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Semi-orthogonal Frame Wavelets and Parseval Frame Wavelets Associated with GMRA

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Abstract

In this paper, we study semi-orthogonal frame wavelets and Parseval frame wavelets (PFWs) in $L^2(\mathbb{R}^d)$ with matrix dilations of form $(Df)(x) = \sqrt{2}f(Ax)$, where A is an arbitrary expanding $d \times d$ matrix with integer coefficients, such that $|\det A| = 2$. Firstly, we obtain a necessary and sufficient condition for a frame wavelet to be a semi-orthogonal frame wavelet. Secondly, we present a necessary condition for the semi-orthogonal frame wavelets. When the frame wavelets are the PFWs, we prove that all PFWs associated with generalized multiresolution analysis (GMRA) are equivalent to a closed subspace W_0 for which $\{T_k \psi : k \in \mathbb{Z}^d\}$ is a PFW. Finally, by showing the relation between principal shift invariant spaces and their bracket function, we discover a property of the PFWs associated with GMRA by the PFWs' minimal vector-filter. In each section, we construct concrete examples.

1. Introduction

The Exceptional Lie groups are applicable to nature because nature is essentially a Cantorian and hierarchal fractal with an octonionic connection. In particular, the Exceptional Lie groups hierarchy is at the very heart of the nature of high energy physical reality [1–3]. Recent research on the exceptional Lie symmetry groups has indicated the existence of tantalizing connections to E-infinity ($\epsilon^{(\infty)}$) [4–7]. From the outset, the Exceptional Lie groups have played a significant role in developing E-infinity theory [8]. The E-infinity view was introduced by Mohamed El Naschie to describe and understand nature because it appears clearly not continuous but self-similar [9–12]. Indeed, Mohamed El Naschie's papers on E-infinity appear to be a new framework. His view about space-time is an infinite dimensional fractal that happens to have $D=4$ as the expectation value for topological dimension. We are making our observation through the energy scale, so in the low energy resolution the topological value $3+1$ means the world appearing four-dimensional to us [13–15]. Wavelets are mathematical functions that take account into the resolutions and the frequencies simultaneously and the wavelet transform can make up for the Fourier's transform disadvantage. Since wavelets could cut up data into different frequency components, people can study each component with a resolution matched

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to its scale. Therefore, wavelets and multiresolution analysis(MRA) are good mathematical tools to support Mohamed El Naschie's picture of the resolution dependence of the observations[25–27].

The "classical" MRA wavelets are probably the most important class of orthonormal wavelets. But they do not produce all the orthogonal wavelets. The concept of frames was introduced a long time ago [16] and has received much attention recently due to the development and study of wavelet theory [17,18]. Frames can be regarded as the generalizations of orthonormal bases of Hilbert spaces. Inspired by these and other applications, many people explored in their articles the theory of frame wavelets. In particular, they are interested in PFWs and their construction via general multiresolution analysis (GMRA) because of the existence of fast implementation algorithm[19–22]. The importance of the concept of an GMRA lies in the fact that each orthonormal wavelet ψ is associated with an GMRA. However, there is an important difference between GMRA's and MRA's. An MRA always admits orthonormal wavelets, but an arbitrary GMRA may not admit associated wavelets. Furthermore, notice that any orthonormal basis in a Hilbert space is a PFW and a PFW is not necessarily an orthonormal basis, so a wavelet need not be associated with an MRA, but it can be made to be associated with an GMRA. Observe that these are new phenomenon in comparison with the MRA case.

In [22], the author investigated the PFWs and got an important result that all semi-orthogonal PFWs coincide with the classes of all GMRA PFWs. Since all GMRA wavelets are contained in the classes of all semi-orthogonal frame wavelets, in this paper, we firstly investigate the semi-orthogonal frame wavelets and show a necessary and sufficient condition for the frame wavelets to be the semi-orthogonal frame wavelets. Then we present a necessary condition for the semi-orthogonal frame wavelets. By focusing our attention to the relation between the PFWs and the GMRA, we derive a necessary and sufficient condition for the PFWs associated with GMRA. This is a different result in comparison with that in [22]. So it extends the theory of the PFWs associated with GMRA. Finally, by showing the property of the bracket function of orthogonal principal shift invariant spaces, we discover an important characteristic of the PFWs associated with GMRA by the PFWs' minimal vector-filter.

The paper is organized as follows. In Section 2 contains some facts about shift invariant spaces and various results on matrices belonging to the class $E_n^{(2)}$. Also, we review some relative notations and definitions. In the next section we prove a necessary and sufficient condition for the frame wavelets to be the semi-orthogonal frame wavelets. Then, we present a necessary condition for the semi-orthogonal frame wavelets. In section 4, we discuss the relation between the PFWs and the GMRA and get a necessary and sufficient condition for the PFWs associated with GMRA. Finally, we derive an important property of the PFWs associated with GMRA by the PFWs' minimal vector-filter. Corresponding examples are given to prove our theorem in each section.

2. Preliminary

Let us now establish some basic notations.

A vector s in R^d induces a unitary translation operator T_s defined by

$$(T_s f)(t) = f(t - s), \forall f \in L^2(R^d), t \in R^d. \quad (2.1)$$

In what follows, the system of all integer translates $\{T_k f : k \in \mathbb{Z}^d\}$ of a function $f \in L^2(R^d)$ will be denoted by $T(f)$.

Let $E_n^{(2)}$ denote the set of all expanding matrices A such that $|\det A| = 2$. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For $A \in E_n^{(2)}$, we denote a unitary operator D acting on $L^2(\mathbb{R}^d)$ defined by

$$(Df)(t) = |\det A|^{\frac{1}{2}} f(At), \quad \forall f \in L^2(\mathbb{R}^d), \quad t \in \mathbb{R}^d. \quad (2.2)$$

The operator D corresponding to a real expansive matrix A is called an A -dilation operator. Let B the transpose of A : $B = A^t$. It is obvious that $B \in E_n^{(2)}$.

If $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then we define its Fourier transform:

$$\widehat{f}(s) = \int_{\mathbb{R}^d} e^{-i2\pi\langle s, t \rangle} f(t) dt, \quad (2.3)$$

where $\langle s, t \rangle$ denotes the standard inner product in $\mathbb{R}^d$. So we have the relative notation:

$$\text{supp} f = \{s \in \mathbb{R}^d : \widehat{f}(s) \neq 0\}. \quad (2.4)$$

It is useful to note the following formulae for any $f \in L^2(\mathbb{R}^d)$:

$$\widehat{T_k f}(\xi) = e^{-2\pi i \langle k, \xi \rangle} \widehat{f}(\xi), \quad \forall k \in \mathbb{Z}^d, \quad (2.5)$$

$$\widehat{D^j f}(\xi) = 2^{-\frac{j}{2}} \widehat{f}(B^{-j} \xi), \quad \forall j \in \mathbb{Z} \quad (2.6)$$

and, in particular,

$$\widehat{D} f(\xi) = 2^{-\frac{1}{2}} \widehat{f}(B^{-1} \xi). \quad (2.7)$$

The following elementary Lemma [24, Lemma 2.2] provides us with a convenient description of $B\mathbb{Z}^n$ for an arbitrary $A \in E_n^{(2)}$, and it will be used in the section 4.

Lemma 2.1

Let $B \in E_n^{(2)}$ be any integer matrix such that $|\det B| = 2$. Then the group $\mathbb{Z}^n/B\mathbb{Z}^n$ is isomorphic to $B^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ and the order of $\mathbb{Z}^n/B\mathbb{Z}^n$ is equal to 2. In particular, if $\alpha \in \mathbb{Z}^n/B\mathbb{Z}^n$ and $\beta = B^{-1}\alpha$, then $\mathbb{Z}^n = B\mathbb{Z}^n \cup (B\mathbb{Z}^n + \alpha)$ and $B^{-1}\mathbb{Z}^n = \mathbb{Z}^n \cup (\mathbb{Z}^n + \beta)$.

Then we introduce some notations and definitions about frame wavelet and shift invariant subspace.

A collection of elements $\{\varphi_j : j \in J\}$ in a Hilbert space H is called a *frame* if there exist constants a and b , $0 < a \leq b < \infty$, such that

$$a \|f\|^2 \leq \sum_{j \in J} |\langle f, \varphi_j \rangle|^2 \leq b \|f\|^2, \quad \forall f \in H. \quad (2.8)$$

Let a_0 be the supremum of all such numbers a and b_0 be the infimum of all such numbers b , then a_0 and b_0 are called the frame bounds of the frame $\{\varphi_j : j \in J\}$. When $a_0 = b_0$ we say that the frame is tight. When $a_0 = b_0 = 1$ we say the frame is a Parseval frame.

We fix an arbitrary matrix $A \in E_d^{(2)}$. For a function $\psi \in L^2(\mathbb{R}^d)$, we will consider the affine system Ψ defined by

$$\Psi = \left\{ \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(A^j x - k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^d \right\}. \quad (2.9)$$

Definition 2.2

We say that a function $\psi \in L^2(\mathbb{R}^d)$ is a frame wavelet if the system (2.9) is a frame for $L^2(\mathbb{R}^d)$.

Definition 2.3

We say that a function $\psi \in L^2(\mathbb{R}^d)$ is a Parseval frame wavelet if the system (2.9) is a Parseval frame for $L^2(\mathbb{R}^d)$.

A frame wavelet is said to be semi-orthogonal if $D^{j_1} T_{k_1} \psi \perp D^{j_2} T_{k_2} \psi$ for all $j_1 \neq j_2$ and for all $k_1, k_2 \in \mathbb{Z}^d$.

Definition 2.4

A closed subspace $V \subset L^2(\mathbb{R}^d)$ is shift invariant if $f \in V$ implies $T_k f \in V$ for any $k \in \mathbb{Z}^d$.

For any subset $\mathfrak{R} \subset L^2(\mathbb{R}^d)$, let

$$S(\mathfrak{R}) = \overline{\text{span}}\{T_k f : f \in \mathfrak{R}, k \in \mathbb{Z}^d\} \quad (2.10)$$

be the shift invariant space generated by $\mathfrak{R}$. If $\mathfrak{R} = \{f\}$ we will also write $S(f) = S(\{f\})$. If $V = S(f)$ for some function f we say V is principal shift invariant (PSI). If $V = S(\mathfrak{R})$ for some finite $\mathfrak{R}$ we say V is finitely generated shift invariant (FSI). We will identify $T^d = \mathbb{R}^d / \mathbb{Z}^d$ with its fundamental domain; that is $T^d = [-\frac{1}{2}, \frac{1}{2}]^d$.

For $f, g \in L^2(\mathbb{R}^d)$, let bracket function $[f, g]$ be the function defined a.e. by

$$[f, g](\xi) = \sum_{k \in \mathbb{Z}^d} f(\xi + k) \overline{g(\xi + k)}. \quad \text{Notice that } [f, g] \in L^1(T^d), \text{ where } L^1(T^d) \text{ denotes the space of all integrable } \mathbb{Z}^d\text{-periodic functions on the } d\text{-dimensional torus } T^d = \mathbb{R}^d / \mathbb{Z}^d. \text{ In particular, we write } \sigma_f = [\hat{f}, \hat{f}]. \text{ For a function } f \in L^2(\mathbb{R}^d), \text{ we denote by } \langle f \rangle \text{ the minimal closed shift invariant subspace that contains } f. \text{ Notice that } \langle f \rangle = \overline{\text{span}} T(f). \text{ If the system } T(f) \text{ is a Parseval frame for } \langle f \rangle, \text{ we say that } \langle f \rangle \text{ is a principal shift invariant space.}$$

Recall that, for $\varphi \in L^2(\mathbb{R}^d)$, the system $T(\varphi)$ is a Parseval frame for the Hilbert space $\langle \varphi \rangle$ if and only if there exists a measurable $\mathbb{Z}^d$ -periodic set $\Omega \subseteq \mathbb{R}^d$ such that $\sigma_\varphi = \chi_\Omega$ a.e., where χ_Ω

denotes the characteristic function of the set Ω . In particular, if $T(\varphi)$ is an orthonormal basis for $\langle \varphi \rangle$, we have $\Omega = R^d$, up to a set of measure zero.

We turn to the concept of the GMRA which is an useful tool in our study.

Definition 2.5. [19]

Let $(V_j), j \in \mathbb{Z}$ be a sequence of closed subspaces of $L^2(R^d)$ satisfying:

1. $V_j \subset V_{j+1}$;
2. $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(R^d)$;
3. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
4. $f \in V_j \Leftrightarrow f(A \cdot) \in V_{j+1}, j \in \mathbb{Z}$;
5. The core space V_0 is shift invariant; i.e. $f \in V_0 \Rightarrow T_k f \in V_0, \forall k \in \mathbb{Z}^d$.

This definition extends naturally the concept of a classical MRA. Recall that a sequence V_j of closed subspace of $L^2(R^d)$ is called a MRA if (V_j) satisfies (1) – (5) from Definition 2.5 and, additionally, there exists a function $\varphi \in V_0$ such that the system $\{T_k \varphi : k \in \mathbb{Z}^d\}$ is an orthonormal basis for V_0 . If this is the case, the function φ is said to be a scaling function for (V_j) .

There is a standard procedure for constructing wavelets from a given GMRA (V_j) .

First, one define $W_j = V_{j+1} \ominus V_j$ for all $j \in \mathbb{Z}$. As an easy consequence of conditions (1) – (4) from Definition 2.5, one obtains $L^2(R^d) = \overline{\bigoplus_{j \in \mathbb{Z}} W_j}$ and $W_{j+1} = DW_j, \forall j \in \mathbb{Z}$. Suppose now that there exists a function $\psi \in W_0$ such that the system $T(\psi)$ is a Parseval frame for W_0 . Then $\{D^j T_k \psi : k \in \mathbb{Z}^d\}$ is a Parseval frame for $W_j, \forall j \in \mathbb{Z}$, and, consequently, $\{D^j T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a Parseval frame for $L^2(R^d)$.

From the construction explained above, we can get that each GMRA wavelet is necessarily semi-orthogonal immediately

Let us recall the definition of GMRA PFW and admissible GMRA.

Definition 2.6

A Parseval frame wavelet ψ is said to be associated with a GMRA (V_j) or GMRA PFW if $T(\psi)$ is a Parseval frame for $W_0 = V_1 \ominus V_0$.

Definition 2.7

A GMRA (V_j) is said to be admissible if there exist Parseval frame wavelets associated with (V_j) .

Then we recall a result from [20, Theorem 3.3] that provides us with the canonical decomposition of shift invariant spaces.

Suppose that $V \subseteq L^2(R^d)$ is a shift invariant space. Then there exist $n \in \mathbb{N} \cap \{\infty\}$ and a (possibly finite) sequence of functions $(\varphi_i), i = 1, 2, \dots$, with the following properties:

1. $V = \bigoplus_{i=1}^n \langle \varphi_i \rangle$;
2. $T(\varphi_i)$ is a Parseval frame for $\langle \varphi_i \rangle$ for all $i = 1, 2, \dots$;
3. $\Omega_1 \supseteq \Omega_2 \supseteq \dots \supseteq \Omega_i \supseteq \dots$, where $\sigma_{\Omega_i} = \chi_{\Omega_i}$ a.e., for all $i = 1, 2, \dots$.

From now on, we will always assume that the core space V_0 of any $\text{GMRA}(V_j)$ is decomposed as above and (φ_i) will be called the sequence of scaling functions of (V_j) . In some situations it will be convenient to specify the number scaling functions of (V_j) . When this is the case, we shall say that (V_j) is an n -GMRA. When (V_j) is an n -GMRA with $n \in \mathbb{N}$, we can write $\varphi_i = 0$ and $\Omega_i = \emptyset$, for all $i > n$. This will enable us to state some results without specifying the effective number of generators of V_0 .

3. Semi-orthogonal Frame Wavelets

The main purpose of this section is to study the semi-orthogonal frame wavelets. We firstly obtain a necessary and sufficient condition for the frame wavelets to be the semi-orthogonal frame wavelets. Then, we get a necessary condition for the semi-orthogonal frame wavelets.

Now we introduce the main results in this section:

Theorem 3.1

Suppose that $\Psi = \{\psi_{j,k}(x) | \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(A^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a frame wavelet for $L^2(\mathbb{R}^d)$ and $T(\psi)$ is the frame wavelet of a closed subspace $W_0 = \{f: \text{supp } \hat{f} \subset M\}$, where $\text{supp } \hat{\psi} = M$. Then Ψ is a semi-orthogonal frame wavelet for $L^2(\mathbb{R}^d)$ if and only if

$$\text{supp } \hat{\psi}_{j,0} \cap \text{supp } \hat{\psi}_{l,0} = \emptyset, \text{ for all } j, l \in \mathbb{Z}, j \neq l \quad (3.1)$$

holds.

Proof

Sufficiency: If (3.1) holds, by (2.5) and (2.6), for all $k, m \in \mathbb{Z}^d$, we have

$$\begin{aligned} 0 &= \langle \hat{\psi}_{j,0}, \hat{\psi}_{l,0} \rangle \\ &= \int_{\mathbb{R}^d} \hat{\psi}(B^{-j}\xi) \cdot \overline{\hat{\psi}(B^{-l}\xi)} d\xi \\ &= \int_{\mathbb{R}^d} e^{-2\pi i \langle B^{-j}\xi, k \rangle} \hat{\psi}(B^{-j}\xi) \cdot \overline{e^{-2\pi i \langle B^{-l}\xi, m \rangle} \hat{\psi}(B^{-l}\xi)} d\xi \\ &= \langle \hat{\psi}_{j,k}, \hat{\psi}_{l,m} \rangle \\ &= \langle \psi_{j,k}, \psi_{l,m} \rangle. \end{aligned} \quad (3.2)$$

Hence, by definition, Ψ is the semi-orthogonal frame wavelet of $L^2(\mathbb{R}^d)$. Sufficiency is proved.

Necessity: Suppose that Ψ is the semi-orthogonal frame wavelet of $L^2(\mathbb{R}^d)$ and $T(\psi)$ is the frame wavelet of a closed subspace $W_0 = \{f: \text{supp } \hat{f} \subset M\}$, where $\text{supp } \hat{\psi} = M$. Hence, it is not difficult to see that $T(\psi_{j,0})$ is the frame wavelet of the closed subspace $W_j = \{f: \text{supp } \hat{f} \subset B^j M\}$. If $\exists j, l \in \mathbb{Z}^d$ and $j \neq l$ such that $\text{supp } \hat{\psi}_{j,0} \cap \text{supp } \hat{\psi}_{l,0} \neq \emptyset$, then there must exist a function $0 \neq f \in L^2(\mathbb{R}^d)$ such that $\text{supp } \hat{f} \in \text{supp } \hat{\psi}_{j,0} \cap \text{supp } \hat{\psi}_{l,0}$. So

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{j,k} \rangle \psi_{j,k}, \quad (3.3)$$

and

$$f = \sum_{m \in \mathbb{Z}^d} \langle f, \psi_{l,m} \rangle \psi_{l,m}. \quad (3.4)$$

hold.

By (3.3) and (3.4), we have

$$\begin{aligned} f &= \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{j,k} \rangle \psi_{j,k} \\ &= \sum_{k \in \mathbb{Z}^d} \langle \sum_{m \in \mathbb{Z}^d} \langle f, \psi_{l,m} \rangle \psi_{l,m}, \psi_{j,k} \rangle \psi_{j,k} \\ &= \sum_{k \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \langle f, \psi_{l,m} \rangle \langle \psi_{l,m}, \psi_{j,k} \rangle \psi_{j,k} \\ &= 0. \end{aligned} \quad (3.5)$$

It contradicts $f \neq 0$, so (3.1) holds. Necessity is proved.

Theorem 3.2

Suppose that $\Psi = \{\psi_{j,k}(x) | \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(A^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is the frame wavelet of $L^2(\mathbb{R}^d)$. If Ψ is a semi-orthogonal frame wavelet for $L^2(\mathbb{R}^d)$, then there exists a closed subspace $W_0 \subset L^2(\mathbb{R}^d)$ such that

$$\{\psi_{0,k}(x) | \psi_{0,k}(x) = \psi(x - k), k \in \mathbb{Z}^d\} \quad (3.6)$$

is the frame wavelet of W_0 .

Proof

We assume that Ψ is a semi-orthogonal frame wavelet. Then

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{j,k} \rangle \psi_{j,k}, \forall f \in L^2(\mathbb{R}^d). \quad (3.7)$$

Denote $f = f_1 + f_2$, $\forall f \in L^2(\mathbb{R}^d)$, where $f_1 = \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{0,k} \rangle \psi_{0,k}$ and $f_2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{j,k} \rangle \psi_{j,k}$. Let $W_0 = \{f_1 : f_1 = f - f_2, \forall f \in L^2(\mathbb{R}^d)\}$, $\overline{W_0} = \{f_2 : f_2 = f - f_1, \forall f \in L^2(\mathbb{R}^d)\}$. By the definition of the semi-orthogonal frame wavelet, we have $|\langle \psi_{0,k}, \psi_{j,m} \rangle|^2 = 0, j \neq 0$, then $\langle f_1, f_2 \rangle = 0$. Again by the definition of W_0 and $\overline{W_0}$, we get $L^2(\mathbb{R}^d) = W_0 \oplus \overline{W_0}$.

Next we will prove that W_0 is a closed subspace of $L^2(\mathbb{R}^d)$. For all $\{f_i\}_{i \in \mathbb{N}} \subset W_0$ be the Cauchy sequence of W_0 , there must exist a function $f \in L^2(\mathbb{R}^d)$ such that $\lim_{i \rightarrow \infty} f_i = f$ under the $L^2(\mathbb{R}^d)$ norm. If $f \in W_0$, the subspace W_0 is closed. However, if $f \notin W_0$, then $f \in \overline{W_0}$. We can write $f = \sum_{j \in \mathbb{Z}, j \neq 0} \sum_{m \in \mathbb{Z}^d} \langle f, \psi_{j,m} \rangle \psi_{j,m}$. Notice $\lim_{i \rightarrow \infty} f_i = f$, so we have $f = \sum_{j \in \mathbb{Z}, j \neq 0} \sum_{m \in \mathbb{Z}^d} \langle \lim_{i \rightarrow \infty} f_i, \psi_{j,m} \rangle \psi_{j,m}$.

Since $f_i = \sum_{k \in \mathbb{Z}^d} \langle f_i, \psi_{0,k} \rangle \psi_{0,k}$, then

$$\begin{aligned}
 f &= \sum_{j \in \mathbb{Z}, j \neq 0} \sum_{m \in \mathbb{Z}^d} |\langle \lim_{t \rightarrow \infty} \sum_{k \in \mathbb{Z}} \langle f_i, \psi_{0,k} \rangle \psi_{0,k}, \psi_{j,m} \rangle|^2 \\
 &= \sum_{j \in \mathbb{Z}, j \neq 0} \sum_{m \in \mathbb{Z}^d} |\lim_{t \rightarrow \infty} \sum_{k \in \mathbb{Z}} \langle f_i, \psi_{0,k} \rangle \langle \psi_{0,k}, \psi_{j,m} \rangle|^2 \\
 &= 0.
 \end{aligned} \tag{3.8}$$

The second equation uses the continuity of inner product in $L^2(\mathbb{R}^d)$. Hence $f \in W_0$. This completes the proof of Theorem 3.2.

In the following, we give several concrete examples.

Example 3.3

Let

$$A^t = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

So $|\det A^t| = 2$. Let F_1 be the square $(-1, 1)^2$ and let $F_2 = A^t(F_1)$. Then F_2 is the rectangle whose four corner points are $(1, 2)$, $(1, -2)$, $(-1, 2)$ and $(-1, -2)$. Let $E_1 = F_2 \setminus F_1$. So E_1 is a Parseval frame wavelet set [23, Theorem 4]. By the definition of Parseval frame wavelet set [23], the function ψ defined by $\hat{\psi}(\xi) = \chi_{E_1}(\xi)$ is a Parseval frame wavelet function. Since $E_1 \subset (-1, 1)^2$, then $T(\psi)$ is a PFW for the closed subspace $W_0 = \{f: f \in L^2(\mathbb{R}^d), \text{supp } \hat{f} \subset E_1\}$ [23, Lemma 3]. Let $E_i = F_{i+1} \setminus F_i$, then $E_i \cap E_j = \emptyset$, $i \neq j$. By Theorem 3.1, the function ψ defined by $\hat{\psi}(\xi) = \chi_{E_1}(\xi)$ is a semi-orthogonal frame wavelet.

Example 3.4

Let A and E_1 be as defined in Example 3.3. We define a set G_1 as the union of $(A^t)^{-1}E_1$ and part of $(A^t)^{-2}E_1$. By [23, Theorem 5], G_1 is a frame wavelet set with lower frame bound 1 and upper frame bound 2. Since $G_2 = A^t G_1$, then G_2 is the union of E_1 and part of $(A^t)^{-1}E_1$. So $G_2 \cap G_1 \neq \emptyset$, by Theorem 3.1, the frame wavelet function ψ defined by $\hat{\psi}(\xi) = \chi_{G_1}(\xi)$ is not a semi-orthogonal frame wavelet.

4. Parseval Frame Wavelets Associated with GMRA

In this section, we discuss the Parseval frame wavelets associated with GMRA. We firstly give a necessary and sufficient condition for the PFWs to be the PFWs associated with GMRA. Then, we show the property of the bracket function of orthogonal principal shift invariant spaces and discover an important characteristic of the PFWs associated with GMRA by their minimal vector-filter.

In the following, we give two important results about the semi-orthogonal Parseval frame wavelet.

Lemma 4.1

Suppose that $\Psi = \{\psi_{j,k}(x) | \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(A^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is the frame wavelet of $L^2(\mathbb{R}^d)$. Then Ψ is semi-orthogonal if and only if the following equation holds.

$$\langle \psi_{0,k}, \psi_{n,m} \rangle = 0, \forall n \in \mathbb{Z}, n \neq 0, \text{ for all } k, m \in \mathbb{Z}^d. \quad (4.1)$$

Proof

By definition, if $\forall l, j \in \mathbb{Z}, l \neq j, k, m \in \mathbb{Z}^d$, we have

$$\begin{aligned} 0 &= \langle \psi_{j,k}, \psi_{l,m} \rangle \\ &= \int_{\mathbb{R}^d} 2^{\frac{j}{2}} \psi(A^j x - k) \cdot \overline{2^{\frac{l}{2}} \psi(A^l x - m)} dx \\ &= \int_{\mathbb{R}^d} 2^{\frac{j}{2}} 2^{\frac{l}{2}} 2^{-j} \psi(x' - k) \cdot \overline{\psi(A^l A^{-j} x - m)} dx' \\ &= \int_{\mathbb{R}^d} 2^{\frac{l-j}{2}} \psi(x' - k) \cdot \overline{\psi(A^{l-j} x' - m)} dx' \\ &= \int_{\mathbb{R}^d} \psi(x' - k) \cdot \overline{2^{\frac{l-j}{2}} \psi(A^{l-j} x' - m)} dx' \\ &= \langle \psi_{0,k}, \psi_{l-j,m} \rangle. \end{aligned} \quad (4.2)$$

Let $l - j = n$, then (4.1) holds.

The converse is clear. So we get the desired result.

Lemma 4.2

Suppose that $\Psi = \{ \psi_{j,k}(x) | \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(A^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d \}$ is the Parseval frame wavelet of $L^2(\mathbb{R}^d)$. Then Ψ is semi-orthogonal if and only if there exists a closed subspace $W_0 \subset L^2(\mathbb{R}^d)$ such that

$$\{ \psi_{0,k}(x) | \psi_{0,k}(x) = \psi(x - k), k \in \mathbb{Z}^d \} \quad (4.3)$$

is the Parseval frame wavelet of W_0 .

Proof

Sufficiency: Suppose that $\{ \psi_{0,k}(x) | \psi_{0,k}(x) = \psi(x - k), k \in \mathbb{Z}^d \}$ is the Parseval frame wavelet of a closed subspace W_0 , we have

$$\|f\|^2 = \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{0,k} \rangle|^2 = \|f\|^2, \forall f \in W_0. \quad (4.4)$$

Since $W_0 \subset L^2(\mathbb{R}^d)$, we get

$$\|f\|^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{j,k} \rangle|^2 = \|f\|^2, \forall f \in W_0. \quad (4.5)$$

Obviously, $|\langle g, \psi_{j,m} \rangle|^2 \geq 0, \forall g \in L^2(\mathbb{R}^d)$. Then, by (4.4) and (4.5), we get $|\langle f, \psi_{j,m} \rangle|^2 = 0, j \neq 0, \forall f \in W_0$. Noting $\{ \psi_{0,k}(x) | \psi_{0,k}(x) = \psi(x - k), k \in \mathbb{Z}^d \} \subset W_0$, then $|\langle \psi_{0,k}, \psi_{j,m} \rangle|^2 = 0, j \neq 0$.

According to the definition of semi-orthogonal frame wavelet, by Lemma 4.1, we have that Ψ is the semi-orthogonal Parseval frame wavelet of $L^2(\mathbb{R}^d)$.

Necessity: By theorem 3.2, we get the desired result immediately.

Then, we introduce one of the main results in this section.

Theorem 4.3

Suppose that ψ is a PFW. Then ψ is a PFW associated with GMRA if and only if there exists a closed subspace $W_0 \subset L^2(\mathbb{R}^d)$ such that (4.3) is the Parseval frame wavelet of W_0 .

Proof

Sufficiency: We assume that ψ is a PFW and there exists a closed subspace $W_0 \subset L^2(\mathbb{R}^d)$ such that (4.3) is the Parseval frame wavelet of W_0 . Define $V_0 = \bigcup_{i \in \mathbb{Z}, i < 0} W_i$, where

$$W_i = \overline{\text{span}} \left\{ 2^{\frac{i}{2}} \psi_{i,k}(x) | \psi_{i,k}(x) = \psi(A^i x - k), k \in \mathbb{Z}^d \right\}. \quad (4.6)$$

So

$$\left\{ \psi_{i,k}(x) | \psi_{i,k}(x) = 2^{\frac{i}{2}} \psi(A^i x - k), k \in \mathbb{Z}^d \right\} \quad (4.7)$$

is the PFW of the closed subspace W_i . By Lemma 4.2, we have $W_i \cap W_h = \emptyset, i \neq h$. Hence $V_0 = \bigoplus_{i \in \mathbb{Z}, i < 0} W_i$ and $\Psi = \left\{ \psi_{i,k}(x) | \psi_{i,k}(x) = 2^{\frac{i}{2}} \psi(A^i x - k), i \in \mathbb{Z}, i < 0, k \in \mathbb{Z}^d \right\}$ is the PFW of V_0 .

For $\forall f \in V_0$, we have

$$f(x) = \sum_{i \in \mathbb{Z}, i < 0} \sum_{k \in \mathbb{Z}^d} \langle f(x), \psi_{i,k}(x) \rangle \psi_{i,k}(x). \quad (4.8)$$

Let $x = x' + m, m \in \mathbb{Z}^d$, then

$$\begin{aligned} f(x' + m) &= \sum_{i \in \mathbb{Z}, i < 0} \sum_{k \in \mathbb{Z}^d} \langle f(x' + m), \psi_{i,k}(x' + m) \rangle \psi_{i,k}(x' + m) \\ &= \sum_{i \in \mathbb{Z}, i < 0} \sum_{k \in \mathbb{Z}^d} \langle f(x' + m), \psi_{i,k-m}(x') \rangle \psi_{i,k-m}(x') \\ &= \sum_{i \in \mathbb{Z}, i < 0} \sum_{k' \in \mathbb{Z}^d} \langle f(x' + m), \psi_{i,k'}(x') \rangle \psi_{i,k'}(x'). \end{aligned} \quad (4.9)$$

So $f(x' + m) \in V_0$. By definition, V_0 is a shift invariant subspace and satisfies the condition (5) in definition 2.5.

If $j \in \mathbb{Z}, j > 0$, we define $V_j = \bigoplus_{i < j, i \in \mathbb{Z}} W_i$. It is easy to check that (1) and (2) in definition 2.5 hold. For condition (3), if $\bigcap_{j \in \mathbb{Z}} V_j \neq \{0\}$, then there exists a function $f \neq 0, f \in \bigcap_{j \in \mathbb{Z}} V_j$. Since $\bigoplus_{i \in \mathbb{Z}} W_i = L^2(\mathbb{R}^d)$, so there exists a set $Q = \{i_1, i_2, \dots, i_n, \dots\} \subset \mathbb{Z}$ such that $f \in \bigoplus_{i \in Q} W_i$. But $(\bigoplus_{i \in Q} W_i) \cap (\bigcap_{j \in \mathbb{Z}} V_j) = \{0\}$, this is a contradiction. So $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ is true.

At last, we just need to prove that the condition (4) in definition 2.5 holds.

For $\forall f \in V_j$,

$$\begin{aligned} f(Ax) &= \sum_{i \in \mathbb{Z}} \sum_{i < jk \in \mathbb{Z}^d} \langle f(Ax), \psi_{i,k}(Ax) \rangle \psi_{i,k}(Ax) \\ &= \sum_{i \in \mathbb{Z}} \sum_{i < jk \in \mathbb{Z}^d} \langle f(x'), \psi_{i+1,k}(x') \rangle \psi_{i+1,k}(x') \\ &= \sum_{i' \in \mathbb{Z}} \sum_{i' < jk \in \mathbb{Z}^d} \langle f(x'), \psi_{i',k}(x') \rangle \psi_{i',k}(x'), \end{aligned} \quad (4.10)$$

then $f(Ax) \in V_{j+1}$.

For $\forall f \in V_{j+1}$,

$$\begin{aligned} f(A^{-1}x) &= \sum_{i \in \mathbb{Z}} \sum_{i \leq jk \in \mathbb{Z}^d} \langle f(A^{-1}x), \psi_{i,k}(A^{-1}x) \rangle \psi_{i,k}(A^{-1}x) \\ &= \sum_{i \in \mathbb{Z}} \sum_{i \leq jk \in \mathbb{Z}^d} \langle f(x'), \psi_{i-1,k}(x') \rangle \psi_{i-1,k}(x') \\ &= \sum_{i' \in \mathbb{Z}} \sum_{i' < jk \in \mathbb{Z}^d} \langle f(x'), \psi_{i',k}(x') \rangle \psi_{i',k}(x'), \end{aligned} \quad (4.11)$$

then $f(A^{-1}x) \in V_j$. Hence the condition 4 in definition 2.5 holds, (V_j) is an GMRA.

Necessity: If ψ is a PFW associated with GMRA, then the PFW must be semi-orthogonal. By Theorem 3.2, there exists a closed subspace $W_0 \subset L^2(\mathbb{R}^d)$ such that (4.3) is the Parseval frame wavelet of W_0 .

Remark

Due to Lemma 4.2 and Theorem 4.3, we get that all the semi-orthonormal PFWs coincide with the classes of all GMRA PFWs. This is the Theorem 1.6 in [22].

In order to prove another main result in this section, we start with an useful Lemma [22, Remark 2.1] concerned with principal shift invariant spaces.

Lemma 4.4

Let $f, g \in L^2(\mathbb{R}^d)$. Then $\langle f \rangle \perp \langle g \rangle$ if and only if $[\hat{f}, \hat{g}] = 0$ a.e..

We also need the following Lemma [22, Lemma 2.3].

Lemma 4.5

Let (V_j) be an n-GMRA with $n \in \mathbb{N}$. Then

$$V_j = \left\{ f \in L^2(\mathbb{R}^d) : \widehat{f}(B^j \xi) = \sum_{i=1}^n t_i(\xi) \widehat{\varphi}_i(\xi), t_i \in L^2(\mathbb{T}^d), \forall i \right\}, \quad \forall j \in \mathbb{Z}. \quad (4.12)$$

Moreover, for every $j \in \mathbb{Z}$, and for each $f \in V_j$, there exist uniquely determined functions $t_i \in$

$$L^2(\mathbb{T}^d) \text{ such that } \widehat{f}(B^j \xi) = \sum_{i=1}^n t_i(\xi) \widehat{\varphi}_i(\xi) \text{ and } t_i(\xi) = 0, \forall \xi \notin \Omega_i, \forall i = 1, \dots, n.$$

By the preceding Lemma 4.5, for each $f \in V_1$, there exists a unique n -tuple $t = (t_1, \dots, t_n)$, $t_i \in L^2(T^d)$, such that

$$\widehat{f}(B\xi) = \sum_{i=1}^n t_i(\xi) \widehat{\varphi_i}(\xi), \quad t_i(\xi) = 0, \forall \xi \notin \Omega_i, \quad \forall i = 1, \dots, n. \quad (4.13)$$

In the sequel, we shall write $\mu(f) = t$ and $\mu(f)$ will be called the minimal vector-filter for $f \in V_1$.

Since $W_i \subset V_1$, $i \leq 0$, $i \in \mathbb{Z}$, this also applies to $\{\psi_{i,0}\}_{i \leq 0, i \in \mathbb{Z}}$. For each $i \leq 0$, we will write $\mu(\psi_{i,0}) = t^i = (t_1^i, \dots, t_n^i)$.

Then we can prove another main result in this section.

Theorem 4.6

Let (V_j) be an n -GMRA, $n \in \mathbb{N}$ with scaling functions $\varphi_1, \dots, \varphi_n$; put $\mu(\psi_{s,0}) = (t_1^s, \dots, t_n^s)$, $s \leq 0$. If φ is a PFW associated with (V_j) , then

$$\sum_{i=1}^n t_i^l(\xi) \overline{t_i^m(\xi)} + \sum_{i=1}^n t_i^l(\xi + \beta) \overline{t_i^m(\xi + \beta)} = 0, \text{ for all } m, l \in \mathbb{Z}, m, l \leq 0, m \neq l. \quad (4.14)$$

Proof

Suppose that ψ is a PFW associated with GMRA (V_j) , then ψ is a semi-orthogonal PFW. So $\{\psi_{i,k}\}_{k \in \mathbb{Z}^d}$ is a PFW for $\langle \psi_{i,0} \rangle$ and $\langle \psi_{l,0} \rangle \perp \langle \psi_{m,0} \rangle$ when $l \neq m$.

By Lemma 4.4 and Lemma 4.5, we have

$$\begin{aligned} 0 &= [\widehat{\psi}_{l,0}, \widehat{\psi}_{m,0}](B\xi) \\ &= \sum_{k \in \mathbb{Z}^d} \widehat{\psi}_{l,0}(B\xi + k) \overline{\widehat{\psi}_{m,0}(B\xi + k)} \\ &= \sum_{k \in \mathbb{Z}^d} \widehat{\psi}_{l,0}(B(\xi + B^{-1}k)) \overline{\widehat{\psi}_{m,0}(B(\xi + B^{-1}k))} \\ &= \sum_{k \in \mathbb{Z}^d} \left(\sum_{i=1}^n t_i^l(\xi + k) \widehat{\varphi_i}(\xi + k) \right) \overline{\left(\sum_{s=1}^n t_s^m(\xi + k) \widehat{\varphi_s}(\xi + k) \right)} \\ &\quad + \sum_{k \in \mathbb{Z}^d} \left(\sum_{i=1}^n t_i^l(\xi + \beta + k) \widehat{\varphi_i}(\xi + \beta + k) \right) \overline{\left(\sum_{s=1}^n t_s^m(\xi + \beta + k) \widehat{\varphi_s}(\xi + \beta + k) \right)} \\ &= \sum_{i=1}^n \sum_{s=1}^n t_i^l(\xi) \overline{t_s^m(\xi)} [\widehat{\varphi_i}, \widehat{\varphi_s}](\xi) + \sum_{i=1}^n \sum_{s=1}^n t_i^l(\xi + \beta) \overline{t_s^m(\xi + \beta)} [\widehat{\varphi_i}, \widehat{\varphi_s}](\xi + \beta) \\ &= \sum_{i=1}^n t_i^l(\xi) \overline{t_i^m(\xi)} \chi_{\Omega_i}(\xi) + \sum_{i=1}^n t_i^l(\xi + \beta) \overline{t_i^m(\xi + \beta)} \chi_{\Omega_i}(\xi + \beta) \\ &= \sum_{i=1}^n t_i^l(\xi) \overline{t_i^m(\xi)} + \sum_{i=1}^n t_i^l(\xi + \beta) \overline{t_i^m(\xi + \beta)}, \end{aligned} \quad (4.15)$$

where the fourth equality is deduced by Lemma 2.1, the fifth equality uses the $\mathbb{Z}^d$ -periodicity of t_i^l 's and t_s^m 's and the last equality is obtained by Lemma 4.5. Hence (4.14) holds.

Finally, we will construct several examples to prove our results.

Example 4.7

Define the function ψ by its Fourier transform:

$$\widehat{\psi}(\xi) = \chi_I(\xi), \text{ where } I = [-2, -1) \cup (1, 2].$$

We have $\widehat{\psi}(2^j \xi) = \chi_{I_j}(\xi)$, where

$$I_j = [-2^{-j+1}, -2^{-j}) \cup (2^{-j}, 2^{-j+1}].$$

Since the I'_j s are disjoint and their union, over $j \geq 1$, coincide with $[-1, 0) \cup (0, 1]$, we can take $\widehat{\varphi}(\xi) = \chi_{[-1, 1]}(\xi)$, so $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$. Choose V_j as the closed span of $\{\varphi_{j,k} = 2^{\frac{j}{2}} \varphi(2^j \cdot - k) : k \in \mathbb{Z}\}$ for all $j \in \mathbb{Z}$. Then $\{V_j : j \in \mathbb{Z}\}$ is a multiresolution analysis if we show that $\frac{1}{2} \varphi(\frac{1}{2}x)$ is an element of V_0 . We can get

$$t^j(\xi) = \begin{cases} 1 & \text{if } -2^{-j+1} \leq \xi < -2^{-j} \text{ and } 2^{-j} < \xi \leq 2^{-j+1}, \\ 0 & \text{if other} \end{cases}.$$

The proof is trivial. It is easy to check that

$$t^l(\xi) \overline{t^m(\xi)} + t^l(\xi + \beta) \overline{t^m(\xi + \beta)} = 0, \quad m, l \in \mathbb{Z}, m, l \leq 0, j \neq l.$$

Hence, this example testifies Theorem 4.6.

Example 4.8

Let A be the quincunx matrix $Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in E_n^{(2)}$, then we get $B = Q^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in E_n^{(2)}$. Furthermore, we have $B^{-1}C \subseteq C$, where C is the standard unit square in $\mathbb{R}^2$. It is well known that the family $\{B^j(C \setminus B^{-1}C) : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^2$. Define the function ψ by $\psi(\omega) = \chi_{B \setminus C}$. So we get a PFW [23, Theorem 4]. By the definition of function ψ , we have $\text{supp} \widehat{\psi}_{j,0} \cup \text{supp} \widehat{\psi}_{l,0} = \emptyset, j \neq l, \text{ for all } j, l \in \mathbb{Z}$. According to Lemma 3 in [23], Theorem 3.1 and Theorem 3.2, we have that there exists a closed subspace W_0 for which $\{\psi_{0,k} : k \in \mathbb{Z}^2\}$ is a PFW. Again by Theorem 4.3, we derive that the PFW function ψ associates with GMRA. In fact, the scaling function is defined by $\widehat{\varphi}(\xi) = [-\frac{1}{2}, \frac{1}{2}]^2$, $\varphi(x)$ serves as an analog of the Shannon wavelet $\psi(\omega) = \chi_{2I}$, where I denotes the unit interval $[-\frac{1}{2}, \frac{1}{2}]^2$.

5. Conclusion

In this paper, we study semi-orthogonal frame wavelets and PFWs associated with GMRA. Firstly, we present a necessary and sufficient condition for the frame wavelets to be the semi-orthogonal frame wavelets. Secondly, we obtain a necessary condition for the semi-orthogonal frame wavelets. Then, by studying the relation between the PFWs and the GMRA, we derive a necessary and sufficient condition for the PFWs associated with GMRA. Finally, we show the characteristic of the bracket function of orthogonal principal shift invariant spaces and

derive an important property of the PFWs associated with GMRA by their minimal vector-filter. In each section, some concrete examples are constructed to prove our theory.

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