NOTE ON ELECTRIC AND MAGNETIC POLARIZATION IN MOVING MEDIA*

BY JOHN CARSTOIU

INTERNATIONAL CONSULTANT SCIENTISTS CORPORATION, BROOKLINE, MASSACHUSETTS

Communicated by L. Brillouin, April 7, 1967

1. *Introduction.*—The considerations presented in this note concern a difficult but favorite subject of classical electrodynamics. This is carefully discussed in Sommerfeld's *Electrodynamics*¹ and a paper by Sydney Goldstein.² As Sommerfeld said, "The literature on this subject is voluminous and by no means free of contradictions. . . ." Unfortunately, not many references can be given here; the reader will find more on the subject in the work quoted. In this paper we study particularly problems involving rotations, whereby great difficulties arise, as pointed out by Sommerfeld. In fact, Sommerfeld simply concluded that Minkowski's theory of moving media is not directly applicable to these problems. This conclusion has been refuted by Goldstein.²

New ideas on the subject will be found in this note. Mathematical refinements are omitted. Instead, we shall postulate at the outset a system of equations to be tested by its consequences. This system complements Maxwell's equations. To be sure, one finds here and there in the literature³ pieces of these equations, but the whole system appears in this note.

The special materials investigated here present conservation properties of great interest. The idealized model of a rigid rotator for nonmagnetic materials is studied in detail, and the polarization problem is practically solved for stationary motions. The formulas obtained may find direct applications to problems concerning atmospheric electricity phenomena depending essentially on rotations.

2. *Equations Governing the Polarization.*—The electric and magnetic polarization vectors $\mathbf{P}$ and $\mathbf{M}$ are defined by

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}, \quad \mathbf{M} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{H}. \quad (1), (2)$$

It may then be assumed that there exist two current density vectors $\mathbf{J}'$ and $\mathbf{J}_m$ associated with a distribution of matter, and which vanish in free space; these shall, respectively, be called densities of "induced electric current" and "magnetic current." We shall postulate that the four vectors in question are subject to the following equations:

$$\text{curl } \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} = \mathbf{J}', \quad -c^2 \text{curl } \mathbf{P} + \frac{\partial \mathbf{M}}{\partial t} = \mathbf{J}_m. \quad (3), (4)$$

We now compare these equations with the Maxwell equations

$$\text{curl } \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \quad \text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (5), (6)$$

By virtue of equations (3) and (4), the latter can be rewritten:

$$\text{curl } \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 (\mathbf{J} + \mathbf{J}'), \quad c^2 \text{curl } \mathbf{D} + \frac{\partial \mathbf{H}}{\partial t} = -\mathbf{J}_m. \quad (7), (8)$$

3. Divergence of Polarization Vectors.—The divergence equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = \rho_e, \quad (9), (10)$$

are usually included as part of Maxwell's system. As pointed out by Stratton,⁵ one must note, however, that if one assumes the conservation of charge, i.e.,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0, \quad (11)$$

then equations (9) and (10) are not independent relations.

Let us now see which are the divergence equations for our new system. Without elaborating, it is evident that if we put

$$\nabla \cdot \mathbf{J}' + \frac{\partial \rho_e'}{\partial t} = 0, \quad \nabla \cdot \mathbf{J}_m + \frac{\partial \rho_m}{\partial t} = 0, \quad (12), (13)$$

the equations (3) and (4) yield

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{P} + \rho_e') = 0, \quad \frac{\partial}{\partial t} (\nabla \cdot \mathbf{M} + \rho_m) = 0. \quad (14), (15)$$

Hence, we can admit that

$$\nabla \cdot \mathbf{P} = -\rho_e', \quad \nabla \cdot \mathbf{M} = -\rho_m. \quad (16), (17)$$

Two new charge densities ρ_e' and ρ_m appear. The latter has been discussed especially by Sommerfeld (ref. 1, pp. 38, 40, 41), and he calls it "magnetic density;" the former will be called "induced electric density."

4. *Charges in Motion.*—We shall be dealing here exclusively with charges in motion, whereby convection currents $\rho_e \mathbf{v}$, $\rho_e' \mathbf{v}$, and $\rho_m \mathbf{v}$ alone are active; hence

$$\mathbf{J} = \rho_e \mathbf{v}, \quad \mathbf{J}' = \rho_e' \mathbf{v}, \quad \mathbf{J}_m = \rho_m \mathbf{v}. \quad (18)-(20)$$

Equations (11), (12), and (13) become

$$\frac{1}{\rho_e} \frac{D \rho_e}{Dt} + \nabla \cdot \mathbf{v} = \frac{1}{\rho_e'} \frac{D \rho_e'}{Dt} + \nabla \cdot \mathbf{v} = \frac{1}{\rho_m} \frac{D \rho_m}{Dt} + \nabla \cdot \mathbf{v} = 0, \quad (21)$$

which are supplemented by the equation of continuity for the material density ρ

$$\frac{1}{\rho} \frac{D \rho}{Dt} + \nabla \cdot \mathbf{v} = 0. \quad (22)$$

It is obvious that the ratios ρ_e/ρ , ρ_e'/ρ , ρ_m/ρ , etc., remain constant along each trajectory. This is true in particular for the quantity

$$S = \frac{\rho_m}{\rho_e'}, \quad (23)$$

which has dimensions of a velocity which we propose to call the *Sommerfeld velocity*.

5. *Minkowski's Equations and Polarization.*—The following equations for moving media, first derived by Minkowski (ref. 1, p. 281)

$$\mathbf{D} + \frac{1}{c^2} \mathbf{v} \times \mathbf{H} = \epsilon(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (24)$$

$$\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} = \mu(\mathbf{H} - \mathbf{v} \times \mathbf{D}), \quad (25)$$

have considerable importance. The space here does not permit a delicate discussion about their range of validity, and we again refer to Sommerfeld's book¹ (see especially footnote 3, p. 280) and Goldstein's paper² for appropriate details. We shall come back to the subject at a later time.

Now, when the values of $\mathbf{P}$ and $\mathbf{M}$ given by (1) and (2) are substituted therein, we obtain

$$\mathbf{P} - \frac{1}{c^2} \mathbf{v} \times \mathbf{M} = \epsilon_0 \chi_e (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (26)$$

$$\mathbf{M} + \mathbf{v} \times \mathbf{P} = \chi_m (\mathbf{H} - \mathbf{v} \times \mathbf{D}), \quad (27)$$

where χ_e and χ_m denote the electric and magnetic susceptibilities, namely

$$\chi_e = \frac{\epsilon}{\epsilon_0} - 1, \quad \chi_m = \frac{\mu}{\mu_0} - 1. \quad (28), (29)$$

This is our result; it has important consequences, as will appear below.

6. *Limiting Cases: Field-Preserving and Polarization-Preserving Motions.*—Each equation (26) and (27) yields two limiting cases of significance:

(I) $\chi_e \rightarrow \infty$ and $\chi_e \rightarrow 0$: (a) $\chi_e \rightarrow \infty$; equation (26) gives

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \quad (30)$$

which when coupled with Maxwell's equations (6) yields

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl} (\mathbf{v} \times \mathbf{B}). \quad (31)$$

If equation (22) is used, then one has

$$\frac{D}{Dt} \left(\frac{\mathbf{B}}{\rho} \right) = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla \right) \mathbf{v}, \quad (32)$$

which shows that the quantity $\rho^{-1}\mathbf{B}$ is frozen into the material. These are the *superconductors*.

(b) $\chi_e \rightarrow 0$, that is, $\epsilon \rightarrow \epsilon_0$; equation (26) gives

$$\mathbf{P} = \frac{1}{c^2} \mathbf{v} \times \mathbf{M}. \quad (33)$$

Taking the curl of equation (33) and using equations (4), (17), (20), and (22), we get

$$\frac{D}{Dt} \left(\frac{\mathbf{M}}{\rho} \right) = \left(\frac{\mathbf{M}}{\rho} \cdot \nabla \right) \mathbf{v}, \quad (34)$$

which shows that the quantity $\rho^{-1}\mathbf{M}$ is frozen into the material. This material is not yet apparently identified in laboratory and current literature.

(II) $\chi_m \rightarrow \infty$ and $\chi_m \rightarrow 0$: (c) $\chi_m \rightarrow \infty$; equation (27) gives

$$\mathbf{H} - \mathbf{v} \times \mathbf{D} = 0, \quad (35)$$

which when coupled with Maxwell's equation (5) (where $\mathbf{J} = \rho_e \mathbf{v}$) yields

$$\frac{\partial \mathbf{D}}{\partial t} + \rho_e \mathbf{v} = \text{curl} (\mathbf{v} \times \mathbf{D}); \quad (36)$$

that is, in view of equations (10) and (22),

$$\frac{D}{Dt} \left(\frac{\mathbf{D}}{\rho} \right) = \left(\frac{\mathbf{D}}{\rho} \cdot \nabla \right) \mathbf{v}, \quad (37)$$

which shows that this time the vector $\rho^{-1} \mathbf{D}$ is frozen into the material. One may regard this material as the analogue of the superconductor and call it *superpermeable material*. However, in distinction to the superconductor, whereby only $\chi_e \rightarrow \infty$ is required, we now have two conditions to be fulfilled: $\chi_m \rightarrow \infty$ and $\sigma \rightarrow 0$, the latter following from Ohm's law for moving media. We must assume that the product $\sigma \chi_m$ has a finite value, such that a magnetic Reynolds number $\mathcal{R}_m$ can be defined; the latter is to be compared with the mechanical Reynolds number $\mathcal{R}$. The number

$$\varepsilon = \frac{\mathcal{R}_m}{\mathcal{R}}, \quad (38)$$

introduced by Elsasser,⁶ plays a crucial role here as it gives the ratio of the electromagnetic and mechanical dissipations, and so it decides which dissipation becomes predominant in the phenomena under consideration.

It may be observed that materials of the ferromagnetic group exhibit an enormous magnetic susceptibility. The author^{4d} inferred that the appearance of electric fields in a tornado might be explained by the presence therein of solid ferromagnetic particles. Indeed, in the author's opinion, a tornado might represent a state of mixture of two charged fluids: air (or water droplets) and dust particles of high magnetic susceptibility which interact with each other in a strange fashion. This concept is, of course, a matter of intuition, rather than a rigorous demonstration; at this stage, it needs experimental evidence.

(d) $\chi_m \rightarrow 0$, that is, $\mu \rightarrow \mu_0$; we now have the *nonmagnetic materials*. In this case, equation (27) gives

$$\mathbf{M} = -\mathbf{v} \times \mathbf{P}. \quad (39)$$

Hence, by virtue of equations (3), (16), (19), and (22),

$$\frac{D}{Dt} \left(\frac{\mathbf{P}}{\rho} \right) = \left(\frac{\mathbf{P}}{\rho} \cdot \nabla \right) \mathbf{v}, \quad (40)$$

showing that the vector $\rho^{-1} \mathbf{P}$ is frozen into a nonmagnetic material. Peculiar to these materials *at rest* there is no magnetic density: $\rho_m = 0$; this follows at once from Sommerfeld's argument [ref. 1, p. 41, eq. (9a)]. For consistency, we must also put $\rho'_e = 0$. One may assume that the motion does not introduce detectable terms in ρ_m and ρ'_e , which, therefore, may be considered to remain vanishingly small as

long as v is much less than c . However, this supposition needs close examination and we leave the question open here.

7. *Rotation of a Nonmagnetic Material.*—For convenience, we now use tensor notation. Equation (40) can be rewritten

$$\frac{D}{Dt} \left(\frac{P_i}{\rho} \right) = \frac{P_j}{\rho} \omega_{ij} + \frac{P_j}{\rho} e_{ij}, \quad (41)$$

where ω_{ij} and e_{ij} are the vorticity and the rate of deformation.

The case of a rigid rotator is of interest. Since a rigid rotation is characterized by $e_{ij} = 0$, or equivalently $v_i = \Omega_{ij}x_j$ (neglecting a translation), equation (41) simplifies to

$$\frac{DP_i}{Dt} = P_j \Omega_{ij}. \quad (42)$$

Assuming a stationary motion $\left(\frac{\partial}{\partial t} = 0 \right)$ and taking the Ox_3 axis in the direction of rotation, equation (42) is resolved into

$$\left. \begin{aligned} x_2 \frac{\partial P_1}{\partial x_1} - x_1 \frac{\partial P_1}{\partial x_2} &= P_2, \\ x_2 \frac{\partial P_2}{\partial x_1} - x_1 \frac{\partial P_2}{\partial x_2} &= -P_1, \\ x_2 \frac{\partial P_3}{\partial x_1} - x_1 \frac{\partial P_3}{\partial x_2} &= 0. \end{aligned} \right\} \quad (43)$$

We assume that $\rho_m = \rho'_e = 0$. Then, taking the divergence of equation (39), we get

$$P_3 = 0. \quad (44)$$

On the other hand, it is a matter of simple verification to see that if we take

$$P_1 = C \frac{x_1}{r^2}, \quad P_2 = C \frac{x_2}{r^2}, \quad (45)$$

where $r^2 = x_1^2 + x_2^2$ and C is a constant, all equations involved are identically satisfied. Finally, equation (39) gives

$$M_1 = M_2 = 0, \quad M_3 = C\Omega, \quad (46)$$

Ω being the magnitude of the rotation. Therefore, subject to the approximation permitted, the problem is practically solved. Nonetheless, these results can be improved by reiteration.

I wish to thank Dr. L. Brillouin and Professor Sydney Goldstein for helpful suggestions and courtesies of all kinds. My thanks are again due to the officials and staff of the Environmental Science Services Administration and the Office of Naval Research for continued support.

* This note generalizes and deepens some of the author's previous research.⁴ As before, the meter-kilogram-second-coulomb system of units is used; the notation employed is classical.

Supported by contracts E22-99-67(N) and N00014-66-C0217.

¹ Sommerfeld, A., *Electrodynamics* (New York: Academic Press, Inc., 1952), pp. 280-290 and 359-363 (problem IV, 1).

² Goldstein, S., in *Proceedings of the Symposium on Electromagnetics and Fluid Dynamics of Gaseous Plasma* (Brooklyn, N.Y.: Polytechnic Press, Polytechnic Institute, 1962), pp. 65-80.

³ See, in particular, (a) Stratton, J. A., *Electromagnetic Theory* (New York: McGraw-Hill Book Co., 1941), pp. 11, 12, 30, 183, 464, and 465; (b) Landau, L. D., and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Reading, Mass.: Addison-Wesley Publishing Co., 1960), pp. 36, 37, 113, and 114.

⁴ (a) Carstoiu, J., *Compt. Rend., Ser. A*, **262**, 1263-1266 (1966); (b) *Ser. A*, **263**, 631-634 (1966); (c) *Ser. B*, **263**, 1050-1053 (1966); (d) *Ser. B*, **263**, 1157-1159 (1966); see also our memoir, "Essay on fundamentals of electrodynamics of moving media and applications," to be published in *J. Math. Pures Appl.*

⁵ Ref. 3a, p.6.

⁶ Elsasser, W. M., in *Magnetohydrodynamics*, Lockheed Symposium, ed. R. K. M. Landshoff (Stanford, Calif.: Stanford University Press, 1957), pp. 16-28; see p. 21.