WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

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Communicated by Einar Hille, February 3, 1967

1. Let H be a Hilbert space. Consider in H a closed linear operator A with dense domain D_A and its adjoint A^* with domain D_{A^*} . The domains D_A and D_{A^*} are Hilbert spaces under their respective graph-scalar products. We denote by $\mathfrak{D}(R)$, $\mathfrak{D}(R; D_{A^*})$, $\mathcal{S}'(R; H)$, and $\mathfrak{D}'(R; H)$ the spaces of infinitely differentiable functions with compact support, infinitely differentiable D_{A^*} -valued functions with compact support, H -valued temperate distributions, and H -valued distributions, respectively, on R with their usual topologies.³ By $\mathcal{K}^s(R; H)$, s real, we mean the space of all $u \in \mathcal{S}'(R; H)$ whose Fourier transform $\hat{u}(t)$ is a function (H -valued) and

$$\int_R (1 + |t|^{2s}) |\hat{u}(t)|^2 dt < \infty.$$

Let $\mathcal{K}_{\text{loc}}^s(R; H)$ be the space of those $u \in \mathfrak{D}'(R; H)$ for which $\phi u \in \mathcal{K}^s(R; H)$ for every $\phi \in \mathfrak{D}(R)$.

We denote by $L_{\text{loc}}^2(R; H)$ the space of H -valued functions f such that the norm $|f(t)|$ is square integrable on every compact set of R . For nonnegative integers, we present another definition of $\mathcal{K}_{\text{loc}}^{-s}(R; H)$ by

$$\mathcal{K}_{\text{loc}}^{-s}(R; H) = \left\{ \sum_{k=0}^s D^{(k)} T_k ; T_k \in L_{\text{loc}}^2(R; H) \right\};$$

$D^{(k)} = \frac{d^k}{dt^k}$ is a weak differential operator of order k .

The equivalence of these two definitions of $\mathcal{K}_{\text{loc}}^{-s}(R; H)$ for any nonnegative integer s may be verified using the partitions of unity.²

2. We need the following definitions in view of imposing conditions on the operator A .

Definition: Let $\mathfrak{F}$ be a family of parallel lines $\{(Im \lambda = \zeta_n)^{\infty}_{-\infty} : \zeta_n \rightarrow \infty \text{ as } n \rightarrow \infty ; \zeta_n \rightarrow -\infty \text{ as } n \rightarrow -\infty\}$ in the complex λ -plane. Let r be a positive real number and j, m be integers. We shall say that the resolvent $R(\lambda; A^*) = (\lambda I - A^*)^{-1}$ is of (j, r, m) -growth on $\mathfrak{F}$ if $R(\lambda; A^*)$ exists for λ (let $|\lambda| > 1$) outside j intervals of length r on every line of $\mathfrak{F}$ and for these λ

$$|R(\lambda; A^*)| < \text{const. } |\lambda|^m.$$

For $m = 0$ in the definition, see reference 1, p. 131, where S. Agmon and L. Nirenberg have defined (j, r) -bounded resolvent on $\mathfrak{F}_+$ ($\mathfrak{F}$ being in the upper half plane).

The aim of this note is to announce a generalization of a result due to S. Zaidman⁴ in the following

THEOREM. *If $R(\lambda; A^*)$ is of (j, r, m) -growth on $\mathfrak{F}$, then for every $f \in \mathcal{K}_{\text{loc}}^{m-s}(R; H)$ the abstract differential equation $\frac{1}{i} \frac{du}{dt} - Au = f$ has at least one weak solution $u \in \mathcal{K}_{\text{loc}}^{-s}(R; H)$, i.e.,*

$$\langle u, \frac{1}{i} \frac{d\phi}{dt} - A^* \phi \rangle = \langle f, \phi \rangle$$

for all $\phi \in \mathcal{D}(R; D_{A^*})$; s and $s-m$ are nonnegative integers.

For $m = s = 0$, we have the theorem of Zaidman.

¹ Agmon, S., and L. Nirenberg, "Properties of solutions of ordinary differential equations in Banach space," *Comm. Pure Appl. Math.*, **16**, 121-239 (1963).

² Gel'fand, I. M., and G. E. Shilov, *Generalized Functions* (New York: Academic Press, 1964).

³ Hörmander, L., *Linear Partial Differential Operators* (New York: Academic Press, 1963).

⁴ Zaidman, S., "A global existence theorem for some differential equations in Hilbert spaces," these PROCEEDINGS, **51**, 1019-1022 (1964).