Published in final edited form as:

Signal Processing. 2008 May ; 88(5): 1165–1175.

A General Scheme for Velocity Tomography

Hengyong Yu and Ge Wang

Biomedical Imaging Division, VT-WFU School of Biomedical Engineering and Science, Virginia Polytechnic Institute and State University, Blacksburg, VA, 240601, USA

Abstract

With the rapid development of x-ray source and detector technologies, multi-source scanners become a hot topic in the CT field, which can acquire several projections simultaneously. Aided with the ECG-gating technique, the multi-source scanner can collect sufficient projections to reconstruct one or more specific phases of a beating heart. Hence, we are motivated to develop velocity tomography as a new dynamic imaging mode to recover the velocity field from the projections. First, we derive a velocity field constraint equation subject to the mass conservation. Then, we present a two-step general scheme to estimate the velocity field. The first step directly or indirectly computes partial derivatives. The second step iteratively determines the velocity field subject to the constraint equation and other conditions. Finally, we describe numerical experiments in the fan-beam geometry to demonstrate the correctness and utility of our scheme.

Keywords

Computed tomography (CT); velocity tomography (VT); velocity field constraint equation; cardiac imaging

I. Introduction

Due to the cardiac and respiratory motions, there may be severely artifacts in the cardiac and lung computed tomography (CT) with the current methods and protocols. To reduce the motion artifacts and improve imaging quality, four-dimensional CT (4DCT) is being actively developed [1-3]. In 4DCT, periodical signals associated with the cardiac and respiratory motions, such as electrocardiogram (ECG), can be used to reconstruct images in terms of different states of a beating heart, which are also referred to as phases in the CT field. With the rapid development of x-ray source and detector technologies, multi-source CT scanners are advantageous to acquire several projections simultaneously [4,5]. Aided by the ECG-gating techniques, multi-source CT scanners can collect sufficient projections to reconstruct one or more specific phases of a beating heart.

While the current algorithms reconstruct a dynamic image series, here we propose to reconstruct the velocity field directly from projections, which we call velocity tomography. The whole paper is organized as the follows. In Section II, we present the framework of velocity tomography. In Section III, we derive a velocity field constraint equation subject to the mass conservation. In Section IV, we give a general two-step scheme for velocity tomography. In

hengyong-yu@ieee.org , ge-wang@ieee.org

Publisher's Disclaimer: This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

section V, we demonstrate how to perform velocity tomography and present preliminary results. Finally, in section VI we discuss relevant issues and conclude the paper.

II. Framework

In the CT field, the general goal is to reconstruct a linear attenuation coefficient distribution of an object from its x-ray projections. Let $\mathbb{R}^3$ be the 3D space and $\mathbb{S}^2$ be the corresponding unit sphere. Let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ be a point in the compact support of a dynamic object and t be time. Let $f(\mathbf{x}, t)$ represent the dynamic subject. Correspondingly, the cone-beam transform of $f(\mathbf{x}, t)$ at the time t can be defined as:

$$D_f(\mathbf{a}, \theta, t) = \int_0^\infty f(\mathbf{a} + \lambda\theta, t) d\lambda, \mathbf{a} \in \mathbb{R}^3, \theta \in \mathbb{S}^2. \quad (1)$$

It is well known that a volumetric image of a motionless object can be exactly and efficiently reconstructed from cone-beam data collected along a rather general scanning trajectory [6-9]. However, here our focus is on the dynamic reconstruction, such as for cardiac CT. That is, we want to recover the velocity field $\mathbf{u}(\mathbf{x}, t) = (u, v, w) = \partial\mathbf{x}/\partial t$ from $D_f(\mathbf{a}, \theta, t)$. Since the reconstruction of the velocity field is a typical tomographic problem, we call our approach velocity tomography (VT).

To reconstruct a vector field, the vector tomography was proposed in the last century [10-12]. Specifically, let $\mathbf{v}(\mathbf{x})$ be a general vector field, an inner product measurement D_v^p of the vector field with respect to the “probe” $\mathbf{p}$ is defined as

$$D_v^p(\mathbf{a}, \theta) = \int_0^\infty \mathbf{p}(\mathbf{a}, \theta) \bullet \mathbf{v}(\mathbf{a} + \lambda\theta) d\lambda, \mathbf{a} \in \mathbb{R}^3, \theta \in \mathbb{S}^2, \quad (2)$$

where $\mathbf{p}(\mathbf{a}, \theta)$ is a known vector function [10]. The basic idea of the vector tomography is to recover the vector field $\mathbf{v}(\mathbf{x})$ from the inner product measurements D_v^p . Clearly, our proposed velocity tomography is different from vector tomography, although both the goals are to reconstruct a vector field. While the measured dataset consists of direct integrals of a certain component of vector field in vector tomography [10, 12], the input data to our proposed velocity tomography is projections of an underlying attenuation coefficient distribution, which is scalar. While the vector tomography problem can be solved using an analytic method based on the Radon inversion theory [12], our velocity tomography will be enabled by combining the tomographic imaging and optical flow techniques [13, 14].

III. Velocity Field Constraint Equation

Assume a mass point $f(\mathbf{x}, t)$ is moved by $\delta\mathbf{x} = (\delta x, \delta y, \delta z)$ to $f(\mathbf{x} + \delta\mathbf{x}, t + \delta t)$ over a time interval δt . Since $f(\mathbf{x}, t)$ and $f(\mathbf{x} + \delta\mathbf{x}, t + \delta t)$ are the attenuation coefficients of the same tissue point, they should be equal if there is no compression or expansion. Because the systole or diastole stages are unavoidable in cardiac imaging, we introduce a coefficient $R(\delta t)$ to adjust the relationship between $f(\mathbf{x}, t)$ and $f(\mathbf{x} + \delta\mathbf{x}, t + \delta t)$. That is,

$$f(\mathbf{x} + \delta\mathbf{x}, t + \delta t) = R(\delta t) f(\mathbf{x}, t). \quad (3)$$

Performing the first order Taylor series expansion, we have

$$f(\mathbf{x} + \delta\mathbf{x}, t + \delta t) = f(\mathbf{x}, t) + \frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}} \bullet \delta\mathbf{x} + \frac{\partial f(\mathbf{x}, t)}{\partial t} \bullet \delta t + H.O.T., \quad (4)$$

where $\frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}} = \left(\frac{\partial f(\mathbf{x}, t)}{\partial x}, \frac{\partial f(\mathbf{x}, t)}{\partial y}, \frac{\partial f(\mathbf{x}, t)}{\partial z} \right)$, $H.O.T.$ denotes the Higher Order Terms which can be ignored. Combining Eqs. (3) and (4), we arrive at:

$$\frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}} \bullet \mathbf{u} + \frac{\partial f(\mathbf{x}, t)}{\partial t} + rf(\mathbf{x}, t) = 0, \quad (5)$$

where

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = \lim_{\delta t \rightarrow 0} \frac{\delta \mathbf{x}}{\delta t} = \left(\lim_{\delta t \rightarrow 0} \frac{\delta x}{\delta t}, \lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t}, \lim_{\delta t \rightarrow 0} \frac{\delta z}{\delta t} \right) = (u, v, w) \quad (6)$$

and

$$r = \lim_{\delta t \rightarrow 0} \frac{1 - R(\delta t)}{\delta t}. \quad (7)$$

Based on the mass conservation, in the appendix A we show

$$r = u_x + v_y + w_z, \quad (8)$$

we have $u_x = \frac{\partial u(\mathbf{x}, t)}{\partial x}$, $v_y = \frac{\partial v(\mathbf{x}, t)}{\partial y}$ and $w_z = \frac{\partial w(\mathbf{x}, t)}{\partial z}$. Letting $f_x = \frac{\partial f(\mathbf{x}, t)}{\partial x}$, $f_y = \frac{\partial f(\mathbf{x}, t)}{\partial y}$, $f_z = \frac{\partial f(\mathbf{x}, t)}{\partial z}$ and $f_t = \frac{\partial f(\mathbf{x}, t)}{\partial t}$, we finally arrive at the velocity field constraint equation (VFCE):

$$f_x u + f_y v + f_z w + f_t + (u_x + v_y + w_z) f = 0. \quad (9)$$

In the 2D case, it is easy to express the corresponding VFCE as

$$f_x u + f_y v + f_t + (u_x + v_y) f = 0. \quad (10)$$

When the object motion is rigid, $R(\delta t) = 1$ holds and leads to $r = 0$ and

$$f_x u + f_y v + f_t = 0. \quad (11)$$

Although the 2D VFCE is essentially the same as the optical flow constraint equation (OFCE) in the computer vision field [15], it should be pointed out that VFCE is different from OFCE in the following aspects. First, the function f represents the x-ray attenuation coefficient distribution in VFCE, while it denotes the brightness of an image in OFCE. Second, the velocity field in VFCE is the original 2D velocity field which can be recovered directly, while the optical flow field in OFCE is generally the 2D projection of the corresponding 3D velocity field. Third, the measured data of velocity tomography are fan-beam/cone-beam integral projections, while measured data of the optical flow problem are 2D image sequences. Fourth, the 2D VFCE can only be used for the 2D motion while the corresponding OFCE can be employed to study a general optical flow distribution. By relating the motion and structural parameters to the optical flow, the 3D motion can be recovered from the estimated optical flow [16]. Thus, the VFCE and OFCE are two complementary equations for different problems.

IV. Two-step Reconstruction Scheme

Because the vector field is constrained by the VFCE in our velocity tomography, we propose the following two-step general reconstruction scheme: Step 1 is to reconstruct the partial

derivatives $\frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}}$ and $\frac{\partial f(\mathbf{x}, t)}{\partial t}$ from measured projections; and Step 2 is to solve the VFCE for the final velocity field. Because VFCE has the structure similar to that of OFCE, the second step can be implemented using an optical flow estimation method [13,14]. Regarding Step 1, we can either first reconstruct the attenuation coefficient $f(\mathbf{x}, t)$ then compute the partial derivatives (the indirect method), or reconstruct the partial derivatives directly from measured datasets (the direct method). Since the indirect method is not as accurate as the direct method, in this proof of concept paper we will focus on the direct method.

In the CT field, the 2D parallel-beam is the simplest case. It is well known that any reconstruction method in the parallel-beam geometry can be extended to the fan-beam and

cone-beam cases. Without loss of generality, let us consider the planar motion in the 2D parallel-beam geometry. In this 2D case, we denote the object function and projection as $f(x, y, t)$ and $p(l, \phi, t)$ as shown in Figure 1. Assuming that $f(x, y, t)$ be a smooth dynamic function, we express the Radon transform and inverse Radon transform as [17]

$$p(l, \phi, t) = \iint_{\mathbb{R}^2} f(x, y, t) \delta(x \cos \phi + y \sin \phi - l) dx dy, \quad (12)$$

$$f(x, y, t) = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{1}{x \cos \phi + y \sin \phi - l} \frac{\partial p(l, \phi, t)}{\partial l} dl d\phi. \quad (13)$$

The inverse Radon Transform is in a filtered backprojection (FBP) format, which means that $f(x, y, t)$ can be reconstructed from $p(l, \phi, t)$ in the three steps: computing the partial derivative of $p(l, \phi, t)$ with respect to l , performing a Hilbert Filtering and backprojecting the filtered data. The first two steps can be combined into a Ramp filtering process:

$$f(x, y, t) = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{p(l, \phi, t)}{(x \cos \phi + y \sin \phi - l)^2} dl d\phi, \quad (14)$$

because the order of the derivative and convolution operations can be exchanged for a smooth function. Now, let us consider the partial derivatives of $f(x, y, t)$. From Eq.(13), we have

$$\frac{\partial f(x, y, t)}{\partial t} = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{1}{x \cos \phi + y \sin \phi - l} \frac{\partial^2 p(l, \phi, t)}{\partial l \partial t} dl d\phi, \quad (15)$$

$$\frac{\partial f(x, y, t)}{\partial x} = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{-\cos \phi}{(x \cos \phi + y \sin \phi - l)^2} \frac{\partial p(l, \phi, t)}{\partial l} dl d\phi, \quad (16)$$

$$\frac{\partial f(x, y, t)}{\partial y} = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{-\sin \phi}{(x \cos \phi + y \sin \phi - l)^2} \frac{\partial p(l, \phi, t)}{\partial l} dl d\phi. \quad (17)$$

Similar to Eq.(14), Eq.(15) can be rewritten as

$$\frac{\partial f(x, y, t)}{\partial t} = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{1}{(x \cos \phi + y \sin \phi - l)^2} \frac{\partial p(l, \phi, t)}{\partial l} dl d\phi, \quad (18)$$

and Eq.(16) becomes

$$\frac{\partial f(x, y, t)}{\partial x} = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{-\cos \phi}{x \cos \phi + y \sin \phi - l} \frac{\partial^2 p(l, \phi, t)}{\partial l^2} dl d\phi, \quad (19)$$

$$\frac{\partial f(x, y, t)}{\partial x} = \frac{1}{\pi^2} \int_0^\pi \int_{-\infty}^\infty \frac{-\cos \phi p(l, \phi, t)}{(x \cos \phi + y \sin \phi - l)^3} dl d\phi. \quad (20)$$

Since the formula for $\frac{\partial f(x, y, t)}{\partial y}$ is similar to that for $\frac{\partial f(x, y, t)}{\partial x}$ except for the triangular

weighting function, we will not list that for $\frac{\partial f(x, y, t)}{\partial y}$. It should be pointed out that Eqs. (15) and (18) have different physical meanings and associated with different reconstruction methods. The same comments apply to Eqs. (16), (19) and (20). Clearly, all the aforementioned direct computations are in the FBP format. By exchanging the order of the integrations, we can have a backprojected filtration (BPF) counterpart for any FBP method. This means that there are many variants of our scheme.

Furthermore, all the methods can be extended to the fan-beam or cone-beam geometry. For example, one possible extension of Eq.(19) into the fan-beam geometry can be summarized as the following theorem 4.1:

Theorem 4.1

As shown in Figure 2, let L be a chord from $\mathbf{a}(s_1, t)$ to $\mathbf{a}(s_2, t)$ along a 2D differentiable general curve Γ at the time t . Consider a bounded smooth function $f(\mathbf{x}, t)$ within a compact support Ω .

For $\mathbf{x} \in L$ and $\mathbf{x} \notin \Gamma$, we have

$$\begin{aligned} \frac{\partial f(\mathbf{x}, t)}{\partial(\mathbf{x} \bullet \mathbf{d})} = & -\frac{1}{2\pi^2} \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \int_0^{2\pi} \left(\frac{\partial^2}{\partial q^2} (D_f(\mathbf{a}(q, t), \theta(\gamma, t))) \right) \Big|_{q=s} \frac{d\gamma}{\sin \gamma} \\ & + \frac{1}{2\pi^2} \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \frac{(\mathbf{a}'' \bullet \theta^\perp)}{(\mathbf{a}' \bullet \theta^\perp)} \int_0^{2\pi} \left(\frac{\partial}{\partial q} (D_f(\mathbf{a}(q, t), \theta(\gamma, t))) \right) \Big|_{q=s} \frac{d\gamma}{\sin \gamma}, \end{aligned} \quad (21)$$

where $\mathbf{a} = \mathbf{a}(s, t)$,

$$\mathbf{e} = \mathbf{e}(s_1, s_2, t) = \frac{\mathbf{a}(s_2, t) - \mathbf{a}(s_1, t)}{\|\mathbf{a}(s_2, t) - \mathbf{a}(s_1, t)\|}, \quad \theta = \theta(\mathbf{x}, s, t) = \frac{\mathbf{x} - \mathbf{a}(s, t)}{\|\mathbf{x} - \mathbf{a}(s, t)\|}, \quad \mathbf{a}' = \mathbf{a}'(s, t) = \frac{d\mathbf{a}(s, t)}{ds}, \quad \mathbf{a}'' = \mathbf{a}''(s, t) = \frac{d^2\mathbf{a}(s, t)}{ds^2},$$

, $\theta^\perp = \theta^\perp(\mathbf{x}, s, t)$ represents a unit vector perpendicular to $\theta(\mathbf{x}, s, t)$, and $\mathbf{a}' \cdot \theta^\perp \neq 0$.

Note that $(\mathbf{a}' \cdot \theta^\perp)$ may be equal to zero when a line is tangential to the scanning curve and through $\mathbf{x} \in \Omega$. Because this generally does not happen in practice, it is reasonable to assume $\mathbf{a}' \cdot \theta^\perp \neq 0$ in Theorem 4.1. For a given unit vector $\mathbf{d}$, $\frac{\partial f(\mathbf{x}, t)}{\partial(\mathbf{x} \bullet \mathbf{d})}$ is the partial derivative of $f(\mathbf{x}, t)$

$$\begin{aligned} \frac{\partial f(\mathbf{x}, t)}{\partial x} &= \frac{\partial f(\mathbf{x}, t)}{\partial(\mathbf{x} \bullet \mathbf{d})} \Big|_{\mathbf{d}=(1,0)} \text{ and } \\ \frac{\partial f(\mathbf{x}, t)}{\partial y} &= \frac{\partial f(\mathbf{x}, t)}{\partial(\mathbf{x} \bullet \mathbf{d})} \Big|_{\mathbf{d}=(0,1)}. \end{aligned}$$

A detail proof of this theorem is in Appendix B.

V. Numerical Results

To demonstrate the correctness and utility of the proposed two-step scheme, we simulated the 2D case in the fan-bam geometry using Matlab, with all the computationally intensive parts coded in C. The partial derivatives were computed using both the indirect and direct methods. In the indirect method, the partial derivatives were calculated by the 2-points difference formulae from the reconstructed images. In the direct method, the partial derivatives were directly calculated from the projection data according to Theorem 4.1. Because Eq.(21) is similar to our lambda tomography formulae [18], the reader can refer to our previous papers for the detailed implementation [18-21]. To recover the velocity field, we need to solve Eq. (10). However, we cannot determine the velocity field locally from one equation without introducing additional constraints. In this preliminary study, we employed the smoothness constraint developed by Horn and Schunck [13]. Letting

$$\mathcal{E}_b = f_x u + f_y v + f_t + (u_x + v_y) f, \quad (22)$$

$$\mathcal{E}_c^2 = u_x^2 + u_y^2 + v_x^2 + v_y^2, \quad (23)$$

the following total error can be minimized:

$$\mathcal{E}^2 = \int_{\mathbb{R}^2} (\mathcal{E}_b^2 + \alpha^2 \mathcal{E}_c^2) dx dy, \quad (24)$$

where α^2 is a weight factor. The solution of the velocity field can be obtained after the minimization is accomplished. Following the derivation by Horn and Schunck [13], we have

$$\begin{cases} (\alpha^2 + f_x^2 + f_y^2)(\bar{u} - \bar{u}) = -f_x(f_x \bar{u} + f_y \bar{v} + f_t + (u_x + v_y)f) \\ (\alpha^2 + f_x^2 + f_y^2)(\bar{v} - \bar{v}) = -f_y(f_x \bar{u} + f_y \bar{v} + f_t + (u_x + v_y)f), \end{cases} \quad (25)$$

where $\bar{u}$ and $\bar{v}$ are the local average of u and v , which can be approximately estimated by a weighted average operator (see P190 in [13]). Eq.(25) implies an iterative solution for the velocity field

$$\begin{cases} u(k+1) = \bar{u}(k) - f_x(f_x \bar{u}(k) + f_y \bar{v}(k) + f_t + (u_x(k) + v_y(k))f) / (\alpha^2 + f_x^2 + f_y^2) \\ v(k+1) = \bar{v}(k) - f_y(f_x \bar{u}(k) + f_y \bar{v}(k) + f_t + (u_x(k) + v_y(k))f) / (\alpha^2 + f_x^2 + f_y^2), \end{cases} \quad (26)$$

where the integer k is the iteration index.

In our study, we selected a 512×512 goat lung image slice as the testing object $f(\mathbf{x}, t)$ shown in Figure 3. We assumed that the lung image was contained in the center of a circular field of view of radius 10 cm. Given the capability of the multi-source CT scanner and the periodicity of the targeted applications, we also assumed that we could acquire sufficient projection data quickly to reconstruct $f(\mathbf{x}, t)$ at the time t for our VT purpose. Hence, we first acquired a set of projection data along a circular scanning trajectory of radius 57 cm simultaneously. During the acquisition process, the object image was kept motionless. The x-rays transmitted from each source were received by a 48cm linear detector with 360 cells at a distance 114 cm from the source. Totally, 720 projections of full-scan were collected for a given time t . Then, we transformed the object image to another status at time $t + \delta t$ with some specific motion pattern and repeated the data acquisition procedure. The standard filtered backprojection method was employed to reconstruct $f(\mathbf{x}, t)$ and $f(\mathbf{x}, t + \Delta t)$, and all the images were reconstructed on 256×256 matrices which were then used to compute the partial derivatives numerically for use of our

indirect method. Figure 4 present partial derivatives $\frac{\partial f(\mathbf{x}, t)}{\partial x}$ and $\frac{\partial f(\mathbf{x}, t)}{\partial y}$ computed using the direct and indirect methods. Assuming that $\Delta t = 1.0$, three basic motion patterns were simulated which involved translation, rotation and contraction, respectively. In the translation motion mode, the whole image was moved -1.0 pixel along the x-axis and 1.0 pixel along the y-axis. In the rotation motion mode, the whole image was clockwise rotated 0.5° with respect to the center of the circular field of view. In the contraction mode, the object image was linearly scaled from a 512×512 matrix to a 511×511 matrix with respect to the center of the field of view. For use of the iterative procedure to estimate the velocity field, we set the maximum iteration number to 400, and α^2 to the average of $(f_x^2 + f_y^2)$ on the whole support,

$$\alpha^2 = \left(\int_{\Omega} (f_x^2 + f_y^2) dx dy \right) / \int_{\Omega} dx dy. \quad (27)$$

Figure 5 plot the estimated velocity fields. From Figures 4 and 5, we see that the indirect and direct methods for computing the partial derivatives produced similar velocity fields although the direct method generated more accurate partial derivatives.

VI. Discussions & Conclusion

In this paper, we have proposed the concept of velocity tomography and the corresponding reconstruction scheme. Our velocity tomography is a combination of tomographic techniques and optical flow methods. Hence, it is different from the existing results in either the computer vision field [13] or the CT field [10]. To recover a velocity field at a time t , our velocity tomography requires sufficient projection data for reconstruction of $f(\mathbf{x}, t)$. With the rapid development of multi-source CT scanners [4,5], it is already feasible to reconstruct a beating heart. That is, the temporal resolution is sufficiently good to obtain decent images at all cardiac

phases (including diastolic and systolic phases). Therefore, one application of our proposal scheme is to study the dynamics of the heart, which may be compared to the situation with the optical flow analysis on image sequences in the computer vision field, yielding utilities in dynamic image segmentation, data compression, *etc.*

Based on the mass conservation, we have derived the velocity field constraint equation which is similar to but different from the optical flow constraint equation. We have assumed that the x-ray attenuation coefficients are homogeneous and proportional to the mass intensity. In fact, the motion of an object is caused by a force. From the point view of biomechanics, we may further perform an analysis of the stress-strain to make the constraint more practical and more powerful. Furthermore, we may model the attenuation coefficients and the mass intensity and determine the parameters in experiments under different conditions. There are the direct and indirect methods to compute the partial derivatives of the object function. While we have many ways to compute the partial derivatives in the parallel-beam geometry directly, there remains still much work to be done in the fan-beam and cone-beam geometry. Although the partial derivatives are used to compute the velocity field in this paper, we believe our direct method can be applied to other applications as well. In the numerical simulation, we have used the smoothness as the additional constraint [13]. However, this is not the only knowledge available. In fact, we have more options in addition to the smoothness constraint. In the future, we may analyze and optimize the additional penalty terms [14], as well as other application dependent rules. Also, we may perform more quantitative comparison between the direct and indirect methods for computing the partial derivatives.

When reviewing our initial submission, a reviewer brought the related recent work by Taguchi *et al.* [22,23] to our attention. By incorporating smoothness constraints, they proposed a block-matching algorithm (BMA) to estimate 2D components of motion vectors from a sequence of cardiac images. The motion vectors were iteratively estimated by minimizing a cost function including image matching and regularization terms in space and time. Although both Taguchi's BMA and our VT methods are to estimate a velocity field, they are totally different. The reasons are as follows. First, the BMA method does not take into account the change of object attenuation coefficients during the dynamic procedure, while the VFCE of the VT scheme incorporates these changes under the constraint of the mass conservation. Second, the image matching term of the cost function in the BMA method is constructed based on the image itself, while our VT scheme employs the VFCE based on the partial derivatives of the image. Third, the BMA method iteratively estimates the vector field by minimizing the cost function using an optimization technology, while our VT scheme offers an estimate of the vector field by solving the corresponding partial differential equation (PDE) using an iterative method.

In conclusion, we have proposed the concept of velocity tomography, formulated a two-step reconstruction scheme, and demonstrated its correctness and utility. Our work represents a combination of tomographic image reconstruction and optical flow recovery. Further research efforts are needed along this new direction towards novel biomedical applications, such as CT elastography.

Acknowledgment

This work is partially supported by the NIH grants (EB002667, EB004287 and EB007288). The authors also express their gratitude to the anonymous reviewers for their valuable comments and suggestions.

Appendix A: Derivative of Eq. (8) Based on the Mass Conservation

As shown in Figure 6, let us consider a fixed point $\mathbf{x}(t) = (x, y, z)$ and a small cubic neighbor region $\Omega_c(t)$ at the time t . Without loss of generality, we assume any point in the region can be

written as $(\mathbf{x} + \delta\mathbf{x})(t)$ with $\delta\mathbf{x} = (\delta x, \delta y, \delta z)$, $-\varepsilon < \delta x < \varepsilon$, $-\varepsilon < \delta y < \varepsilon$ and $-\varepsilon < \delta z < \varepsilon$. Evidently, the volume of this small region is:

$$\delta V(t) = \int \int \int_{\Omega(t)} d\delta\mathbf{x} = 8\varepsilon^3, \quad (\text{A1})$$

After a small time duration δt , all the points in $\Omega_\varepsilon(t)$ move to $(\mathbf{x} + \delta\mathbf{x})(t + \delta t)$ within $\Omega_\varepsilon(t + \delta t)$. Since δt is very small, we can write $(\mathbf{x} + \delta\mathbf{x})(t + \delta t)$ as:

$$\begin{aligned} (\mathbf{x} + \delta\mathbf{x})(t + \delta t) &= (\mathbf{x} + \delta\mathbf{x})(t) + \mathbf{u}(\mathbf{x} + \delta\mathbf{x}, t) \delta t + H.O.T. \\ &= (\mathbf{x} + \delta\mathbf{x})(t) + \left(\mathbf{u}(\mathbf{x}, t) + \frac{\mathbf{u}(\mathbf{x}, t)}{\partial \mathbf{x}} \bullet \delta\mathbf{x} \right) \delta t + H.O.T. \\ &= \mathbf{x}(t) + \mathbf{u}(\mathbf{x}, t) \delta t + \delta\mathbf{x} + \frac{\mathbf{u}(\mathbf{x}, t)}{\partial \mathbf{x}} \bullet \delta\mathbf{x} \delta t + H.O.T. \end{aligned} \quad (\text{A2})$$

The volume of $\Omega_\varepsilon(t + \delta t)$ can be computed as

$$\begin{aligned} \delta V(t + \delta t) &= \int \int \int_{\Omega_\varepsilon(t + \delta t)} d\left(\delta\mathbf{x} + \frac{\mathbf{u}(\mathbf{x}, t)}{\partial \mathbf{x}} \bullet \delta\mathbf{x} \delta t \right) \\ &= \int \int \int_{\Omega_\varepsilon(t)} J(\delta t) d\delta\mathbf{x} = J(\delta t) \delta V(t), \end{aligned} \quad (\text{A3})$$

where $J(\delta t)$ is the Jacobi factor

$$J(\delta t) = \det \begin{vmatrix} 1 + \frac{\partial u(\mathbf{x}, t)}{\partial x} \delta t & \frac{\partial u(\mathbf{x}, t)}{\partial y} \delta t & \frac{\partial u(\mathbf{x}, t)}{\partial z} \delta t \\ \frac{\partial v(\mathbf{x}, t)}{\partial x} \delta t & 1 + \frac{\partial v(\mathbf{x}, t)}{\partial y} \delta t & \frac{\partial v(\mathbf{x}, t)}{\partial z} \delta t \\ \frac{\partial w(\mathbf{x}, t)}{\partial x} \delta t & \frac{\partial w(\mathbf{x}, t)}{\partial y} \delta t & 1 + \frac{\partial w(\mathbf{x}, t)}{\partial z} \delta t \end{vmatrix}. \quad (\text{A4})$$

In the small regions $\Omega_\varepsilon(t)$ and $\Omega_\varepsilon(t + \delta t)$, we can further assume that the attenuation coefficients are homogeneous and proportional to the mass intensity. Based on the principle of mass conservation, we can determine the ratio of the linear attenuation coefficients in this small region as:

$$R(\delta t) = \frac{\delta V(t)}{\delta V(t + \delta t)} = \frac{1}{J(\delta t)}. \quad (\text{A5})$$

Therefore, we have

$$r = \lim_{\delta t \rightarrow 0} \frac{1 - R(\delta t)}{\delta t} = \frac{\partial u(\mathbf{x}, t)}{\partial x} + \frac{\partial v(\mathbf{x}, t)}{\partial y} + \frac{\partial w(\mathbf{x}, t)}{\partial z} = u_x + v_y + w_z. \quad (\text{A6})$$

Appendix B: Proof of Theorem 4.1

Because all the variables in Theorem 4.1 depend on the time t , we will omit t in this proof for simplicity. First, let us define the Fourier Transform and inverse Fourier Transform of a 2D bounded smooth function $f(\mathbf{x})$ as

$$\begin{cases} \widehat{f}(\xi) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-i\mathbf{x} \bullet \xi} d\mathbf{x} \\ f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{f}(\xi) e^{i\mathbf{x} \bullet \xi} d\xi \end{cases}. \quad (\text{B1})$$

Hence,

$$\begin{aligned} & \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \int_0^{2\pi} \left(\frac{\partial^2}{\partial q^2} (D_f(\mathbf{a}(q), \theta(\gamma))) \right) \Big|_{q=s} \frac{d\gamma}{\sin \gamma} \\ &= \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \left(\int_0^{2\pi} \frac{\partial}{\partial^2 q} \left(\frac{1}{(2\pi)^2} \int_0^\infty d\lambda \int_{\mathbb{R}^2} d\xi \widehat{f}(\xi) e^{i\xi \bullet (\mathbf{a}(q) + \lambda \theta(\gamma))} \right) \Big|_{q=s} \frac{d\gamma}{\sin \gamma} \right) \\ &= \frac{1}{(2\pi)^2} \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \left(\int_0^{2\pi} \int_0^\infty d\lambda \int_{\mathbb{R}^2} d\xi (i\xi \bullet \mathbf{a}'' - (\xi \bullet \mathbf{a}')^2) \widehat{f}(\xi) e^{i\xi \bullet (\mathbf{a}(s) + \lambda \theta(\gamma))} \frac{d\gamma}{\sin \gamma} \right) \\ &= \frac{1}{(2\pi)^2} \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \left(\int_{\mathbb{R}^2} d\xi (i\xi \bullet \mathbf{a}'' - (\xi \bullet \mathbf{a}')^2) \widehat{f}(\xi) e^{i\xi \bullet \mathbf{a}} \int_0^{2\pi} \int_0^\infty d\lambda e^{i\xi \bullet (\lambda \theta(\gamma))} \frac{d\gamma}{\sin \gamma} \right) \\ &= \frac{1}{(2\pi)^2} \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \left(\int_{\mathbb{R}^2} d\xi (i\xi \bullet \mathbf{a}'' - (\xi \bullet \mathbf{a}')^2) \widehat{f}(\xi) e^{i\xi \bullet \mathbf{a}} (2\pi \delta(\xi \bullet \theta)) (\pi i \text{sgn}(\xi \bullet \theta^\perp)) \right) \\ &= \frac{1}{(2\pi)^2} \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \text{sgn}(\mathbf{e} \bullet \theta^\perp)}{\mathbf{a}' \bullet \theta^\perp} \left(\int_{\mathbb{R}^2} d\xi (i\xi \bullet \mathbf{a}'' - (\xi \bullet \mathbf{a}')^2) \widehat{f}(\xi) e^{i\xi \bullet \mathbf{a}} (2\pi \delta(\xi \bullet (\mathbf{x} - \mathbf{a}))) (\pi i \text{sgn}(\xi \bullet \theta^\perp)) \right), \end{aligned} \quad (\text{B2})$$

where the following result has been used

$$\begin{aligned}
 & \int_0^{2\pi} \int_0^\infty d\lambda e^{i\xi \bullet (\lambda \theta(\gamma))} \frac{d\gamma}{\sin \gamma} = \int_0^{2\pi} \int_0^\infty d\lambda e^{i\lambda \xi \bullet (\theta \cos \gamma + \theta^\perp \sin \gamma)} \frac{d\gamma}{\sin \gamma} \\
 & = \int_0^{2\pi} \int_0^\infty d\lambda e^{i\lambda \xi \bullet (\theta \cos \gamma + \theta^\perp \sin \gamma)} \frac{\lambda d\gamma}{\lambda \sin \gamma} \\
 & = \int_{-\infty}^\infty e^{i(\xi \bullet \theta)u} du \int_{-\infty}^\infty e^{i(\xi \bullet \theta^\perp)v} \frac{dv}{v} \Big|_{u=\lambda \cos \gamma, v=\lambda \sin \gamma} \\
 & = (2\pi \delta(\xi \bullet \theta)) (\pi i \operatorname{sgn}(\xi \bullet \theta^\perp)).
 \end{aligned} \tag{B3}$$

Due to $\delta(\xi \bullet (\mathbf{x} - \mathbf{a}(t)))$, we set $\xi \bullet \mathbf{x} = \xi \bullet \mathbf{a}(t)$. Because both $\theta^\perp$ and ξ are perpendicular to the vector θ in the 2D space, we have $\theta^\perp = \pm \xi / \|\xi\|$. Consequently, (B2) can be simplified as

$$\begin{aligned}
 & \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \operatorname{sgn}(\mathbf{e} \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \int_0^{2\pi} \left(\frac{\partial^2}{\partial q^2} (D_f(\mathbf{a}(q), \theta(\gamma))) \Big|_{q=s} \right) \frac{d\gamma}{\sin \gamma} \\
 & = \frac{1}{(2\pi)^2} \int_{s_1}^{s_2} ds \frac{(\xi \bullet \mathbf{d}) \operatorname{sgn}(\mathbf{e} \bullet \xi)}{\mathbf{a}' \bullet \xi} \left(\int_{\mathbb{R}^2} d\xi \left(i\xi \bullet \mathbf{a}'' - (\xi \bullet \mathbf{a}')^2 \right) \widehat{f}(\xi) e^{i\xi \bullet \mathbf{a}} (2\pi \delta(\xi \bullet (\mathbf{x} - \mathbf{a}))) (\pi i \operatorname{sgn}(\xi \bullet \xi)) \right) \\
 & = \frac{2\pi^2}{(2\pi)^2} \int_{\mathbb{R}^2} d\xi \widehat{f}(\xi) (i\xi \bullet \mathbf{d}) e^{i\xi \bullet \mathbf{x}} \operatorname{sgn}(\mathbf{e} \bullet \xi) \int_{s_1}^{s_2} ds \left(\frac{i\xi \bullet \mathbf{a}''}{\mathbf{a}' \bullet \xi} - \xi \bullet \mathbf{a}' \right) \delta(\xi \bullet (\mathbf{x} - \mathbf{a})).
 \end{aligned} \tag{B4}$$

On the other hand, we have

$$\begin{aligned}
 & \int_{s_1}^{s_2} ds \frac{(\theta^\perp \bullet \mathbf{d}) \operatorname{sgn}(\mathbf{e} \bullet \theta^\perp) (\mathbf{a}' \bullet \theta^\perp)}{\|\mathbf{x} - \mathbf{a}\| \cdot (\mathbf{a}' \bullet \theta^\perp)} \int_0^{2\pi} \left(\frac{\partial}{\partial q} (D_f(\mathbf{a}(q, t), \theta(\gamma, t))) \Big|_{q=s} \right) \frac{d\gamma}{\sin \gamma} \\
 & = \frac{2\pi^2}{(2\pi)^2} \int_{\mathbb{R}^2} d\xi \widehat{f}(\xi) (i\xi \bullet \mathbf{d}) e^{i\xi \bullet \mathbf{x}} \operatorname{sgn}(\mathbf{e} \bullet \xi) \int_{s_1}^{s_2} ds \left(\frac{i\xi \bullet \mathbf{a}''}{\mathbf{a}' \bullet \xi} \right) \delta(\xi \bullet (\mathbf{x} - \mathbf{a})).
 \end{aligned} \tag{B5}$$

By the relationship

$$\begin{aligned}
 & \int_{s_1}^{s_2} ds (\xi \bullet \mathbf{a}'(s)) \delta(\xi \bullet (\mathbf{x} - \mathbf{a}(s))) \\
 & = \frac{1}{2\pi} \int_{s_1}^{s_2} ds (\xi \bullet \mathbf{a}'(s)) \int_{-\infty}^\infty d\lambda e^{i\lambda \xi \bullet (\mathbf{x} - \mathbf{a}(s))} = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \int_{s_1}^{s_2} ds (\xi \bullet \mathbf{a}'(s)) e^{i\lambda \xi \bullet (\mathbf{x} - \mathbf{a}(s))} \\
 & = -\frac{1}{2\pi} \int_{-\infty}^\infty \int_{(\mathbf{x} - \mathbf{a}(s_1)) \bullet \xi}^{(\mathbf{x} - \mathbf{a}(s_2)) \bullet \xi} e^{i\lambda u} du d\lambda = \frac{i}{2\pi} \int_{-\infty}^\infty \frac{e^{i\lambda (\mathbf{x} - \mathbf{a}(s_2)) \bullet \xi} - e^{i\lambda (\mathbf{x} - \mathbf{a}(s_1)) \bullet \xi}}{\lambda} d\lambda \\
 & = \frac{\operatorname{sgn}((\mathbf{x} - \mathbf{a}(s_1)) \bullet \xi) - \operatorname{sgn}((\mathbf{x} - \mathbf{a}(s_2)) \bullet \xi)}{2} = \operatorname{sgn}(\mathbf{e} \bullet \xi),
 \end{aligned} \tag{B6}$$

and (B5) being subtracted from (B4), the right side of Eq.(21) becomes

$$\begin{aligned}
 & = \frac{2\pi^2}{(2\pi)^2} \int_{\mathbb{R}^2} d\xi \widehat{f}(\xi) (i\xi \bullet \mathbf{d}) e^{i\xi \bullet \mathbf{x}} \operatorname{sgn}(\mathbf{e} \bullet \xi) \int_{s_1}^{s_2} ds (\xi \bullet \mathbf{a}') \delta(\xi \bullet (\mathbf{x} - \mathbf{a})) \\
 & = \frac{2\pi^2}{(2\pi)^2} \int_{\mathbb{R}^2} d\xi \widehat{f}(\xi) (i\xi \bullet \mathbf{d}) e^{i\xi \bullet \mathbf{x}} \operatorname{sgn}(\mathbf{e} \bullet \xi) \operatorname{sgn}(\mathbf{e} \bullet \xi) \\
 & = \frac{2\pi^2}{(2\pi)^2} \int_{\mathbb{R}^2} d\xi \widehat{f}(\xi) (i\xi \bullet \mathbf{d}) e^{i\xi \bullet \mathbf{x}} = 2\pi^2 \frac{\partial f(\mathbf{x})}{\partial(\mathbf{x} \bullet \mathbf{d})}.
 \end{aligned} \tag{B7}$$

This completes the proof of Theorem 4.1.

References

1. Ehrhardt J, et al. An optical flow based method for improved reconstruction of 4D CT data sets acquired during free breathing. *Medical Physics* 2007;34(2):711–721. [PubMed: 17388189]
2. Pan T, et al. 4D-CT imaging of a volume influenced by respiratory motion on multi-slice CT. *Medical Physics* 2004;31(2):333–340. [PubMed: 15000619]
3. Wang G, Zhao S, Heuscher D. A knowledge-based cone-beam x-ray CT algorithm for dynamic volumetric cardiac imaging. *Med. Phys* 2002;29(8):1807–1822. [PubMed: 12201428]
4. DeMan, B.; Basu, S. Method and apparatus for employing multiple axial-sources. US Patent 2005/0135550 A1. 2005.
5. Sukovie, P., et al. Multiple source CT scanner. US Patent 2006/0285633 A1. 2006.
6. Wang G, Ye Y, Yu H. Appropriate and exact cone-beam reconstruction with standard and nonstandard spiral scanning. *Physics in Medicine and Biology* 2007;52(6):R1–R13. [PubMed: 17327647]
7. Ye YB, Wang G. Filtered backprojection formula for exact image reconstruction from cone-beam data along a general scanning curve. *Medical Physics* 2005;32(1):42–48. [PubMed: 15719953]
8. Ye YB, et al. A General Exact Reconstruction for Cone-Beam CT via Backprojection-Filtration. *IEEE Transactions on Medical Imaging* 2005;24(9):1190–1198. [PubMed: 16156356]

9. Zhao S, Yu H, Wang G. A unified framework for exact cone-beam reconstruction formulas. *Medical Physics* 2005;32(6):1712–1721. [PubMed: 16013729]
10. Prince JL. Tomographic Reconstruction Of 3-D Vector-Fields Using Inner-Product Probes. *Ieee Transactions On Image Processing* 1994;3(2):216–219. [PubMed: 18291921]
11. Osman NF, Prince JL. 3D vector tomography on bounded domains. *Inverse Problems* 1998;14(1):185–196.
12. Braun H, Hauck A. Tomographic Reconstruction Of Vector-Fields. *Ieee Transactions On Signal Processing* 1991;39(2):464–471.
13. Horn BKP, Schunck BG. Determining optical flow. *Artificial Intelligence* 1981;17:185–203.
14. Beauchemin SS, Barron JL. The computation of optical flow. *Acm Computing Surveys* 1995;27(3):433–467.
15. Berezat D, Berroir JP. Motion estimation on meteorological infrared data using a total brightness invariance hypothesis. *Environmental Modelling & Software* 2000;15(67):513–519.
16. Verri A, Poggio T. Motion Field And Optical-Flow - Qualitative Properties. *Ieee Transactions On Pattern Analysis And Machine Intelligence* 1989;11(5):490–498.
17. Radon J. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Ber. Verh. Sachs. Akad. Wiss. Leipzig- Math.-Natur* 1917;69:262–277.
18. Yu H, Wang G. A general formula for fan-beam lambda tomography. *International Journal of Biomedical Imaging* 2006;2006(1)Article ID: 10427, 9 pages
19. Yu H, Wang G. Studies on implementation of the Katsevich algorithm for spiral cone-beam CT. *Journal of X-ray Science and Technology* 2004;12(2):96–117.
20. Yu H, Ye Y, Wang G. Practical Cone-beam Lambda Tomography. *Medical Physics* 2006;33(10):3640–3646. [PubMed: 17089830]
21. Yu H, et al. Fan-beam and cone-beam lambda tomography for discontinuous scanning loci. *Physics In Medicine And Biology* 2007;52(14):4331–4344. [PubMed: 17664611]
22. Taguchi, K., et al. Toward Time Resolved Cardiac CT Images with Patient Dose Reduction: Image-based Motion Estimation; *IEEE Nuclear Science Symposium Conference*; 2006. p. P2029-2032.
23. Taguchi, K., et al. Image-domain motion compensated time-resolved 4D cardiac CT; *Medical Imaging 2007: Physics of Medical Imaging, Proc. of SPIE Vol. 6510*; 2007. Paper ID: 651016

Figure 1.
Radon Transform in the parallel-beam geometry.

Figure 2.
Coordinate system for the proof of Theorem 4.1.

Figure 3.
Representative goat lung image used in our experimental study with the display window [0,2].

Figure 4.

Partial derivatives computed using the direct and indirect methods, respectively. The left column images were computed using the direct method, while the right ones using the indirect method. The top row images are the derivatives along the horizontal direction, while the bottom ones along the vertical direction. The display windows were always set to $[-6,6]$.

Figure 5. Estimated velocity fields using our velocity tomography approach. The top, middle and bottom rows show the estimated velocity fields due to translation, rotation and contraction, respectively. The left column plots were computed from the partial derivatives obtained using the direct method, while the right column counterparts using the indirect method. Arrows were automatically scaled to fit within the grid.

Figure 6.

Differential motion in a small neighbor region of $\mathbf{x}(t)$.