-beam tomography. *Phys. Med. Biol.* 1998; 43:797–810. [PubMed: 9572505]
25. Katsevich A. Image reconstruction for the circle-and-arc trajectory. *Phys. Med. Biol.* 2005; 50:2249–2265. [PubMed: 15876665]
26. Zou Y, Pan X, Sidky E. Theory and algorithms for image reconstruction on chords and within regions of interest. *J. Opt. Soc. Am.* 2005; 22:2372–2384.
27. Zeng K, Yu H, Fajardo LL, Wang G. Cone-beam mammo-computed tomography from data along two tilting arcs. *Med. Phys.* 2006; 33:3621–3633. [PubMed: 17089827]
28. Tang X, Ning R. A cone beam filtered backprojection (CB-FBP) reconstruction algorithm for a circle-plus-two-arc orbit. *Med. Phys.* 2001; 28:1042–1055. [PubMed: 11439474]
29. Zhuang T-L, Nett BE, Leng S, Chen G-H. A shift-invariant filtered backprojection (FBP) cone-beam reconstruction algorithm for the source trajectory of two concentric circles using an equal weighting scheme. *Phys. Med. Biol.* 2006; 51:3189–3210. [PubMed: 16757871]
30. Yang H, Li M, Koizumi K, Kudo H. Exact cone beam reconstruction for a saddle trajectory. *Phys. Med. Biol.* 2006; 51:1157–1172. [PubMed: 16481685]
31. Yang H, Li M, Koizumi K, Kudo H. View-independent reconstruction algorithms for cone beam CT with general saddle trajectory. *Phys. Med. Biol.* 2006; 51:3865–3884. [PubMed: 16861786]
32. Yang, H.; Li, M.; Koizumi, K.; Kudo, H. Application of Pack and Noo's cone-beam inversion formula to a wide class of trajectories. *IEEE Medical Imaging Conference Record*; San Diego, California. 2006. p. M14-450.
33. Cho, S.; Xia, D.; Pan, X. Exact image reconstruction in reverse helical cone-beam CT. *Proceeding of the Fully 3D Meeting and HPIR Workshop*; Lindau, Germany. 2007. p. 84-87.
34. Xia, D.; Cho, S.; Pan, X. Image reconstruction for a reduced scan in circular sinusoidal cone-beam CT. *Proceeding of the Fully 3D Meeting and HPIR Workshop*; Lindau, Germany. 2007. p. 213-216.
35. Feldkamp LA, Davis LC, Kress JW. Practical cone-beam algorithm. *J. Opt. Soc. Am.* 1984; A1:612–619.
36. Nett BE, Zhuang T-L, Leng S, Chen G-H. Arc based cone-beam reconstruction algorithm using an equal weighting scheme. *J. X-Ray Sci. and Tech.* 2007; 15:19–48.
37. Noo F, Pack J, Heuscher D. Exact helical reconstruction using native cone-beam geometries. *Phys. Med. Biol.* 2003; 48:3787–3818. [PubMed: 14703159]
38. Kak, AC.; Slaney, M. *Principles of Computerized Tomographic Imaging*. IEEE; Bellingham, WA: 1988.

Figure 1.
Demonstration of the volume within which DM-lines do not exist for an image point.

Figure 2.

(a) An example of the CC-geometry. (b) An example of the rectangle $ABCD$ which is parallel to the xz plane.

Figure 3.

(a) Demonstration of pattern of the filtering lines. (b) Demonstration of the regions $S(t)$, $S_1(t)$, $S_2(t)$, $P(t)$ and typical filtering lines on the detector plane. The region $S(t)$ is divided into $S_1(t)$ and $S_2(t)$ by the solid line. The filtering line drawn by dash-dotted line is completely inside the region $S_1(t)$ while dotted line is partially inside.

Figure 4.

An example of the choice of the detector point $E(\hat{e}, t)$ and the filtering line passing through that point.

Figure 5.

Reconstructed images of a high-contrast mathematical Defrise phantom at $y=0.2$ using CC-scan with formula (12) (center) and a single-arc-scan with FDK algorithm (right), comparing with the expected result (left).

Figure 6.

Reconstructed images for (a) a short Shepp-Logan phantom and, (b) a long Shepp-Logan phantom using the algorithm given by Equation (12)(central row of each figure) and the algorithm given by Equation (16)(bottom row of each figure) comparing with the phantom data (top row of each figure) at $z_0 = -0.25$ (left column of each figure), $y_0 = 0.2$ (central column of each figure), and $x_0 = -0.11$ (right column of each figure). The display window is $[0.95, 1.05]$ for all images.

Figure 7.

Reconstructed images of the physical Defrise phantom at $y=19.92$ mm using CC-scan with formula (12) (left), CC-scan with formula (16) (center), and a single-arc-scan with FDK algorithm (right).

Figure 8.

Reconstructed images of the Rando phantom at $y_0=23.05$ mm, (a) using the CC-scan with formula (12) (left) and formula (16) (right), (b) using the CC-scan with formula (16) (left) and using a single-arc-scan with FDK algorithm (right).

Figure 9.
Cone-beam projection of the cylindrical object on the detector plane.