ERGODIC THEORY AND TRANSLATION-INVARIANT OPERATORS*

By A. P. CALDERÓN

UNIVERSITY OF CHICAGO

Communicated by Antoni Zygmund, December 13, 1967

It has been known for a long time that there is a close connection between the ergodic theorem and certain facts about functions on the real line, such as the maximal theory of Hardy and Littlewood and the differentiation theorem for indefinite integrals. In more recent times Cotlar¹ has shown that theorems on singular integrals can be generalized and put in the framework of ergodic theory. The purpose of this note is to exhibit a class of results on translation-invariant operators which can be automatically rephrased in such a framework. In this light the maximal ergodic theorem appears as an immediate consequence of the maximal theorem of Hardy and Littlewood, and the results of Cotlar as following from the ordinary theorems on singular integrals. For the sake of simplicity we shall consider only groups of measure-preserving transformations depending on a single real parameter t ; as the reader will immediately see, our statements and their proofs extend without modification to continuous or discrete abelian groups depending on finitely many parameters. Extension of these results to semigroups is discussed briefly at the end of the paper.

Let $\mathfrak{M}$ be a measure space which is totally σ -finite and U^t a one-parameter group of measure-preserving transformations of $\mathfrak{M}$. As usual it will be assumed that for every measurable f on $\mathfrak{M}$ the function $f(U^t x)$ is measurable in the product of $\mathfrak{M}$ with the real line. We will denote by T an operator defined on the space of locally integrable functions on the real line and which has the following properties: the values of T are continuous functions on the real line, T is sublinear and commutes with translations, and finally T is semilocal in the sense that there exists a positive number ϵ such that the support of Tf is always contained in an ϵ -neighborhood of the support of f (for the definition of sublinearity and of weak type, as used below, see ref. 2, vol. 2, page 111). With such an operator T we shall associate an operator $T^\sharp$ on functions on $\mathfrak{M}$ as follows. Given a function $f(x)$ on $\mathfrak{M}$ we let $F(t, x) = f(U^t x)$. If f is the sum of two functions which are bounded and integrable, respectively, then $F(t, x)$ is a locally integrable function of t for almost all x and therefore $G(t, x) = T\{F(t, x)\}$ is a well-defined continuous function of t for almost all x . Thus $g(x) = G(0, x)$ has a meaning and we define $T^\sharp f = g(x)$. Then we have the following

THEOREM 1. *Let T_n be a sequence of operators as above and suppose that the operator $Sf = \sup |T_n f|$ is of strong or weak type (p, p) , $1 \leq p \leq \infty$. Then the same holds for the operator $S^\sharp f = \sup |T_n^\sharp f|$ and $\|S^\sharp\| < \|S\|$.*

If we set

$$(Sf)(t) = \sup_s |s^{-1} \int_0^s f(t+u) du|,$$

then $S^\sharp$ is the maximal ergodic operator, and the preceding theorem asserts that, as a consequence of the maximal theorem of Hardy and Littlewood, this operator

is of weak type (p, p) for $p \geq 1$, and of strong type (p, p) for $p > 1$. If we set

$$(Sf)(t) = \sup_{0 < \alpha, \beta} \left| \int_{\alpha < |u| < \beta} u^{-1} f(t + u) du \right|,$$

then $S^\#$ is the maximal ergodic Hilbert operator of Cotlar, and on the basis of the known properties of the ordinary maximal Hilbert operator, the preceding theorem asserts that the maximal ergodic Hilbert operator is of weak type (p, p) for $1 \leq p < \infty$, and of strong type (p, p) for $1 < p < \infty$.

To derive results on almost everywhere convergence we have the following theorem.

THEOREM 2. *Let $T_n f = k_n * f$, where k_n is bounded and has bounded support. Suppose that $Sf = \sup_n |T_n f|$ is of weak type (p, p) , $1 < p < \infty$, and that $\int k_n(t) dt$ converges and $k_n * \phi$ converges in L^1 , as $n \rightarrow \infty$, for every infinitely differentiable ϕ with compact support and vanishing integral. Then $T_n^\# f$ converges almost everywhere in $\mathfrak{M}$ for every f in $L^p(\mathfrak{M})$.*

If we set $k_n = (1/n)\gamma_n$, where γ_n is the characteristic function of the interval $(-n, 0)$, we obtain the almost everywhere convergence of the ergodic averages. If we set $k_n(t) = 1/t$ for $1/n < |t| < n$ and $k_n(t) = 0$ elsewhere, we obtain the almost everywhere convergence of the truncated ergodic Hilbert transform.

The proof of these results is relatively simple. Let us start with Theorem 1. Without loss of generality we may assume that the sequence T_n is finite, for if the theorem is established in this case, the general case follows by a passage to the limit. Under this assumption the operator S has the same properties as the operators T above. As we did before, we let $G(t, x) = S\{F(t, x)\}$. We note that $F(t, U^s x) = F(t + s, x)$, which means that for any two given values t_1, t_2 of t , $F(t_1, x)$ and $F(t_2, x)$ are equimeasurable functions of x . On the other hand, due to the translation invariance of S , the function $G(t, x)$ has the same property. In fact we have

$$G(t, U^s x) = S\{F(t, U^s x)\} = S\{F(t + s, x)\} = G(t + s, x).$$

Let now $F_a(t, x) = F(t, x)$ if $|t| < a$, $F_a(t, x) = 0$ otherwise, and let $G_a(t, x) = S\{F_a(t, x)\}$. Since S is positive (i.e., its values are nonnegative functions) and sublinear, we have

$$G(t, x) = S\{F\} = S\{F_{a+\epsilon} + (F - F_{a+\epsilon})\} \leq S\{F_{a+\epsilon}\} + S\{F - F_{a+\epsilon}\}$$

and since $F - F_{a+\epsilon}$ has support in $|t| > a + \epsilon$, and S is semilocal, the last term on the right vanishes for $|t| \leq a$ for ϵ sufficiently large, independently of a . Thus we have $G \leq G_{a+\epsilon}$ for $|t| \leq a$. Suppose now that S is of strong type (p, p) . Then since $G(0, x)$ and $G(t, x)$ are equimeasurable functions of x , we have

$$\begin{aligned} 2 \int_{\mathfrak{M}} G(0, x)^p dx &= (1/a) \int_{|t| < a} dt \int_{\mathfrak{M}} G(t, x)^p dx \leq (1/a) \int_{|t| < a} dt \int_{\mathfrak{M}} G_{a+\epsilon}(t, x)^p dx \\ &= (1/a) \int_{\mathfrak{M}} dx \int_{|t| < a} G_{a+\epsilon}(t, x)^p dt \end{aligned}$$

and since $SF_{a+\epsilon} = G_{a+\epsilon}$

$$\int_{|t|<a} G_{a+\epsilon}(t,x)^p dt \leq \|S\|^p \int |F_{a+\epsilon}(t,x)|^p dt,$$

whence substituting above we obtain

$$2 \int_{\mathfrak{M}} G(0,x)^p dx \leq (1/a) \|S\|^p \int_{\mathfrak{M}} dx \int |F_{a+\epsilon}(t,x)|^p dt$$

and again, since $F(0,x)$ and $F(t,x)$ are equimeasurable, the last integral is equal to

$$2(a + \epsilon) \int_{\mathfrak{M}} |F(0,x)|^p dx$$

and

$$\int_{\mathfrak{M}} G(0,x)^p dx \leq (1/a)(a + \epsilon) \|S\|^p \int_{\mathfrak{M}} |F(0,x)|^p dx.$$

Letting a tend to infinity, we obtain the desired result.

Suppose now that S is of weak type (p,p) . For any given $\lambda > 0$, let E and $\bar{E}$ be the sets of points where $G(0,x) > \lambda$ and $G_{a+\epsilon}(t,x) > \lambda$, respectively, and $\bar{E}_x$ the intersection of $\bar{E}$ with the set $\{(t,x)|x=y\}$. Then we have

$$2a|E| \leq |\bar{E}| = \int_{\mathfrak{M}} |\bar{E}_x| dx$$

but since S is of weak type (p,p) , we have

$$|\bar{E}_x| \leq (1/\lambda)^p \|S\|^p \int |F_{a+\epsilon}(t,x)|^p dt.$$

Substituting above and using the fact that $F(0,x)$ and $F(t,x)$ are equimeasurable, we obtain

$$a|E| \leq (1/\lambda)^p \|S\|^p (a + \epsilon) \int |F(0,x)|^p dx$$

and letting a tend to infinity the desired result follows. The preceding argument applies to the case $p < \infty$. If $p = \infty$, the assertion of the theorem is immediate.

To establish Theorem 2 it is sufficient to show that $T_n^* f$ converges almost everywhere for every f in a dense subset of $L^p(\mathfrak{M})$. From this and the fact that S is of weak type (p,p) , the almost everywhere convergence of $T_n^* f$ for every f in $L^p(\mathfrak{M})$ follows by a well-known argument.

Consider functions on $\mathfrak{M}$ of the form

$$h(x) = \int_{-\infty}^{+\infty} f(U^t x) \phi(t) dt,$$

where ϕ is infinitely differentiable and has compact support and vanishing integral, and f is bounded and has support of finite measure. Then

$$\begin{aligned} (T_n^* h)(x) &= \int_{-\infty}^{+\infty} k_n(-t) h(U^t x) dt = \int_{-\infty}^{+\infty} k_n(-t) \int_{-\infty}^{+\infty} \phi(s) f(U^{t+s} x) ds dt \\ &= \int_{-\infty}^{+\infty} f(U^t x) \int_{-\infty}^{+\infty} k_n(s-t) \phi(s) ds dt. \end{aligned}$$

But for almost all x , $f(U^t x)$ is a bounded measurable function of t and according to our assumptions, the inner integral in the last expression converges in L^1 . Consequently, $T_n^\# h$ converges almost everywhere in $\mathfrak{M}$.

Let now $l(x)$ be in $L^p(\mathfrak{M})$ and be invariant with respect to U^t , i.e., such that $l(U^t x) = l'(x)$ for almost all (t, x) . Then for almost all x , $l(U^t x)$ coincides with a constant for almost all t and

$$(T_n^\# l)(x) = \int_{-\infty}^{+\infty} k_n(-t) l(U^t x) dt = l'(x) \int_{-\infty}^{+\infty} k_n(-t) dt$$

which again, according to our assumptions, converges as n tends to infinity. Thus $T_n^\# f$ converges almost everywhere in the linear span of the functions l and h . Hence if we show that this linear span is dense in $L^p(\mathfrak{M})$, our theorem will be established. Suppose that the function $g(x)$ in $L^q(\mathfrak{M})$, $1/q = 1 - 1/p$, is orthogonal to all h . Then

$$\begin{aligned} \int_{\mathfrak{M}} g(x) h(x) dx &= \int_{\mathfrak{M}} g(x) \int_{-\infty}^{+\infty} \phi(t) f(U^t x) dt dx \\ &= \int_{-\infty}^{+\infty} \phi(t) \int_{\mathfrak{M}} g(x) f(U^t x) dx dt = 0. \end{aligned} \quad (1)$$

Setting $U^t x = y$ in the inner integral we obtain

$$\int_{-\infty}^{+\infty} \phi(t) \int_{\mathfrak{M}} g(U^{-t} y) f(y) dy dt = \int_{\mathfrak{M}} f(y) \int_{-\infty}^{+\infty} \phi(t) g(U^{-t} y) dt dy = 0.$$

Since this holds for all bounded f with support of finite measure and all ϕ , we conclude that

$$\int_{-\infty}^{+\infty} \phi(t) g(U^{-t} y) dt$$

vanishes for almost all y , simultaneously for any countable family of functions ϕ , which implies that for almost all y , $g(U^{-t} y)$ coincides almost everywhere in t with a constant $g'(y)$. The function g is thus an invariant function and so is $\psi(g)$ for every bounded continuous ψ . But if ψ vanishes near the origin, then $\psi(g)$ belongs to $L^p(\mathfrak{M})$; consequently, if g is also orthogonal to every invariant function in $L^p(\mathfrak{M})$, then it must be orthogonal to $\psi(g)$ for every bounded ψ vanishing near the origin. But this evidently implies that g vanishes almost everywhere. Thus, we have shown that every function g which is orthogonal to all h and all l vanishes almost everywhere. Consequently the linear span of the functions h and l is dense in $L^p(\mathfrak{M})$, and the proof of Theorem 2 is complete.

For the almost everywhere convergence of $T_n^\# f$ for functions f which are merely integrable, we have the following result.

THEOREM 3. *If the operator S in Theorem 2 is in addition of weak type (1,1), then $T_n^\# f$ converges almost everywhere in $\mathfrak{M}$ for every integrable f .*

Since we know now that $T_n^\# f$ converges almost everywhere for every bounded f vanishing outside a set of finite measure, since S is of weak type (1,1), the convergence of $T_n^\# f$ for every integrable f follows by a standard argument, as was pointed out in the proof of Theorem 2.

In the case of semigroups U^t , $t \geq 0$, very similar results hold. In fact one sees readily that Theorem 1 and its proof apply without change to this case. As for almost everywhere convergence, we have the following theorem.

THEOREM 4. *Suppose that in Theorem 2, U^t , $t \geq 0$, is a semigroup of measure-preserving transformations with the property that*

$$\int_{\mathfrak{M}} g(x)f(U^t x)dx$$

is a continuous function of t in $t \geq 0$ for f in $L^p(\mathfrak{M})$ and g in $L^q(\mathfrak{M})$, $1/q = 1 - 1/p$. Suppose furthermore that the functions k_n have support in $t \leq 0$. Then $T_n^\# f$ converges almost everywhere for every f in $L^p(\mathfrak{M})$. If, in addition, S is of weak type $(1,1)$, then $T_n^\# f$ converges almost everywhere for every integrable f .

This result is obtained by modifying the last part of the proof of Theorem 2. Let g be a function which is orthogonal to all functions h and l introduced in that proof. Then we showed that (1) holds for all ϕ and all bounded f with support of finite measure. But this implies that the inner integral on the left-hand side of (1), as a function of t , coincides almost everywhere with a constant, and since according to our assumptions this integral is a continuous function of t , it follows that

$$\int_{\mathfrak{M}} g(x)f(U^t x)dx = a, \quad t \geq 0.$$

If we show that $a = 0$ for all f , setting $t = 0$ it will follow that g is orthogonal to all bounded f vanishing outside a set of finite measure, and that, consequently, g vanishes almost everywhere. From this, the desired conclusion will follow as in the proof of Theorem 2. For this purpose let us consider the sequence of functions

$$(1/n) \int_0^n f(U^t x)dt.$$

This sequence is bounded in $L^p(M)$, and consequently, an appropriate subsequence converges weakly to a limit $l(x)$. As is readily verified, l is invariant and

$$\int_{\mathfrak{M}} g(x)l(x)dx = a.$$

But since g was assumed to be orthogonal to all invariant functions l , the integral above vanishes and therefore we have $a = 0$. This concludes the proof of the first part of the theorem. The convergence of $T_n^\# f$ for integrable f is obtained as in Theorem 3.

* This research was partially supported by NSF grant GP-5628.

¹ Cotlar, M., "A unified theory of Hilbert transforms and ergodic theorems," *Rev. Mat. Cuyana*, 1, 105-167 (1955).

² Zygmund, A., *Trigonometric Series* (Cambridge University Press, 1959).