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A stochastic adding machine and complex dynamics

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Abstract

This paper considers properties of a Markov chain on the natural numbers which models a binary adding machine in which there a non-zero probability of failure each time a register attempts to increment the succeeding register and resets. This chain has a family of natural quotient Markov chains, and extends naturally to a chain on the 2-adic integers. The transition operators of these chains have a self-similar structure, and have a spectrum which is, variously, the Julia set or filled Julia set of a quadratic map of the complex plane.

1. Introduction

In this paper we consider a family of discrete time Markov chains $\zeta(t)$ on the natural numbers which model a process of counting by a (binary) adding machine which evolves by addition of 1 upon each input. Stochasticity enters through fallibility of communication between one register and the next; each time a register rolls over there is a non-zero probability of failing to increment the next register. If the input derives from a pacemaker the models have applications in the theory of biological clocks [4,7–11]. This paper may be seen as a mathematical counterpoint to [8]. Models of biological clocks considered previously, such as the Poisson counter, failed to reproduce a key aspect of the phenomenology, which is that the standard deviation grows linearly with the mean (Weber's law). The stochastic adding machine model reproduces Weber's law as well as local deviations from it which are consistent with the data.

The Markov chain $\zeta(t)$ exhibits a number of fascinating structural properties, including a self-similarity of the transition matrix. A consequence of this self-similarity turns out to be an equivalence between a quadratic function of the transition matrix and the direct sum of the transition matrix with itself. The consequences of this relationship imply that the spectrum of the transition matrix is a Julia set. Although the probability theory, operator theory and dynamical systems theory in this paper each are used at a rather elementary level, the natural way in which they fit together in this problem provides answers to questions about the spectrum which would be more difficult to obtain by other means. A web search on the keywords 'fractal' and 'spectra' reveals that there is a considerable literature on self-adjoint operators, in particular Schrödinger operators, with fractal or Cantor spectrum. This situation is distinguished from the current case in that self-adjoint operators have a real spectrum. The spectral properties of the stochastic adding machine model are also quite different from those of many commonly

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Dedicated to the memory of Domingo Herrero, who thought about such things.

used stochastic models; for instance, a translation-invariant random walk on the integers has a spectrum which is a smooth curve in the complex plane.

2. Preview of results

In section 3 we detail the transition structure of the stochastic adding machine chain, and have as a consequence that the transition operator is double stochastic (i.e. the transposed matrix is also a Markov matrix) and that the chain is null recurrent. Another consequence is the following result regarding the self-similar structure of the allowed transitions of this chain.

Proposition 1

The transition probabilities $\mathbf{P}(\xi(t+1) = n | \xi(t) = m)$ satisfy the translation invariance $\mathbf{P}(\xi(t+1) = n | \xi(t) = m) = \mathbf{P}(\xi(t+1) = n + k2^i | \xi(t) = m + k2^i)$ for all k and i such that $n, m < 2^i$.

As a consequence of this proposition we obtain the unusual property that there is an infinite family of functions of this chain which are also Markov chains.

Corollary 2

The stochastic process $x_n(t) = \xi(t) \bmod 2^n$ is a Markov chain for all n .

Recall that the 2-adic topology on the natural numbers regards two natural numbers n, m as ‘close’ if their binary representations agree up to a large power of 2, i.e. if $n \equiv m \pmod{2^k}$ for k large. The completion of the natural numbers with respect to the 2-adic topology is $\mathbb{Z}_2$, the 2-adic integers; this is a compact topological space which is homeomorphic to the Cantor set, or equivalently to the set of infinite binary sequences with its product topology. As a consequence of corollary 2, we have

Corollary 3

The transition function extends in a natural way to a transition function on the 2-adic integers. For Markov chains obtained from this transition function, when the initial distribution is supported in the natural numbers, the distribution at all future times t is supported in the natural numbers and the Markov process obtained may be identified with $\xi(t)$.

These results are considered in section 4. In section 5 we discuss properties of the transition matrix as an operator on certain spaces of functions and measures, as well as relationships between various parts of the spectrum of the transition matrix on these spaces.

In section 6, building on the foundation laid in sections 3–5, we apply properties of quadratic maps of the complex plane to obtain a characterization of the spectrum of the transition operator. Recall that if $f(z)$ is a rational function on the complex plane, the Julia set of f is defined, for example, to be the closure of the set of unstable periodic points, while the filled Julia set is defined as the set of points with bounded orbit under f . In the case of quadratic maps, which are all conjugate to maps of the form $f_c(z) = z^2 + c$, there is a dichotomy: either the Julia set is connected or it is a Cantor set. When the Julia set is a Cantor set, the filled Julia set is always equal to the Julia set. When the Julia set is connected, the Julia set is the boundary of the filled Julia set; in this circumstance the Julia set is variously equal to, or strictly contained in, the filled Julia set, depending on the choice of c .

The set of values c for which the Julia set is connected is known as the Mandelbrot set. These ideas are pertinent to the study of the transition operator P due to the following result. Let p be the probability that a register succeeds in incrementing the next register as it rolls over.

Proposition 4

Let

$$f(z) = \frac{(z - (1 - p))^2}{p^2}.$$

Then the operator $f(P)$ is conjugate to $P \oplus P$

As it happens, the Julia set of $f(z)$ is connected for $p \geq \frac{1}{2}$, and is a Cantor set for $p < \frac{1}{2}$. We have:

Proposition 5

The spectrum of the transition operator is forward and backward invariant under f . The spectrum is contained in the filled Julia set, and contains the Julia set. On $\mathbb{C}(\mathbb{Z}_2)$ the spectrum is equal the Julia set of $f(z)$. On the $l^\infty(\mathbb{N})$, when $p > 0$ the spectrum of the transition operator is the filled Julia set, and this set consists of point spectrum.

To add intuition to these results, in a suitable Cartesian coordinate system the following image represents the Julia set of $f(z)$ when $p = 0.7$, and consists of the 2^9 points in the Julia set which are in the inverse image $f^{-9}(\{1\})$; 1 is a hyperbolic fixed point of f for all p .

3. Transition probabilities

The state of a binary adding machine may be described in terms of a left-infinite sequence of registers $\dots r_k, r_{k-1}, \dots, r_2, r_1, r_0$ in which each register takes the value 0 or 1, and at most a finite number of registers are non-zero. Such a sequence of registers uniquely represents a

natural number n according to the rule $n = \sum_{i=0}^{\infty} r_i 2^i$. To motivate a formal treatment of the operation of addition of 1 in a perfect adding machine, consider the process of addition of 1 to the number $215 = (11010111)_{bin}$. We have

$$\begin{array}{r} 1 \ 1 \ 0 \ 1 \ 1^0 \ 1^1 \ 1^1 \ 1^1 \\ \hline 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \end{array}$$

where we have added 1 to the rightmost 1 in $(11010111)_{bin}$, obtained $(10)_{bin}$, written 0 below the line and *carried* 1 to the next bit, denoted as a superscripted 1 to the left of the second bit, etc. We may abstract this process in terms of a system of evolution equations by introducing

an auxiliary binary ‘carry’ variable $c_k(t)$ for each register $r_k(t)$. In terms of these variables, the state evolves according to the rule,

$$\begin{aligned} r_k(t+1) &= r_k(t) + c_{k-1}(t+1) \bmod 2 \\ c_k(t+1) &= \lfloor \frac{r_k(t) + c_{k-1}(t+1)}{2} \rfloor \end{aligned} \quad (3.1)$$

for all $k \geq 0$. In this equation $\lfloor x \rfloor$ denotes the greatest integer less than or equal x , and we utilize the convention $c_{-1}(t) = 1$ for all t . Note that $c_k(t+1) = 0$ implies $c_j(t+1) = 0$ for all $j > k$ and that $c_k(t+1) = 1$ if and only if $r_j(t) = 1$ for all $j \leq k$. We may generalize this description to include fallible adding machines by requiring instead that the variables evolve according to

$$\begin{aligned} r_k(t+1) &= r_k(t) + e_k(t)c_{k-1}(t+1) \bmod 2 \\ c_k(t+1) &= \lfloor \frac{r_k(t) + e_k(t)c_{k-1}(t+1)}{2} \rfloor \end{aligned} \quad (3.2)$$

where $e_k(t)$ is an iid family of random variables parametrized by natural numbers k, t which each take the value 0 with probability $(1-p)$ and the value 1 with probability p . Again, $c_k(t+1) = 0$ implies $c_j(t+1) = 0$ for all $j > k$. In addition $c_k(t+1) = 1$ if and only if $r_j(t) = 1$ for all $j \leq k$, $c_j(t) = 1$ for all $j < k$ and $e_k(t) = 1$. Since these events imply that $e_j(t) = 1$ for $0 \leq j \leq k$, which occurs with probability p^{k+1} , it follows that a number having binary representation of

the form $\underbrace{11 \dots 1}_k$ transitions to $\underbrace{00 \dots 0}_k$ with probability p^{k+1} and a number having binary representation $\underbrace{11 \dots 1}_k$ transitions to $\underbrace{00 \dots 0}_k$ with probability $p^k(1-p)$. All transitions are of one of these two types. Note that for all $k \geq 0$, $c_k(t+1)$ is independent of $c_j(t)$ for all $j \geq 0$. It follows that the binary-sequence-valued process (t) process defined by the sequence of registers $\dots r_k(t), r_{k-1}(t), \dots, r_1(t), r_0(t)$ is Markov, for any choice of initial condition or initial distribution on the registers. Equivalently, we may consider $\xi(t)$ as the process evolving on the natural numbers given at each t by

$$\xi(t) = \sum_{i=0}^{\infty} r_i(t) 2^i \quad (3.3)$$

where the fact that there are only a finite number of non-zero terms implies convergence. Consequently, we have

Lemma 6

The process $\xi(t)$ has the following properties:

- From every integer n the probability of a transition to n is $(1-p)$.
- From every even integer $n = 2k$ the probability of a transition to $n+1$ is p .
- From every odd integer n , if k is the largest integer such that $n \equiv 2^k - 1 \pmod{2^k}$ then the transition probability to $n - 2^k + 1$ is $p^k(1-p)$ and the transition to $n+1$ has probability p^{k+1} .
- The transition graph and transition matrix P of $\xi(t)$ take the following forms:

$$\begin{pmatrix}
 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & \dots \\
 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & \dots \\
 p^2(1-p) & 0 & p(1-p) & 1-p & p^3 & 0 & 0 & 0 & \dots \\
 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & \dots \\
 0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0 & \dots \\
 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & \dots \\
 p^3(1-p) & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p & \ddots \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots
 \end{pmatrix}.$$

Note that both the column and row sums are equal to 1: thus by definition the transition matrix is doubly stochastic. This statement is equivalent to the statement that the counting measure on $\mathbb{N}$ is a stationary (non-normalized) distribution for this chain. Since the stochastic adding machine is manifestly irreducible and closed (e.g. [5]), it follows easily that the chain is null recurrent. Note also the evident self-similarity of the P in the $2^k \times 2^k$ diagonal blocks, which may be formalized in the following lemma.

Lemma 7

The transition probabilities $\mathbf{P}(\xi(t+1) = n | \xi(t) = m)$ satisfy the translation invariance $\mathbf{P}(\xi(t+1) = n | \xi(t) = m) = \mathbf{P}(\xi(t+1) = n + k2^i | \xi(t) = m + k2^i)$ for all $k \geq 0$ and i such that $n, m < 2^i$.

Proof

For natural numbers n (respectively, m) $< 2^i$ the natural numbers $n + k2^i$ (respectively, $m + k2^i$) for $k \geq 0$ have the same binary expansion up to the i th bit. Thus for all $k \geq 0$ the transition from $m + k2^i$ to $n + k2^i$ has the same probability as the transition from m to n .

4. Quotient and extension processes

Note that the transition matrix P is zero above the superdiagonal. This reflects that $\xi(t+1) \leq \xi(t) + 1$, i.e. $\xi(t)$ increases by at most one as t increases by one. Using lemmas 6 and 7, one may check that for all $k > 0$ the transition from $k2^i - 1$ into the set of natural numbers congruent to 0 mod 2^i occurs with the probability p^i . Thus, mod 2^i , there is an unambiguously defined transition from $2^i - 1$ to 0 with probability p^i . This fact and lemma 7 give us the following.

Lemma 8

Let $x_i(t) = \xi(t) \bmod 2^i$. Then

- $x_i(t)$ is a Markov chain on $\mathbb{Z}/(2^i \mathbb{Z})$, the integers mod 2^i .
- For every $0 < j < i$, $x_j(t) = x_i(t) \bmod 2^j$.
- The transition matrix P_i of $x_i(t)$ is given by the $2^i \times 2^i$ submatrix in the upper left-hand corner of P with the transition probability from $2^i - 1$ to 0 replaced by p^i .

For example, the transition matrix P_1 is the 2×2 matrix $\begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$, while P_3 is the 8×8 matrix

$$\begin{pmatrix} 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 \\ p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 \\ p^2(1-p) & 0 & p(1-p) & 1-p & p^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p \\ p^3 & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p \end{pmatrix}$$

which is equivalent to the transition graph

Recall that the 2-adic topology on $\mathbb{N}$ regards two natural numbers n, m as ‘close’ if $n \equiv m \pmod{2^k}$ for all k less than a large number l . The completion of $\mathbb{N}$ with respect to this topology is called the 2-adic integers, denoted by $\mathbb{Z}_2$. This completion may be identified with the set of all left-infinite binary sequences with its weak topology; this is a compact, uncountable space in which the natural numbers are identified with the dense subset consisting of binary sequences having at most a finite number of non-zero entries. The operation of addition extends in a natural way from $\mathbb{N}$ to $\mathbb{Z}_2$; the binary addition algorithm embodied by equation (3.2) defines an addition of pairs of left-infinite binary sequences. This addition is continuous in the 2-adic topology, the zero sequence is an additive identity in $\mathbb{Z}_2$, and it happens that the left-infinite sequence consisting of 1’s in all places is an additive inverse of 1. This is to say that $\mathbb{Z}_2$ is a topological group, and the inclusion map $\iota : \mathbb{N} \rightarrow \mathbb{Z}_2$ extends to an injective group homomorphism of $\mathbb{Z}$ into $\mathbb{Z}_2$.

Define the mapping $\tau : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ by

$$\tau(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Note that τ acts as the shift map on the binary representation of 2-adic numbers. Define two 2-adic numbers m and n to have the same tail if there is some power N such that $\tau^N(n) = \tau^N(m)$, or equivalently, if there are natural numbers k, l such that $n-k = m-l$. Note that the stochastic evolution given by equation (3.2), which was used to define the process $\zeta(t)$, does not require the assumption imposed there, that the initial string of registers $\dots, r_k(0), r_{k-1}(0), \dots, r_1(0), r_0(0)$ contain only a finite number of non-zero entries (although equation (3.3) does use this for convergence). The evolution is perfectly well defined on the set of all binary sequences, or equivalently on the 2-adic integers. Let $\zeta_2(t)$ denote this process for some choice of initial condition or distribution.

Lemma 9

- a. The transition probability $\mathbf{P}(\xi_2(t+1) = n | \xi_2(t) = m)$ is zero unless n, m have the same tail.
- b. For every choice of initial distribution on $\mathbb{Z}_2$, $\xi_2(t)$ is a Markov process on $\mathbb{Z}_2$.
- c. Suppose that the binary representation of m is $\dots, r_k, r_{k-1}, \dots, r_0$, and that $r_j = 1$ for $0 \leq j < k$ but that $r_k = 0$. Then the probability of the transition to $m+1$ is p^k , and the probabilities of the transitions to $m-2^i+1$ is $p^i(1-p)$ for $0 \leq i \leq k$; these probabilities add to 1. If m has no zero entries then m has the binary representation consisting of all 1's (i.e. is -1), and the transition to the sequence with k 0's followed on the left by an infinite string of 1's (i.e. -2^k) has probability $p^k(1-p)$ for $k \geq 0$; the sum of these probabilities is equal to 1.
- d. $\xi_2(t)$ and $\xi(t)$ are a.s. equal when $\mathbf{P}(\xi_2(t_0) \in \mathbb{N}) = 1$ and $\mathbf{P}(\xi_2(t_0) \in E) = \mathbf{P}(\xi(t_0) \in E)$ for every subset $E \subseteq \mathbb{N}$.

5. Function spaces and operators

Denote by $C(\mathbb{N})$ the set of bounded functions on $\mathbb{N}$ (i.e. bounded sequences) and by $C_0(\mathbb{N})$ the set $\{f \in C(\mathbb{N}) : \lim_{n \rightarrow \infty} f(n) = 0\}$. Endow $\mathbb{N}$ with the discrete topology; note that in this case $C(\mathbb{N})$ is the Banach space of continuous functions under the sup norm, and $C_0(\mathbb{N})$ is the closed subspace of continuous functions ‘vanishing at infinity’. Denote the Banach space of bounded Borel measures on $\mathbb{N}$ by $B(\mathbb{N})$ with its total variation norm; this linear space may be identified with the space of summable sequences and also with the space of continuous linear functionals on $C_0(\mathbb{N})$. Represent functions on $\mathbb{N}$ as infinite column vectors, bounded Borel measures may be identified with summable row vectors; distributions are then identified with non-negative row vectors which sum to one. Then the transition matrix P acts by right multiplication on $C(\mathbb{N})$ and by left multiplication on $B(\mathbb{N})$.

The probability of transition from a given number $m > 2^k$ to a number $n < 2^{k-1}$ is less than or equal to $p^k(1-p)$; since this goes to zero as $k \rightarrow \infty$, $C_0(\mathbb{N})$ is invariant under the action of P on $C(\mathbb{N})$. Since $B(\mathbb{N})$ is the dual space $C_0(\mathbb{N})$, the left action of P on $B(\mathbb{N})$ may be identified as the dual of the right action on $C_0(\mathbb{N})$. It follows from the Markov property that the norm of P as a linear operator on any of $B(\mathbb{N})$, $C(\mathbb{N})$, $C_0(\mathbb{N})$ is one. Since $B(\mathbb{N})$ is also $l^1(\mathbb{N})$, its dual space is $l^\infty(\mathbb{N})$ which is also $C(\mathbb{N})$. It follows from this that the spectrum of P on $C(\mathbb{N})$ and $C_0(\mathbb{N})$ is identical, since the spectrum of the dual of an operator is the complex conjugate of the spectrum of the operator.

Let π_n denote the canonical projection $\pi_n(i) = i \bmod 2^n$ from $\mathbb{Z}$ onto $\mathbb{Z}/2^n$, for $m > n$ let $\pi_{n,m}$ denote the canonical projection from $\mathbb{Z}/2^m$ onto $\mathbb{Z}/2^n$, and let Π_n (respectively, $\Pi_{n,m}$) denote the induced map from distributions on $\mathbb{Z}$ (respectively, on $\mathbb{Z}/2^m$) to distributions on $\mathbb{Z}/2^n$. Under the representation of distributions as row vectors Π_n may be represented as the left action of an $\infty \times 2^n$ matrix with entries $(\Pi_n)_{ij} = 1$ when $i = j \bmod 2^n$ and $(\Pi_n)_{ij} = 0$ otherwise. The dual action by right multiplication on column vectors maps $C(\mathbb{Z}/2^n)$ into $C(\mathbb{N})$ by $(\Pi_n f)(i) = f(i \bmod 2^n)$. We may rephrase lemma 8, items (a) and (b) as the equations

$$\begin{aligned}\Pi_n P_n &= P \Pi_n \\ \Pi_{n,m} P_n &= P_m \Pi_{n,m}.\end{aligned}\tag{5.1}$$

Since $\Pi_n^T \Pi_n$ and $\Pi_{n,m}^T \Pi_{n,m}$ are the $n \times n$ identity matrix, these imply the equations

$$\begin{aligned} P_n &= \Pi_n^T P \Pi_n \\ P_n &= \Pi_{n,m}^T P_m \Pi_{n,m}. \end{aligned} \quad (5.2)$$

As a consequence we have the following

Lemma 10

Let λ be an eigenvalue of P_n , corresponding to the right eigenvector u . Then λ is also an eigenvalue of P (respectively, P_m) and its corresponding right eigenvector is $\Pi_n u$ (respectively, $\Pi_{n,m} u$). Likewise, if μ is a left eigenvector of P (respectively, P_m) for the eigenvalue λ and if μ is not in the left kernel of Π_n (respectively, $\Pi_{n,m}$) then $\mu \Pi_n$ (respectively, $\mu \Pi_{n,m}$) is a left eigenvector of P_n and λ is its eigenvalue.

Proof

We have $\Pi_n P_n u = \lambda \Pi_n u = P \Pi_n u$ and $\mu \Pi_n P_n = \mu P \Pi_n = \lambda \mu \Pi_n$

Let $P^{(2)}$ denote the transition operator on $\mathbb{Z}_2$. Note that continuous functions on $\mathbb{Z}_2$ restrict to continuous functions on $\mathbb{N}$. Since $\mathbb{N}$ is dense in $\mathbb{Z}_2$, the restriction operation is an injection, in fact an isometry, of $C(\mathbb{Z}_2)$ into $C(\mathbb{N})$; the image of $C(\mathbb{Z}_2)$ is a closed proper P -invariant subspace of $C(\mathbb{N})$. Likewise, the space $\mathbf{B}(\mathbb{Z}_2)$ of Borel measures on $\mathbb{Z}_2$ contains $\mathbf{B}(\mathbb{N})$ as a norm closed proper (dual) $P^{(2)}$ -invariant subspace, which is also weak* dense. Recall that the point spectrum $\sigma_p(P)$ of P acting on a Banach space V consists of the set of eigenvalues of P , that the continuous spectrum $\sigma_c(P)$ consists of the set of complex numbers λ for which $(P - \lambda I)^{-1}$ exists as a densely defined closed unbounded operator, and that the residual spectrum $\sigma_r(P)$ is the point spectrum of the adjoint action of P on the dual Banach space V' of continuous linear functionals on V ; the spectrum of P on V is the union of these three sets (e.g. [12]). We have

Lemma 11

The point and continuous spectrum $\sigma_p(P) \cup \sigma_c(P)$ of the dual action of P on $\mathbf{B}(\mathbb{N})$ is contained in the point and continuous spectrum of $\sigma_p(P^{(2)}) \cup \sigma_c(P^{(2)})$ of the dual action of $P^{(2)}$ on $\mathbf{B}(\mathbb{Z}_2)$; the residual spectrum of P acting on $\mathbf{C}_0(\mathbb{N})$ is contained in the residual spectrum of $P^{(2)}$ acting on $\mathbf{C}(\mathbb{Z}_2)$. The point spectrum and continuous spectrum $\sigma_p(P^{(2)}) \cup \sigma_c(P^{(2)})$ of $P^{(2)}$ acting on $\mathbf{C}(\mathbb{Z}_2)$ is contained in the point and continuous spectrum of P acting on $\mathbf{C}(\mathbb{N})$.

Proof

Clearly, any right eigenvector of $P^{(2)}$ in $\mathbf{C}(\mathbb{Z}_2)$ is also a right eigenvector of P in $\mathbf{C}(\mathbb{N})$, and any left eigenvector of P in $\mathbf{B}(\mathbb{N})$ is a left eigenvector of $P^{(2)}$ in $\mathbf{B}(\mathbb{Z}_2)$. Likewise, if λ is in the continuous spectrum of $P^{(2)}$, there is a sequence of elements $\{u_n\}_{n=0}^\infty \subset \mathbf{C}(\mathbb{Z}_2)$ such that $(\lambda - P^{(2)}) u_n / \|u_n\| \rightarrow 0$ as $n \rightarrow \infty$. The same sequence, considered as a subset of $\mathbf{C}(\mathbb{N})$, suffices to show that λ is in the (point or continuous) spectrum of P on $\mathbf{C}(\mathbb{N})$. Similar reasoning shows that the continuous spectrum of P acting on $\mathbf{B}(\mathbb{N})$ is contained in the point or continuous spectrum of $P^{(2)}$ acting on $\mathbf{B}(\mathbb{Z}_2)$. Now the residual spectrum of P acting on $\mathbf{C}_0(\mathbb{N})$ is the complex conjugate of the point spectrum of the (dual) action of P on $\mathbf{B}(\mathbb{N})$. By the above, this is contained in the point spectrum of the (dual) action of $P^{(2)}$ on $\mathbf{B}(\mathbb{Z}_2)$, which is the residual spectrum of $P^{(2)}$ acting on $\mathbf{C}(\mathbb{Z}_2)$.

6. Complex dynamics and the spectrum

Note that for all $m \in \mathbb{Z}_2$ the probability $P(m|m) = 1 - p$. It follows that

$$\sum_{n \neq m} P(n|m) = p.$$

Denote by $\tilde{P}$ the operator $\frac{1}{p}(P - (1-p)I)$ or $\frac{1}{p}(P^{(2)} - (1-p)I)$; which operator is meant will be clear from the context.

Lemma 12

- $\tilde{P}$ is a Markov operator.
- The Markov chain defined by $\tilde{P}$ has period two. The two-step transition matrix $\tilde{P}^2$ has transitions only between numbers of the same parity, i.e. even goes to even and odd to odd.

Proof

By lemma 6 the diagonal elements of P (respectively, $P^{(2)}$) are all equal to $(1-p)$, so that the diagonal elements of $P - (1-p)I$ (respectively, $P^{(2)} - (1-p)I$) are all equal to zero and all row sums are equal to p . It follows that the row sums of $\tilde{P}$ are equal to one. Since the only transitions of P between numbers of the same parity are self-transitions from n to n , $\tilde{P}$ has no transitions between numbers of the same parity, and every odd number has probability $p(1-p)$ of a transition to the previous even number, the period of $\tilde{P}$ is two.

The transition graph corresponding to $\tilde{P}$ is the following:

It follows that $\tilde{P}^2$ has transitions only from even numbers to even numbers, and odd numbers to odd numbers. Note that, by following the above transition diagram through two steps, one obtains a transition diagram for $\tilde{P}^2$ which decomposes into two transition diagrams, one on the even natural numbers and one on the odd natural numbers, each isomorphic to the transition diagram of P . One may obtain a formal proof by letting $\tilde{\xi}_t$ denote a Markov chain (on $\mathbb{N}$ or on $\mathbb{Z}_2$) evolving according to the transition probabilities given by $\tilde{P}$. We may describe $\tilde{\xi}_t$ as the natural number valued (or 2-adic number valued) process which corresponds to the register process which evolves according to the stochastic evolution equation

$$\begin{aligned} \tilde{r}_k(t+1) &= \tilde{r}_k(t) + \tilde{e}_k(t) \tilde{c}_{k-1}(t+1) \bmod 2 \\ \tilde{c}_k(t+1) &= \lfloor \frac{\tilde{r}_k(t) + \tilde{e}_k(t) \tilde{c}_{k-1}(t+1)}{2} \rfloor \end{aligned} \quad (6.1)$$

where for $k > 0$, $t \geq 0$, exactly as before $\tilde{e}_k(t)$ is an iid family of random variables which takes the value 0 with probability $(1-p)$ and the value 1 with probability p , but now for all $t \geq 0$, $\tilde{e}_0(t) = 1$ with probability 1.

Lemma 13

Let $\zeta_t = \tilde{\xi}_{2t}$ for some choice of initial distribution; note that this is a Markov process. Then, for all natural numbers j and k the transition probabilities $P(\zeta_t = 2k | \zeta_t = 2j)$ (respectively, $P(\zeta_t = 2k+1 | \zeta_t = 2j+1)$) are equal to the transition probabilities of $\tilde{\xi}_t$, $P(\tilde{\xi}_t = k | \tilde{\xi}_t = j)$.

Proof

An even number corresponds to the register $r_0 = 0$. The two-parameter process $(\tilde{r}_k(t), \tilde{c}_k(t))$ evolves exactly as does $(r_k(t), c_k(t))$ for $k > 0$, and for $k = 0$ evolves in a deterministic fashion, attempting to increment the register $\tilde{r}_1$ on every second increment of t . However, this is exactly what happens to $r_0(t)$ on every increment of t . Hence on every second increment of t , the evolution of the left-infinite sequence $\dots, \tilde{r}_k(t), \tilde{r}_{k-1}(t), \dots, \tilde{r}_2(t), \tilde{r}_1(t)$ is the same as the corresponding evolution of $\dots, r_k(t), r_{k-1}(t), \dots, r_1(t), r_0(t)$.

Thus, when the initial distribution of ζ_t lies in the even (respectively, odd) natural numbers, the transitions from $m = 2k$ (respectively, $m = 2k + 1$) to $n = 2j$ (respectively, $n = 2j + 1$) occurs with the probability P_{kj} . Note that the transition matrix of ζ_t is $\tilde{P}^2$.

Corollary 14

The transition matrix $\tilde{P}^2$ may be identified with the direct sum of two copies of the transition matrix P , one copy acting on the continuous functions on the even numbers, the other acting on functions on the odd numbers.

Proof

The direct sum corresponds to the splitting of $\mathbf{B}(\mathbb{N})$, $\mathbf{C}(\mathbb{N})$ or (respectively, $\mathbf{C}(\mathbb{Z}_2)$) into the direct sum of subspaces supported on the even and odd natural (respectively, 2-adic) numbers.

Definition 15

Let E (respectively, O) denote the operator that maps a real-valued function $h(n)$ on the natural or 2-adic numbers into

$$(Eh)(n) = \begin{cases} h(n/2) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

respectively,

$$(Oh)(n) = \begin{cases} h([n-1]/2) & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

note that both E and O are isometries of $\mathbf{C}_0(\mathbb{N})$, (respectively, $\mathbf{C}(\mathbb{N})$, respectively, $\mathbf{C}(\mathbb{Z}_2)$) into itself.

Lemma 16

Suppose $v \in \mathbf{C}_0(\mathbb{N})$ (respectively $\mathbf{C}(\mathbb{N})$, respectively, $\mathbf{C}(\mathbb{Z}_2)$). Then $\tilde{P} Ov = Ev$ and $\tilde{P} Ev = Ov$. In consequence, if v is an eigenvector, $Pv = \lambda v$, then $\tilde{P} Ev = (p\lambda + (1-p))Ov$, so that $\pm \sqrt{p\lambda + (1-p)}$ is also eigenvalue of $\tilde{P}$ with corresponding eigenvector $v_{\pm} = Ev \pm \sqrt{p\lambda + (1-p)} Ov$. More generally, if $\lambda \in \sigma(\tilde{P})$ then $\pm \sqrt{p\lambda + (1-p)} \in \sigma(\tilde{P})$.

Proof

For all bounded functions h ,

$$(\tilde{P} h)(i) = \sum_{j=0}^{\infty} (\tilde{P})_{ij} h(j).$$

When i is even, $(\tilde{P})_{i,i+1} = 1$ so $\tilde{P} Ov = Ev$. Thus $\tilde{P}^2 Ov = \tilde{P} Ev$. By corollary 14, $\tilde{P}^2 Ov = O Pv$, so $\tilde{P} Ev = O Pv$. If λ is an eigenvalue, the two-dimensional subspace spanned by Ev and

Ov is an invariant subspace on which the matrix of $\tilde{P}$ is $\begin{pmatrix} 0 & 1 \\ \delta & 0 \end{pmatrix}$; one may easily compute that the eigenvalues of this matrix are $\pm \sqrt{\delta}$ and the corresponding eigenvectors are

$v_{\pm} = Ev \pm \sqrt{(p\lambda + (1-p))} Ov$. In terms of the direct sum decomposition $C_0(\mathbb{N}) = EC_0(\mathbb{N}) \oplus OC_0(\mathbb{N})$ (respectively, $C(\mathbb{N}) = EC(\mathbb{N}) \oplus OC(\mathbb{N})$, respectively, $C(Z_2) = EC(Z_2) \oplus OC(Z_2)$),

in view of the above, $\tilde{P}$ takes the form $\begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix}$, where I denotes the identity matrix. Thus, the

resolvent $(\tilde{P} - \lambda I)^{-1}$ takes the form $\begin{pmatrix} -\lambda I & I \\ P & -\lambda I \end{pmatrix}^{-1}$. However, one may compute that this is

equal $(\lambda^2 I - P)^{-1} \begin{pmatrix} -\lambda I & -I \\ -P & -\lambda I \end{pmatrix}$, so that spectrum of $\tilde{P}$ consists exactly of the square roots of the points in the spectrum of P , i.e. of the points $\pm \sqrt{(p\lambda + (1-p))}$ where λ is in the spectrum of P .

Lemma 17

The spectrum of P on $C(\mathbb{N})$ (respectively, $P^{(2)}$ on $C(Z_2)$) is strictly invariant under application of the self mapping of the complex plane $f: \mathbb{C} \rightarrow \mathbb{C}$ given by

$$f(z) = \frac{(z - (1-p))^2}{p^2}.$$

In other words, a complex number $\lambda \in \sigma(P)$ iff there is $\alpha \in \sigma(P)$, such $f(\lambda) = \alpha$.

Proof

We have that

$$\frac{(P - (1-p)I)^2}{p^2}$$

respectively,

$$\frac{(P^{(2)} - (1-p)I)^2}{p^2}$$

is similar to $P \oplus P$ (respectively, $P^{(2)} \oplus P^{(2)}$), and hence has the same spectrum as P (respectively, $P^{(2)}$) (although with doubled multiplicity). However, the spectral mapping theorem [3] asserts that this is the image under f of the spectrum of P (respectively, $P^{(2)}$), i.e.

$f(\sigma(P)) = \sigma(P)$ (respectively, $f(\sigma(P^{(2)})) = \sigma(P^{(2)})$). Thus the spectrum of P is forward invariant under f . On the other hand, by lemma 16, the spectrum of P (respectively, $P^{(2)}$) is also backward invariant under f .

The set of points with bounded forward orbits is called the filled Julia set; this set has the Julia set as its boundary.

Corollary 18

The spectrum of P (respectively, $P^{(2)}$) contains the Julia set of $f(z)$, and is contained in the filled Julia set.

Proof

By the Markov property, the operator P (respectively, $P^{(2)}$) is a contraction, hence has spectral radius one. However, by lemma 17, any point of the spectrum in the stable set of ∞ has an orbit under iteration by f which converges to infinity and which is contained in the spectrum. This contradicts the bound on the spectral radius. On the other hand, 1 is always an eigenvalue of a Markov operator. The fixed points of f are the two points 1 and $(1-p)^2$, the former hyperbolic and hence in the Julia set. The inverse images of 1 are dense in the Julia set [6], and by lemma 17, contained in the spectrum of P (respectively, $P^{(2)}$). Since the spectrum is closed, it must contain the Julia set.

Thus, an issue remaining to be resolved is how much of the interior of the filled Julia set is in the spectrum.

Corollary 19

The spectrum of P (respectively, $P^{(2)}$) is a compact forward- and backward-invariant subset of the filled Julia set which contains the Julia set.

For a quadratic map, the filled Julia set, and also the Julia set, is either connected or a Cantor set [1]. When the Julia set is connected, the stable set of ∞ is the complement of the Julia set. In this regard, note also that under the linear map $h(z) = p^2z + (1-p)$, $f(z)$ is conjugate to the quadratic map $z^2 - \frac{1-p}{p^2}$. As p decreases from 1 to 0, $-\frac{1-p}{p^2}$ decreases from 0 to $-\infty$, and takes the value -2 when $p = \frac{1}{2}$. When the Julia set is not connected, it is a Cantor set, and is equal to the filled Julia set; in this case the stable set of infinity contains all points not in the Julia set [1].

Corollary 20

The spectrum of P is a Cantor set for $p < \frac{1}{2}$.

Lemma 21

The spectrum of P_n is contained in the Julia set and is equal to the set of points in the complex plane which are mapped into 1 by f^n .

Proof

Define the matrix

$$\tilde{P}_n = \frac{P_n - (1-p)I}{p}.$$

Define the operators E_n and O_n as operators from $C(\mathbb{Z}/2^n)$ to $C(\mathbb{Z}/2^{n+1})$ in analogy with definition 15; E_n (respectively, O_n) maps functions on $\mathbb{Z}/2^n$ functions on $\mathbb{Z}/2^{n+1}$ supported on the even (respectively, odd) numbers mod 2^{n+1} . In analogy with lemmas 12, 13, 17 and

corollary 14, $\tilde{P}_n^2$ is isomorphic to $\tilde{P}_{n-1} \oplus \tilde{P}_{n-1}$, $\Pi_n E_{n-1} = E \Pi_{n-1}$ and $\Pi_n O_{n-1} = O \Pi_{n-1}$, $\tilde{P}_n O_{n-1} = E_{n-1}$ and $\tilde{P}_n E_{n-1} = P_n O_{n-1}$. One concludes that the spectrum of P_n , $\sigma(P_n)$, is forward invariant under f , contains the eigenvalue 1, and is equal to $f^{-1}(\sigma(P_{n-1}))$. From this, $\sigma(P_n)$ is equal to $f^{-n}(1)$; in particular the spectrum has 2^n distinct eigenvalues.

Note that $\mathbb{N}$ is dense in $\mathbb{Z}_2$ and the map $\pi_n : \mathbb{N} \rightarrow \mathbb{Z}/2^n$ is continuous in the 2-adic topology, hence extends by continuity to a map $\pi_n^{(2)} : \mathbb{Z}_2 \rightarrow \mathbb{Z}/2^n$. Denote the natural pullback operation by $\Pi_n^{(2)}$; this is the natural inclusion of $C(\mathbb{Z}/2^n)$ into $C(\mathbb{Z}_2)$.

Proposition 22

The spectrum of $P^{(2)}$ on $C(\mathbb{Z}_2)$ is equal to the Julia set.

Proof

The linear manifold of functions $L + \{h = \Pi_n^{(2)} h_n : h_n \in C(\mathbb{Z}/2^n), n \geq 1\}$ is dense in $C(\mathbb{Z}_2)$. Thus for every λ not in the Julia set of f , there exists a $c > 0$ such that the $\|(P^{(2)} - \lambda I)h\| > c$ for all $h \in L$; and hence for all $h \in C(\mathbb{Z}_2)$. Thus λ is not in the point or continuous spectrum of $P^{(2)}$ on $C(\mathbb{Z}_2)$. By the same token, if μ is an eigenvector of the dual operator of $P^{(2)}$ in $B(\mathbb{Z}_2)$, i.e. a left eigenvector of $P^{(2)}$, then μ cannot be in the left null space of $\Pi_n^{(2)}$ for all $n > 0$, else $\mu L = 0$, so that L is not dense. Thus there is an $n \geq 1$ such that $\mu \Pi_n^{(2)}$ is a left eigenvector of P_n , so that the conjugate of the corresponding left eigenvalue is in the Julia set.

Corollary 23

For $1 \geq p \geq \frac{1}{2}$ the spectrum of $P^{(2)}$ is connected.

Lemma 24

A complex number λ in the filled Julia set of f is an eigenvalue of P acting on $C(\mathbb{N})$ if the orbit of λ under f is contained in the closed disc D of radius p centred at $1 - p$ for all n sufficiently large. Conversely, if the orbit is excluded from an open neighbourhood of D for an infinite number of n , then λ is not an eigenvalue.

Proof

From lemma 6 note that $P - \lambda I$ has just one entry above the diagonal in every row. Thus, as long as $p > 0$, the k th entry in the column vector equation $(P - \lambda I)v = 0$ may be interpreted as a linear equation in the variables $v_0, v_1, \dots, v_{k+1}$ in which the coefficient of the variable v_{k+1} is non-zero, which is the same as to say an $(n$ -dependent) homogeneous linear recursion for v_{k+1} in terms of $v_0, v_1, \dots, v_k$. Thus, given an initial condition v_0 there is a unique solution for the infinite column vector v ; this need not belong to $C(\mathbb{N})$. From the transition matrix, it is easy to see that the coefficients of the recursion for v_k are polynomials in $\frac{1}{p}$ and linear in λ . For instance, one may see from the transition matrix that

$$v_{2^n} = \frac{-1}{p^{n+1}} \left((1 - p - \lambda)v_{2^n-1} + \left(\sum_{i=1}^{n-1} p^i (1 - p)v_{2^n-2^i} \right) + p^n (1 - p)v_0 \right). \quad (6.2)$$

For the general index k , iterating the recursion shows that $v_k = q_k(z, \lambda) v_0$, where $z = \frac{1}{p}$ and $q_k(z, \lambda)$ is a (in general, rather complicated) polynomial in z and λ . However, from the self-similar structure of the transition matrix, the transition probability from i to j for $2^{n-1} \leq i, j < 2^n$ is equal the probability from $i - 2^{n-1}$ to $j - 2^{n-1}$, so that $v_{2^n} = z q_{2^n-1}(z, \lambda) v_{2^n-1} - (z - 1) v_0$, so that

$$q_{2^n}(z, \lambda) = z q_{2^n-1}(z, \lambda)^2 - (z - 1). \quad (6.3)$$

This is to say that q_{2^n} evolves with n according to the quadratic map

$$g_p(q) = \frac{q^2}{p} - \frac{1-p}{p}$$

with the initial condition

$$q_1 = q_{2^0} = -\frac{1-\lambda-p}{p}.$$

The map $g_p(q)$ is conjugate to $f(z)$ by the map

$$h(q) = \frac{q}{p} - \frac{1-p}{p}; h^{-1}(g_p(h(z))) = f(z).$$

In this case the initial condition

$$q_{2^0} = -\frac{1-\lambda-p}{p}$$

amounts to the initial condition $z_0 = h^{-1}(q_{2^0}) = \lambda$ for $f(z)$. It follows that the sequence $n \mapsto q_{2^n}$ is bounded if and only if λ belongs to the filled Julia set of f . Now, by the self-similarity of the transition matrix as described in lemma 7, the values of v_{k+2^n} are in the same proportionality to v_{2^n} as the values of v_k are to v_0 , for $0 < k < 2^n$; i.e. $q_{2^n+k} = q_{2^n} q_k$. Thus, if $|q_{2^n}(z, \lambda)| \leq 1$ for all n sufficiently large, the inequality $|v_{k+2^n}| \leq |v_k|$ is valid for $0 < k < 2^n$, so that $\max_{k \leq 2^{n+1}} |v_k| \leq \max_{k \leq 2^n} |v_k|$. In particular, there is a uniform bound on $k \mapsto |v_k|$, so that v is an eigenvector of P in $C(\mathbb{N})$ and λ is its eigenvalue. However, by the conjugacy with $f(z)$, $|q_{2^n}(z, \lambda)| \leq 1$ for all n sufficiently large if and only if the orbit $n \mapsto f^n(\lambda)$ is contained in the p times the unit disc centred at $\frac{1-p}{p}$, or, in other words, the disc of radius p centred at $1-p$, for all n sufficiently large. Suppose, on the other hand, that $|q_{2^n}(z, \lambda)| > C > 1$ for an infinite number of n . It follows that the inequality $\max_{k \leq 2^{n+1}} |v_k| > C \max_{k \leq 2^n} |v_k|$ is valid for an infinite number of n , hence that v is unbounded, hence that λ is not an eigenvalue of P .

Lemma 25

The closed disc of radius p centred at $1-p$ contains the Julia set of f .

Proof

As in the proof of lemma 24, let

$$g_p(q) = \frac{q^2}{p} - \frac{1-p}{p}.$$

Suppose that $|z| > 1$; we estimate $|g_p(z)|$. Suppose that $z = x + iy$. Then,

$$\begin{aligned} |g_p(z)| &= \frac{\sqrt{(x^2 - y^2 - (1-p))^2 + (2xy)^2}}{p} \\ &= \frac{\sqrt{x^4 + 2x^2y^2 + y^4 - 2(1-p)(x^2 - y^2) + (1-p)^2}}{p}. \end{aligned}$$

Thus, since $|x^2 - y^2| \leq (x^2 + y^2)$, the value of the square root is greater than or equal to $|x^2 + y^2 - (1-p)|$. It follows that

$$\begin{aligned} |g_p(z)| &\geq \frac{x^2 + y^2 - (1-p)}{p} \\ &= \frac{(|z|^2 - 1) + (1 - (1-p))}{p} \\ &= \frac{|z|^2 - 1}{p} + 1 \\ &= \frac{|z| + 1}{p} (|z| - 1) + 1 \\ &> \frac{2}{p} (|z| - 1) + 1. \end{aligned}$$

Thus, when the initial condition satisfies $|z_0| > 1$, it follows that $|z_{k+1}| - 1 > \frac{2}{p} (|z_k| - 1)$, so that the orbit of z_0 is unbounded, hence is in the complement of the filled Julia set of g_p . As in the proof of lemma 24, h^{-1} applied to the unit disc is the disc of radius p centred at $(1-p)$, and f is conjugate to g_p by h .

Lemmas 24 and 25 imply the following.

Corollary 26

For $p > \frac{1}{2}$, the point spectrum of P on $\mathbf{C}(\mathbb{N})$ is equal to the filled Julia set.

Proof

This result is immediate from lemmas 24 and 25

We remark that when $p = 1$, P amounts to the left shift map, and the point spectrum in this case is well known to contain the interior of the unit disc. However, for any $p < 1$ the difference between P and the left shift is a bounded, but non-compact operator; thus establishing the nature of the spectrum of P when $p < 1$ seems not to be accessible by perturbation theory from the spectrum of the left shift.

Corollary 27

The spectrum of P on $\mathbf{C}(\mathbb{N})$ and $\mathbf{C}_0(\mathbb{N})$ is the filled Julia set, as is the spectrum of the left action of P on $\mathbf{B}(\mathbb{N})$.

Proof

For every bounded operator A , the spectrum of the dual operator A' is equal to the complex conjugate of the spectrum of A . From $B(\mathbb{N}) = C_0(\mathbb{N})'$, $C(\mathbb{N}) = B(\mathbb{N})' = C_0(\mathbb{N})''$ and since the real coefficients of the quadratic polynomial $f(z)$ imply that the Julia set is symmetric with respect to complex conjugation, corollary 26 implies the result.

7. Conclusions

We have investigated the properties of a Markov chain which represents a natural model in various applications. This process has a self-similar structure which reflects and idealizes counting systems with errors of this type. These structures are reflected in the spectral dynamics of the transition operator of the Markov chain, resulting in a description of the spectrum in terms of iterated polynomial maps of second order.

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References

1. Branner, B. The Mandelbrot set in chaos and fractals. In: Devaney, R.; Keen, L., editors. Proc Symp Pure and Applied Mathematics. Vol. 39. 1989. p. 75-105.
2. Brolin H. Invariant sets under iteration of rational functions. Ark Mat 1965;6:103-44.
3. Dunford, N.; Schwartz, JT. Linear Operators, Part I. New York: Interscience; 1958 1963.
4. Higa J. Interval timing: is there a clock? Special issues. Behavioral Processes 1998;44Higa J. Interval timing: is there a clock? Special issues. Behavioral Processes 1999;45
5. Hoel, P.; Port, S.; Stone, C. Introduction to Stochastic Processes. Boston, MA: Houghton-Mifflin; 1972.
6. Keen, L. Julia sets in chaos and fractals. In: Devaney, R.; Keen, L., editors. Proc Symp Pure and Applied Mathematics. Vol. 39. 1989. p. 57-74.
7. Killeen PR, Weiss N. Optimal timing and the Weber function. Psychol Rev 1987;94:455-68. [PubMed: 3317471]
8. Killeen PR, Taylor TJ. How the propagation of error through stochastic counters affects time discrimination and other psychophysical judgements. Psychol Rev 2000;107to appear
9. Ratcliff R, McKoon G. A counter model for implicit priming in perceptual word identification. Psychol Rev 1997;104:319-43. [PubMed: 9127584]
10. Roberts WA. Simultaneous numerical and temporal processing in the pigeon. Current Directions Psychol Sci 1995;4:47-51.
11. Schwarz W. A general counter model for refractory phenomena. J Math Psychol 1989;33:473-95.
12. Yosida, K. Functional Analysis. Berlin: Springer; 1980.