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Voxel-shift and Interpolation for Hadamard-encoded MR images

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Abstract

While Fourier gradient phase-encoding and Hadamard radio-frequency encoding are two established spatial MR localization techniques, the absence of voxel-shift and interpolation post-processing algorithms for the latter has always placed it at a discouraging disadvantage. This paper presents a method for voxel-shift and interpolation of Hadamard-encoded data and demonstrates both theoretically and experimentally the similarities of the respective operations between the two localization methods.

Keywords

Fourier encoding; Hadamard encoding; interpolation; voxel-shift

Introduction

Gradient (Fourier) and radio-frequency (RF) phase encodings are two established localization methods for both MRI (1,2) and MR spectroscopic imaging (MRSI) (3–6). Despite mathematical similarities, however, Fourier encoding has become the spatial localization method of choice for most MRI and MRSI applications, due, in part, to the simplicity of its implementation and the convenience of its image interpolation and voxel-shift post-processing operations (7,8). The last two features are especially attractive for MRSI since they allow to arbitrary align the relatively coarse imaging grid with specific anatomical regions within the field-of-view (FOV) in post-processing (9).

Gradient phase-encoding, however, suffers two main drawbacks: its FOV must extend beyond the imaged object to prevent aliasing; and its intrinsic point-spread function leads to intervoxel signal bleed inversely proportional to the number of encoding steps per FOV (9,10). Without loss in signal-to-noise ratio (SNR) both issues can be mitigated with spatial Hadamard RF encoding (5,6,11,12). It offers sharp voxel edges that are insensitive to the number of encoding steps and the ability to minimize its (non-intrinsic) voxel bleed (13–15). Therefore, while gradient and RF encoding seem to be complementary localization methods (11,12), absence of interpolation and voxel-shift analogs for the latter placed it at a disadvantage which presumably contributed to the impediment of its wider use.

In this paper we address these issues by first developing the theory of voxel-shift and interpolation for Hadamard encoding. To demonstrate their effectiveness and show the

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From equation [6]:
$$\mathfrak{I}_m^s = \frac{1}{\delta} \sum_p a_p \int_{m\delta-s}^{(m+1)\delta-s} \phi_p(x) dx = \sum_{n,p} \left(\int_{m\delta-s}^{(m+1)\delta-s} \phi_p(x) dx \right) (H)_{p,n}^{-1} I_n = \sum_n D_{m,n}^s I_n.$$

equivalence of each operation to its Fourier analogue two independent forms of spatial encoding need to be applied in the same direction simultaneously. One of these encoding techniques would be either the Hadamard (*i.e.* RF-encoding) or the Fourier (*i.e.* gradient encoding) while the other should be an artefact-free “reference”. Since neither RF nor gradient encoding methods are available for this purpose, a multi-compartment phantom was built and compounds of different known chemical shifts were used to label the compartments. Consequently, spectroscopic encoding was employed to provide the additional spatial “reference”. As a result, all experiments were conducted using Hadamard spectroscopic imaging (HSI) and Fourier chemical shift imaging (CSI) (3,4). In addition, to isolate reconstruction artefacts from the acquisition ones, an idealized analytic MR signal calculation was used to reveal the fundamental limitations of the two techniques. Although for the sake of simplicity and clarity all operations are demonstrated or compared for the one-dimensional (1D) case, they can be generalized to higher spatial orders.

Theory

First, we briefly review the well-known formulation of the Fourier-encoded MR imaging voxel-shift and interpolation problems. In doing so, we will introduce convenient notations which may seem cumbersome at first, but will prove to be quite beneficial in devising a scheme for Hadamard-encoded imaging voxel interpolation and shift. Then, we will show that the problem of voxel-shift and interpolation of Hadamard-encoded MR images can be formulated, and thus, solved in a manner similar to that of the Fourier-encoded ones.

Let

$$\phi_p(x) = \begin{cases} e^{+2\pi i p x}, & -L/2 \leq x < L/2 \\ 0, & x < -L/2 \text{ or } x \geq L/2 \end{cases} \quad [1]$$

be a set of functions over the field of view L . Then, in Fourier-encoded MR imaging, the spin-density $\rho(x)$ at any point x is related to the complex digitized MR signal a_p by:

$$\rho(x) = \sum_{p=-P/2}^{P/2-1} a_p \phi_p(x) \quad [2]$$

where P is the number of phase-encoding steps. The reconstructed image I_n in the voxel n is simply the average spin-density over the voxel size $\Delta = L/P$:

$$I_n = \frac{1}{\Delta} \int_{n\Delta}^{(n+1)\Delta} \rho(x) dx = \frac{1}{\Delta} \sum_p a_p \int_{n\Delta}^{(n+1)\Delta} \phi_p(x) dx \quad n=0, 1, \dots, P-1 \quad [3]$$

Therefore, it is clear that to construct an image I_n^s , shifted by s with respect to the image I_n , it is enough to shift the range of integration in equation [3]:

$$\mathfrak{I}_n^s = \frac{1}{\Delta} \int_{n\Delta-s}^{(n+1)\Delta-s} \rho(x) dx = \frac{1}{\Delta} \sum_p a_p \int_{n\Delta-s}^{(n+1)\Delta-s} \phi_p(x) dx \quad [4]$$

On the other hand, to construct an interpolated image I_m it is enough to decrease the voxel size from Δ to $\delta < \Delta$ in equation [3]:

$$\Im_m = \frac{1}{\delta} \int_{m\delta}^{(m+1)\delta} \rho(x) dx = \frac{1}{\delta} \sum_p a_p \int_{m\delta}^{(m+1)\delta} \phi_p(x) dx \quad [5]$$

As a simple consequence, the interpolated to voxel size δ and shifted by s image I_m^s is given by:

$$\Im_m^s = \frac{1}{\delta} \int_{m\delta-s}^{(m+1)\delta-s} \rho(x) dx = \frac{1}{\delta} \sum_p a_p \int_{m\delta-s}^{(m+1)\delta-s} \phi_p(x) dx \quad [6]$$

Restoring the explicit form of $\phi_p(x)$ from equation [1], equation [3] becomes:

$$I_n = \frac{1}{\Delta} \sum_p a_p \int_{n\Delta}^{(n+1)\Delta} e^{+2\pi i p x} dx = \sum_{p=-P/2}^{P/2-1} \tilde{a}_p e^{2\pi i p n \Delta} \quad [7]$$

$$\tilde{a}_p = a_p \text{sinc}(\pi \Delta p) e^{i\pi \Delta p} \quad [8]$$

In other words, to reconstruct the image I_n of the average spin-density, it is sufficient to multiply the measured k-space data by the filter of equation [8] and apply the Fourier transformation.

Analogously, according to equation [4], the “voxel-shift” operation is nothing more than a Fourier transform of the filtered k-space data [8] weighted by a phase roll in k-space:

$$\Im_n^s = \frac{1}{\Delta} \sum_p a_p \int_{n\Delta-s}^{(n+1)\Delta-s} e^{+2\pi i p x} dx = \sum_{p=-P/2}^{P/2-1} \tilde{a}_p e^{-2\pi i p s} e^{2\pi i p n \Delta} \quad [9]$$

By the same token, setting the voxel size $\delta = \Delta/2^z$ (z is an integer, greater than unity), the expression for the interpolated image (equation [5]) becomes the familiar “zero-fill-and-transform” operation applied to the filtered k-space data:

$$\Im_m = \frac{1}{\delta} \sum_p a_p \int_{m\delta}^{(m+1)\delta} e^{+2\pi i p x} dx = \sum_{p=-zP/2}^{zP/2-1} \bar{a}_p e^{2\pi i p m \Delta/2^z} \quad [10]$$

$$\bar{a}_p = \begin{cases} a_p \text{sinc}(\pi \delta p) e^{i\pi \delta p}, & p = -\frac{P}{2}, -\frac{P}{2}+1, \dots, \frac{P}{2}-1 \\ 0, & p = -\frac{(z+1)P}{2}, -\frac{(z+1)P}{2}+1, \dots, -\frac{P}{2}-1, \frac{P}{2}, \frac{P}{2}+1, \dots, \frac{(z+1)P}{2}-1 \end{cases} \quad [11]$$

The “zero-fill” factor $z > 1$ is chosen to provide an adequate level of spatial detail.

We have specifically chosen this form of presentation because now, it is easy to see how this problem can be solved for Hadamard-encoded MR images as well. The main conceptual step is realizing that Hadamard-encoded MR image can be written in the form of equation

[2] with the functions $\phi_p(x)$ defined according to the rows of the Hadamard matrix (16) (see Figure 1 for the representation of 2, 4, and 8 order Hadamard matrices) and a_p being the Hadamard expansion coefficients of the spin density (see equation [2]).

Since equations [3], [4], [5] and [6] have been written in terms of the functions $\phi_p(x)$, they will not change their form if Hadamard functions are used in place of the Fourier ones [1]. More specifically, equation [6] can be evaluated analytically to relate the interpolated and shifted image I_m^s to its Hadamard reconstructed progenitor I_n^1 :

$$\mathfrak{I}_m^s = \sum_n D_{m,n}^s I_n \quad [12]$$

$$D_{m,n}^s = \Theta(\Delta - \delta) \bar{\Lambda} \left(\frac{(2m+1)\delta + 2s}{\Delta} - 2n - 1, \frac{\delta}{\Delta} \right) + \frac{\Delta}{\delta} (1 - \Theta(\Delta - \delta)) \bar{\Lambda} \left(\frac{(2m+1)\delta + 2s}{\Delta} - 2n - 1, \frac{\Delta}{\delta} \right) \quad [13]$$

where $\Lambda(x, y)$ is a trapezoid function:

$$\bar{\Lambda}(x, y) = \begin{cases} 1, & |x| \leq (1-y) \\ 1 - \frac{1}{2y}(|x| - 1+y), & (1-y) < |x| \leq (1+y) \\ 0, & |x| > (1+y) \end{cases} \quad [14]$$

and

$$\Theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad [15]$$

is the Heaviside step-function.

Therefore, the procedure of interpolation and voxel-shift for Hadamard-encoded MR images is: take the Hadamard-reconstructed image I_n and multiply it by the interpolation/shift matrix D from equation [13].

Methods

Phantom

Assessing the effectiveness of the shift and interpolation operations requires a well characterized structure where the correct results are known, a condition rarely encountered in vivo. Therefore, a cylindrical phantom with “puck-like” compartments was built from polycarbonate resin thermoplastic (LEXAN®), as shown in Figure 2. Each compartment, a cylinder with internal diameter of ϕ 121 mm and 11.5 mm thickness with 1.0 mm walls between the compartments (Accurate Lock, Stamford, CT), represents a distinct voxel in 1D if viewed along the phantom’s axis, as shown in Figure 2. To enable observation of voxel bleeds and other reconstruction artifacts the compartments were “labeled” by filling each of them with 100 mM (protons) chemical solutions of a distinct singlet resonance frequency

¹Equation [3] has a form of $I_n = \sum_p H_{n,p} a_p$ where $H_{n,p}$ is the Hadamard matrix, then: $a_p = \sum_n (H)_{p,n}^{-1} I_n$.

(Methanol at 3.4 ppm; Na-acetate at 1.9 ppm; tert-butanol at 1.2 ppm; Na-3-Methyl-Silyl propionate at 0 ppm) and performing spectroscopic data acquisition. Thanks to the one-to-one correspondence between frequencies and compartments, spectral line intensity at a specific frequency reveals the magnitude of the artifact. The water signal at 4.7 ppm, common to all compartments, was suppressed during the acquisition with WET (17) and removed during post-processing (18).

Experiments

All experiments were performed in a 3 T Magnetom Trio whole-body clinical MRI scanner (Siemens AG, Erlangen, Germany) equipped with a $\varnothing 25$ cm diameter TEM (19) transmit-receive head coil (TEM3000, MRInstruments, Minneapolis, MN). To reduce air-water susceptibility artefacts, the phantom was immersed in a larger water-filled tube, as shown in Figure 2. Our custom CSI-based auto-shim procedure yielded a 6 Hz water linewidth from the whole phantom (20) allowing clean separation of the partition “labels” (the singlets), as shown in Figure 2.

Two series of 1D localization experiments were conducted which differed only by the spatial encoding scheme used along the phantom axis: Hadamard (using 1D HSI) and Fourier (using 1D CSI). In each series, a parallelepiped volume-of-interest (VOI) of $12.5 \times 12.5 \times 50$ mm³ was excited with $T_E = 35$ ms and $T_R = 1600$ ms PRESS. The long 50 mm aspect of the VOI was oriented along the phantom axis and was placed once to contain all four phantom partitions exactly and once — shifted by 6.25 mm (half a partition) as shown schematically in Figure 3. Data sampling with 1024 complex points at 500 μ s per point were common to all experiments. The MRSI data were processed off-line using our custom reconstruction software. Residual water was removed from the data in the time domain (18) and the signals were apodized with a 1.3 Hz exponential before transforming from the time domain into the frequency one. Neither frequency nor phase corrections were applied. The spatial data were then 1D Hadamard or Fourier (as appropriate) reconstructed as described in the Theory section.

The first series comprised the fourth and the eighth order 1D-HSI using four and eight different Hadamard RF pulses for the PRESS 90° (21,22) along the 50 mm aspect of the VOI. The PRESS 90° pulse was 5.12 ms long with a 3.8 mT/m slice-select gradient followed by two 10.24 ms long Shinar-Le Roux refocusing 180° pulses under 9 mT/m gradients each. Due to the high slice-selective gradients and narrow span of the resonance frequencies of the compartment labels, the chemical shift displacement due to slice-selective excitation was below 1 mm — the thickness of the wall. The second series consisted of the fourth and the eighth order 1D-CSI along the 50 mm aspect of the VOI.

The results of these experiments are suitable to assess the performance of both voxel-shift and interpolation operations. Indeed, an ideal post-processing software-based voxel-shift operation would translate the result of the experiment schematically presented on panel **c** of Figure 3 into the one presented on panel **b** of the figure and *vice versa*, as indicated by the diagonal arrows. On the other hand, an ideal post-processing interpolation operation would transform results of the experiments schematically presented on panels **c** and **d** of Figure 3 into the ones obtained with the eight-step encoding on panels **e** and **f** of the figure, as indicated by the vertical arrows. Of course, a combination of voxel-shift and interpolation operations should produce panel **e** of Figure 3 from panel **d** and panel **f** from — **c**, as indicated by the diagonal arrows.

Analytical calculation

It is well-known that hardware imperfections (such as B_0 and B_1 inhomogeneities, non-ideal excitation pulses, non-zero chemical displacements, etc) may bias the assessments of the voxel-shift and interpolation operations. Thus, the aim of the synthetic experiments was to generate ideal HSI and CSI data sets (*i.e.* free of the above imperfections) in order to objectively compare the performance of the methods. Hence, analytical calculations were done to analyze the outcome of experiments conducted on a four-partition phantom with a singlet at a distinct chemical shift in each partition for four-step and eight-step HSI and CSI encodings on a FOV equal to the phantom size assuming ideal B_0 and B_1 distributions and excitation pulse profiles (see appendix). Reconstruction of these data was performed according to the Theory section. The FOV was placed to contain all four partitions exactly (as shown in Figure 3, panel **c**) and shifted by half a partition (as in Figure 3, panel **d**).

Results

Results of the analytical calculations for the Hadamard and Fourier methods without and with interpolations are presented in Figures 4 and 5. Their purpose is to demonstrate that even in the absence of hardware imperfections the voxel-shift and interpolation operations do not always deliver the expected results. The figures present two examples of the reconstructed-image alignments: the left column — where the reconstruction grid is aligned with the partitions (see Figure 3**a, e**) and the right column — where the voxels are shifted by half a partition (see Figure 3**b, f**). Figure 4 summarizes the results of the voxel-shift calculations, while Figure 5 presents the results of the interpolation and interpolation coupled with the voxel-shift calculations. The vertical scaling in all the plots of the Figure 4 and 5 was kept consistent, therefore, variability in the peak heights reflects a reconstruction artefact.

The first row in Figures 4 and 5 contains a pictorial description of what would be desired, artefact free spectra in these cases. In the absence of reconstruction artefacts, all images in the columns should be identical to those. When the reconstruction grid is aligned with the partitions, one expects to see one metabolite per voxel. When, on the other hand, the image grid is shifted by half of the partition size, only the higher resolution interpolated image should contain one metabolite per voxel. With the exception of one “edge” voxel which does not have a neighbor and therefore, contains a single spectral peak, all voxels of the shifted non-interpolated image are expected to contain half of the amount of two metabolites from neighboring partitions.

The second and the third rows of the Figures 4 and 5 present spectroscopic images of data synthesized with the FOV and partitions aligned using Hadamard and Fourier encoding respectively. It can be seen that reconstruction of the Hadamard encoded data produces expected, artefact-free images. With the Fourier encoding, despite the FOV-to-partition alignment, the well-known intervoxel “bleed” and “wrap-around” artefacts are apparent.

The last two rows of the figures depict reconstructed spectroscopic images of data synthesized with the FOV shifted by half a partition size using Hadamard and Fourier encodings. Because, the FOV is not aligned with the partitions, neither scheme produces the desired results. Fourier encoded images demonstrate long-range intervoxel bleed and wrap-around artefacts, while Hadamard-encoded ones — reveal short-range intervoxel bleed.

The quantification of the reconstruction artefacts due to the voxel-shift and interpolation operations for Hadamard and Fourier encoded images can be accomplished by reading off the peak heights from the Figures 4 and 5. Indeed, since both encoding and reconstruction schemes are R2.4 linear operations, the spectral leakage may be quantified as the relative

deviation of the peak height from its nominal which can be read off the figures and is summarized in Tables 1 and 2. The negative leakage indicates that the reconstructed peak is smaller than the ideal, while the positive leakage corresponds to artificially large peaks.

Analysis of the Figures 4 and 5 reveals that artefact free images can be obtained when the data are Hadamard encoded with FOV aligned to the partitions (second row of the figures) and when FOV and the reconstruction are shifted by half a partition size (fourth row, right column of Figure 4) only. In all other considered cases, some inter-voxel leakage is present. For Fourier encoded images this leakage may be quite complex. For example, the distortions produced by Fourier encoding with FOV and the reconstruction aligned to the partitions, demonstrated in third row, left column of Figure 4, lead to apparent violation of the mass-conservation law: the sum of the peak areas for each metabolite is 1.11 instead of 1.00. The effect for the shifted images (right column of the same row) is even more dramatic — the sum of the peak areas is 1.21. Figure 5 exposes a similar problem with the mass-conservation and interpolated Fourier-encoded images.

Theoretically, the demonstrated artefacts can be understood by studying the structure of equations [A.3] and [A.5]. These equations describe the reconstructed images in terms of the defined object. Indeed, Hadamard voxel-shift and interpolation operations do not cause any phase distortions and do not violate the mass-conservation law since they are described by matrices with nonnegative (real) elements of equation [A.3]. Furthermore, the inter-compartment “leakage” is of the short-range nature because the non-vanishing elements of the matrices are near the diagonal. The analogous operations for the Fourier-encoded images, however, often cause phase distortions with an associated local mass-conservation law violation since they are described by complex-valued matrices of equation [A.5] and the inter-compartment “leakage” is of the long-range nature because the matrices, generally, do not have vanishing elements (known as the “Gibbs ringing”).

Figures 6 and 7 are experimental analogues of Figures 4 and 5, respectively, and summarize the results of the shift and interpolation operations based on the actual experimental data. Therefore, the top-most “true” row of Figure 6 is missing compared to Figure 4, but the top-most “true” row of Figure 5 is substituted by two rows of acquired double-resolution spectroscopic images in Figure 7. Even though the acquired images contain a combination of both reconstruction artefacts described here and the acquisition artefacts (15) due to B_0 and B_1 inhomogeneities and different spin-relaxation properties of the employed compounds, the experimental spectroscopic images exhibit the general features of the theoretical results presented in Figures 4 and 5.

Discussion

The lower spatial resolution of MRSI (compared with its guiding MRI) makes it nearly impossible to simultaneously optimize the position of the localization grid over several anatomies and pathologies of interest. Therefore, the zero-fill and voxel-shift operations, which are based on the corresponding well known mathematical theorems of the Fourier transform that allow to improve the grid’s granularity and position over any anatomy in post processing, have made CSI particularly attractive for in vivo spectroscopic imaging. The metabolites in that optimized region of interest can then be quantified and the grid, if necessary, shifted to the next region and so on. Consequently, it has become widely accepted that a precise (but potentially time consuming) positioning of the localization grid over the object, during the experiment is of secondary importance as long as all the anatomies of interest are within the FOV.

Although the voxel-bleed performance of Hadamard encoding is often superior to that of Fourier, as demonstrated in the literature (14) and shown here, its lack of a formalism for interpolation and voxel-shift operations was considered a significant weakness. This shortcoming is addressed in this paper which, for the first time, outlines the underlying analogy between these two localization methods and constructs the formalism for these two important operations for Hadamard encoded images as well. Introduction of these methods offers the experimenter the option to base their choice of localization method on their voxel-profile and VOI performance rather than post-processing capabilities and convenience.

It is noteworthy that neither voxel-shift nor interpolation operation for either the Hadamard or the Fourier methods is ideal, *i.e.*, may not yield the desired result from a misaligned grid, as demonstrated in Figures 4, 5, 6 and 7 and Tables 1 and 2. This is not a surprise since the success or failure of the voxel-shift and interpolation procedures depends on the type of functions that are chosen to represent the data — sines and cosines for Fourier encoding, versus rectangles for Hadamard encoding.

In practical sense, the fact that Fourier encoding always results in artifacts that make quantification consistently off may be considered a virtue. By the way of illustration, consider prescribing voxels that either line-up (Figure 3.c) or do not line-up (Figure 3.d) with the desired locations (Figure 3.a). In case **c**, direct reconstruction gets voxels in the desired location (Figure 3.c to Figure 3.a) and in case **d** a voxel shift operation must be applied (Figure 3.d to Figure 3.a). Then the Fourier encoding leads to a -25% leakage in both cases so the results are biased but consistent whereas leakages in Hadamard encoded images are 0% and -50% respectively (see Table 1). On the other hand, in the most simplistic situation where one believes that the VOI alignment is prescribed as needed during acquisition and, thus, no shift operations are applied in the post-processing (cases Figure 3.c to Figure 3.a or Figure 3.d to Figure 3.b), the estimated metabolite amounts are always correct if Hadamard encoding were used and are off by either -25% or +10% if Fourier encoding were employed instead (see Table 1).

It should be emphasized that the extreme case of 1/2 voxel shift was considered and, presumably, smaller shifts would lead to smaller reconstruction artefacts. Also, while the inter-voxel leakage provides some quantification of the artifacts, it does not capture the complexity of the situation in its entirety. There can be spectral phase distortions, amplitude reduction of the expected peaks as well as introduction of fictitious ones on a per-voxel basis (see Results section). Perhaps more importantly, it may appear that since the leakages can be quantified theoretically, this bias may be removed in post-processing. This is misleading because besides the encoding scheme, leakages also depend on the structure of the object. In most practical situation, however, *a priori* structural information is not available. Of course, if such structural information were available, SLIM(23) reconstruction would be the most appropriate and would lead to artefact free results. As a result, it should be recognized that neither Fourier nor Hadamard can completely correct for a miss-aligned acquisition with either an interpolation, a voxel-shift, or their combination without a cost of some image degradation.

It should be noted that even though 4th order encoding was considered in this work, the general features of the developed method will persist for any number of encoding steps as can be seen from the formulae provided in the theory section and the appendix. Nonunique to the method presented here, any patient motion, B_0 and B_1 inhomogeneities may violate the fundamental assumptions of MR imaging and, consequently, may lead to additional unpredictable image degradation.

Conclusion

We have introduced the formalism for the interpolation and voxel-shift operations of Hadamard encoded imaging and shown their equivalence to the well known analogs for Fourier encoding. This formalism reduces the disparity between these two complementary methodologies and aids in providing more choice in matching a localization method to a specific problem at hand. It is shown theoretically and demonstrated experimentally that while the performance of neither operation is perfect, they do get the voxel-shifted and/or interpolated data to better represent the sought image than the measured source for either Fourier or Hadamard encoded images.

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Appendix

The details of the analytic calculation of the MR signal and image reconstruction for the Hadamard and Fourier acquisition/reconstruction pairs are given here.

The spin density $\rho(x)$ in a phantom of N compartments of size Δ with spin density of I_n in the n -th compartment is a weighted sum of “box-car” functions:

$$\rho(x) = \sum_{n=0}^{N-1} I_n \bar{\Lambda} \left(\frac{2x}{\Delta} - 2n - 1, 0 \right) \quad [\text{A.1}]$$

where $\Lambda(x, y)$ was defined in equation [14].

Hadamard encoding

The signal after Hadamard reconstruction due to the true spin-density $\Lambda(x, y)$ ([Eq. A.1]) when the field of view is shifted by S with respect to the phantom partitions is:

$$a_p = \int_0^{N\Delta} \rho(x - S) dx = \sum_n \bar{\Lambda} \left(\frac{(2p+1)\Delta + 2S}{\Delta} - 2n - 1, 1 \right) I_n \quad [\text{A.2}]$$

According to equation [6], the reconstructed image shifted by s and interpolated to the voxel size δ is:

$$\mathfrak{I}_m^s = \sum_n I_n \sum_{p=-P/2}^{P/2-1} \left[\Theta(\Delta - \delta) \bar{\Lambda} \left(\frac{(2p+1)\delta + 2s}{\Delta} - 2m - 1, \frac{\delta}{\Delta} \right) + \frac{\delta}{\Delta} (1 - \Theta(\Delta - \delta)) \bar{\Lambda} \left(\frac{(2p+1)\delta + 2s}{\Delta} - 2m - 1, \frac{\delta}{\delta} \right) \right] \bar{\Lambda} \left(\frac{(2p+1)\Delta + 2S}{\Delta} - 2n - \right. \quad [\text{A.3}]$$

Fourier encoding

The signal due to the true spin-density $\Lambda(x, y)$ ([Eq. A.1]) when the field of view is shifted by $|S| < \Delta$ with respect to the phantom partitions is:

$$a_p = \int_0^{N\Delta} \rho(x - S) e^{-2\pi i p x} dx = \Delta \sum_n \text{sinc}(\pi p \Delta) e^{-2\pi i p S} e^{-\pi i p \Delta} e^{-2\pi i p n \Delta} I_n - |S| \cdot \text{sinc}(\pi p S) e^{-\pi i p S} \left[e^{-2\pi i p S} I_0 \Theta(-S) + I_{N-1} \Theta(S) \right]$$

[A.4]

According to equation [6], the reconstructed image shifted by s and interpolated to the voxel size δ is:

$$\begin{aligned} \mathfrak{S}_m^s &= \sum_{p=-P/2}^{P/2-1} a_p \text{sinc}(\pi \delta p) e^{i\pi \delta p} e^{+2\pi i p s} e^{2\pi i p m \delta} \\ &= \Delta \sum_n I_n \sum_{p=-P/2}^{P/2-1} \text{sinc}(\pi p \Delta) \text{sinc}(\pi p \delta) e^{2\pi i p (m\delta - n\Delta + s - S + \delta/2 - \Delta/2)} \\ &\quad - |S| \sum_{p=-P/2}^{P/2-1} \text{sinc}(\pi p S) \text{sinc}(\pi p \delta) e^{2\pi i p (m\delta + s + \delta/2 - S/2)} \left[e^{-2\pi i p S} I_0 \Theta(-S) + I_{N-1} \Theta(S) \right] \end{aligned}$$

[A.5]

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Figure 1.
Second, fourth and eighth order Hadamard functions and matrices.

Figure 2.
Sagittal section of the phantom (left) and the absolute 3 T spectrum from the phantom (right).

Figure 3.

Schematic representation of the phantom's four partitions (solid lines) and the acquisition alignment (dashed lines on panels **c** and **d**) for each of the experiments. In particular, the acquired volume is aligned with the partitions on panel **c** and is shifted by half the partition size in setup **d**. Reconstructed image voxel-size is equal to the size of the partition on panels **a** and **b** and is half the partition size on panels **e** and **f**. Dashed-boxes on panels **a**, **b** **e** and **f** show the reconstruction grid alignment, while the arrows indicate the transformations which were performed during post-processing to test the quality of the voxel-shift and interpolation operations. For example, application of voxel-shift and interpolation to data acquired according to panel **c**, ideally, should produce an image which could be acquired according to panel **f** directly.

Figure 4.

Voxel-shift operation. Analytically synthesized ^1H MR spectra with reconstructed voxel grid aligned with the partitions (left column) and shifted by half the partition size (right column). The reconstructed voxel size is equal to the size of the partition. Each cell is marked by an arrow representing the encoding type, FOV and reconstruction alignment in accordance with Figure 3.

Figure 5.

Interpolation and voxel-shift operations. Analytically synthesized ^1H MR spectra with reconstructed voxel grid aligned with the partitions (left column) and shifted by half the partition size (right column). The reconstructed voxel size is equal to the half of the size of the partition (interpolated). Each cell is marked by an arrow representing the encoding type, FOV and reconstruction alignment in accordance with Figure 3.

Figure 6.

Voxel-shift operation. The experimentally acquired ^1H MR spectra with reconstructed voxel grid aligned with the phantom partitions (left column) and shifted by half the partition size (right column). The reconstructed voxel size is equal to the size of phantom partition. Each cell is marked by an arrow representing the encoding type, FOV and reconstruction alignment in accordance with Figure 3.

Figure 7.

Interpolation and voxel-shift operation. The experimentally acquired ^1H MR spectra with reconstructed voxel grid aligned with the phantom partitions (left column) and shifted by half the partition size (right column). The reconstructed voxel size is equal to the half of the size of phantom partition (interpolated). Each cell is marked by an arrow representing the encoding type, FOV and reconstruction alignment in accordance with Figure 3.

Table 1

Leakage due to the voxel-shift operation (see Figure 4). The percentages represent the relative deviation of the reconstructed peak-areas from their nominal values based on the spatial overlap of a center voxel of the imaging grid with the phantom partitions. The “-” sign indicates “leakage out” and the “+” — “leakage in”.

		Reconstructed image grid alignment	
		Aligned to partitions (see Fig. 3a)	Shifted by 1/2 of the partition (see Fig. 3b)
Encoding & FOV alignment	Hadamard, FOV aligned to partitions (see Fig. 3c)	0%	0%
	Fourier, FOV aligned to partitions (see Fig. 3c)	-25%	+10%
	Hadamard, FOV shifted by 1/2 of partition (see Fig. 3d)	-50%	0%
	Fourier, FOV shifted by 1/2 of partition (see Fig. 3d)	-25%	+10%

Table 2

Leakage due to interpolation and voxel-shift operations (see Figure 5). The percentages represent the relative deviation of the reconstructed peak-areas from their nominal values based on the spatial overlap of a center voxel of the imaging grid with the phantom partitions. The “-” sign indicates “leakage out” and the “+” — “leakage in”.

		Reconstructed image grid alignment	
		Aligned to partitions (see Fig. 3e)	Shifted by 1/2 of the partition (see Fig. 3f)
Encoding & FOV alignment	Hadamard, FOV aligned to partitions (see Fig. 3c)	0%	0%
	Fourier, FOV aligned to partitions (see Fig. 3c)	-25%	-25%
	Hadamard, FOV shifted by 1/2 of partition (see Fig. 3d)	-50%	-50%
	Fourier, FOV shifted by 1/2 of partition (see Fig. 3d)	-25%	-25%