ON THE CHERN NUMBERS OF CERTAIN COMPLEX AND ALMOST COMPLEX MANIFOLDS

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Let V be a compact, oriented $2n$ -dimensional ($C^\sim$ -)differentiable manifold. If V admits a complex structure, compatible with its differentiable structure, the structural group of the tangent bundle of V can be restricted in a natural way from $GL(2n, \mathbf{R})$ to $GL(n, \mathbf{C})$. This gives for V a necessary condition of topological nature, which led Ch. Ehresmann and H. Hopf some 20 years ago to the concept of an almost complex structure, exactly meaning a restriction of the structural group of the tangent bundle of a differentiable manifold from $GL(2n, \mathbf{R})$ to $GL(n, \mathbf{C})$. An almost complex manifold is a differentiable manifold, provided with an almost complex structure.

A complex manifold is of course an almost complex manifold, and for several years a condition has been known, the Eckmann-Fröhlicher condition, which is necessary and sufficient for a $C^\sim$ almost complex structure to be integrable, that is, to be induced by a complex structure. However, once there exists on a differentiable manifold an almost complex structure, by a slight change of this structure, an infinity of others can be obtained, and it would have been possible, though not likely, that to a given almost complex structure there is always a homotopic one, which is integrable. In particular, no example was known of a compact, differentiable manifold, admitting almost complex structures, but no complex structure whatsoever.

Now to begin with, we would like to point out here that many examples of compact 4-dimensional differentiable manifolds, admitting almost complex structures, but no complex structure, can be obtained by combining a theorem of Ch. Ehresmann and W. W. Wu (in a formulation due to F. Hirzebruch and H. Hopf) with certain results of K. Kodaira. These last results—and consequently the main theorems of this announcement—depend in an essential and up to now unavoidable way on the Atiyah-Singer Riemann-Roch theorem. As will be explained at the end of this note, to obtain specific examples we need, apart from the theorems of Ehresmann-Wu and Atiyah-Singer, only some very general facts about complex surfaces.

But we can get slightly more. To explain this properly, we shall recall the definition of the Chern numbers of a compact almost complex manifold V^{2n} . Let $[V]$ be the fundamental cycle of V , c_i the i th Chern class of V , and $i_1, \dots, i_n$ nonnegative integers with $i_1 + 2i_2 + \dots + ni_n = n$. Then $c_1^{i_1} \cup c_2^{i_2} \cup \dots \cup c_n^{i_n} \in H^{2n}(V, \mathbf{Z})$, and the value of this cohomology class on $[V]$ is an integer. The $\pi(n)$ integers, obtained this way for all $i_1, i_2, \dots, i_n$ with $i_1 + 2i_2 + \dots + ni_n = n$ are the Chern numbers of V . J. Milnor¹ has determined the sets of $\pi(n)$ integers occurring as the Chern numbers of a not necessarily connected V . It turns out in particular that they are the same as those occurring as the Chern numbers of the complex or even the projective algebraic manifolds of the same dimension, but again not necessarily connected. Milnor's methods do not allow us to decide which set

of these sets of $\pi(n)$ integers occur as the Chern numbers of connected almost complex, complex or algebraic manifolds. The case $n = 1$ indicates already that in general there will be less. Now in this note we announce results for $n = 2$. Here we have two Chern numbers, $c_1^2(V)$ and $c_2(V)$. As a special case of Milnor's results, it is known that two integers p and q occur as Chern numbers $c_1^2(V)$ and $c_2(V)$ of a not necessarily connected compact almost complex (or complex, or projective) manifold if and only if $p + q \equiv 0(12)$.

We now come to the main result of this note. Let D_1 and D_2 be the subsets of the (p, q) -plane, respectively, given by

$$D_1 = \{(p, q), \quad p \leq 2q\}.$$

$$D_2 = \{(p, q), \quad p > 2q, \quad p \leq 8q\},$$

and set $D_1 \cup D_2 = D$.

Then we have

THEOREM 1. *For each pair of integers (p, q) , with $p + q \equiv 0(12)$, there exists a compact, connected almost complex manifold V (of real dimension 4), with $c_1^2(V) = p$, $c_2(V) = q$. If $(p, q) \in D_1$, then there is a compact, connected complex (and even projective algebraic) V , with $c_1^2(V) = p$, $c_2(V) = q$. If $(p, q) \notin D$, then there is no compact, connected complex V with $c_1^2(V) = p$, $c_2(V) = q$. For certain pairs $(p, q) \in D_2$, there is a complex (even projective algebraic) V , with $c_1^2(V) = p$, $c_2(V) = q$, but for other pairs $(p, q) \in D_2$, $p + q = 0(12)$, there is no such complex surface V .*

It should be understood that for many pairs $(p, q) \in D_2$ the question has not yet been settled.

As a consequence of Theorem 1 we get a large number of compact, 4-dimensional differentiable manifolds, admitting almost complex structures, but no complex structure.

The phenomenon of a set of integers, occurring as the set of Chern numbers of a connected almost complex manifold, but not of any connected complex manifold, though not present in the case of real dimension 2, is not unexpected, but an example does not seem to be found in the literature.

We give a short sketch of the proof of Theorem 1. The first statement of that theorem is based upon

THEOREM 2. *Let V be a compact, connected, differentiable 4-manifold, and let $h \in H^2(V, \mathbb{Z})$. There exists on V an almost complex structure with first Chern class h if and only if $h \equiv w_2(V) \pmod{2}$ and $h^2 = 3\tau(V) + 2\chi(V)$, where $w_2(V)$ is the second Stiefel-Whitney class, $\tau(V)$ the index, and $\chi(V)$ the Euler-Poincaré characteristic of V .*

This is the theorem of Ehresmann and Wu, which we mentioned before.²

If V_1 and V_2 are compact, connected, oriented n -dimensional differentiable manifolds, we denote by $V_1 + V_2$ a connected sum of V_1 and V_2 .³ $V_1 + V_2$ is again a compact, connected, oriented n -dimensional differentiable manifold.

Let P be the naturally oriented underlying differentiable manifold of the complex projective plane, $\bar{P}$ the same, but with the orientation reversed, and Q the naturally oriented underlying differentiable manifold of the product of a Riemann surface of genus 2 and one of genus 0. Finally, for integers $l, m, n \geq 0$, $l + m + n > 0$, let $W_{l, m, n}$ be a connected sum of l copies of P , m copies of $\bar{P}$, and n copies of Q . Then we can derive from Theorem 2

PROPOSITION 3. *For each pair of integers p, q with $p + q = 0(12)$ there are integers l, m, n such that $W_{l, m, n}$ admits an almost complex structure with $c_1^2(V_{l, m, n}) = p$, $c_2(V_{l, m, n}) = q$.*

The following result is essentially due to F. de Franchis and G. Castelnuovo.⁴

PROPOSITION 4. *If on the compact, connected complex surface V there exist two linearly independent holomorphic 1-forms ω_1 and ω_2 , with $\omega_1 \wedge \omega_2 \equiv 0$ on V , then there is a connected holomorphic map f of V onto a nonsingular algebraic curve W , of genus at least 2, such that ω_1 and ω_2 are pullbacks of holomorphic 1-forms on W .*

Let X be a compact, complex Kähler manifold, and Ω^p the sheaf of germs of holomorphic p -forms on X . If we set, as usual, $\dim H^q(X, \Omega^p) = h^{p, q}$, we have $h^{q, p} = h^{p, q}$. Each closed complex differential form of type p, q determines by the de Rham isomorphism an element of $H^{p+q}(V, \mathbf{C})$. The dimension of the subspace $H^{p, q}$ of those elements of $H^{p+q}(V, \mathbf{C})$, which can be represented in this way by a form of type p, q , equals $h^{p, q}$. Furthermore, $H^n(V, \mathbf{C}) \cong \sum_{p+q=n} H^{p, q}(V, \mathbf{C})$, hence the n th Betti number $b_n(V) = \sum_{p+q=n} h^{p, q}$.⁵

Proposition 4 and the results just summarized enable us to prove the next two propositions.

PROPOSITION 5. *If the compact, connected Kähler surface V does not admit a holomorphic map onto an algebraic curve of genus ≥ 2 , then $h^{2, 0}(V) \geq 2h^{1, 0}(V) - 3$, and $h^{1, 1}(V) \geq 2h^{1, 0}(V) - 1$. If $h^{1, 0}(V) = 2$, then $h^{1, 1}(V) \geq 4$.*

PROPOSITION 6. *If the compact, connected Kähler surface V admits a connected holomorphic map onto a nonsingular algebraic curve W , of genus π , and if $h^{1, 0}(V) \geq \pi + 1$, then $h^{2, 0}(V) \geq h^{1, 0}(V) - 1$, and $h^{1, 1}(V) \geq 2h^{1, 0}(V) - 1$.*

The essence of the next proposition goes back to F. Enriques and L. Campedelli. After we had given a precise formulation and a proof, we discovered that this result also occurs in the recent book of I. R. Šafarevič.⁷

PROPOSITION 7. *If the connected, compact complex surface V admits a connected holomorphic map f onto a nonsingular algebraic curve W , of genus π , and if the generic fiber of f has genus π^1 , then $\chi(V) \geq 4(\pi - 1)(\pi^1 - 1)$.*

Now let V be any compact, connected surface. If V is not projective algebraic, it follows from results of K. Kodaira⁸ that $(c_1^2(V), c_2(V)) \in D_1$. If V is algebraic, and does not admit a holomorphic map onto an algebraic curve of genus ≥ 2 , it follows from Proposition 5 and a short analysis of some special cases that $c_2(V) \geq 2$. From this, Proposition 5, and the Todd-Hirzebruch formula we find that in this case $(c_1^2(V), c_2(V)) \in D$. If V admits a connected holomorphic map onto a nonsingular algebraic curve W , we consider separately the cases where the generic fiber has genus $\pi^1 = 0, 1$ or at least 2. In the first case it can be shown that V is obtained from a holomorphic P_1 -bundle over W by successively blowing up points, hence $(c_1^2(V), c_2(V)) \in D_1$. If $\pi^1 = 1$, it follows from Proposition 7 that $c_2(V) \geq 0$. Since it is not difficult to show that in this case $c_1^2(V) \leq 0$, we have here too that $(c_1^2(V), c_2(V)) \in D_1$. Now let $\pi^1 \geq 2$. Proposition 7 implies that $c_2(V) \geq 4$. If $h^{1, 0}(V) \geq \pi + 1$, we derive from this fact and Proposition 6 in the same way as before that $(c_1^2(V), c_2(V)) \in D$. Finally, if $h^{1, 0}(V) = \pi$, we know from Proposition 7 that $c_2(V) \geq 4(\pi - 1) = 4(h^{1, 0}(V) - 1)$. Using Thoms index formula,⁹ we derive from this

$$c_1^2(V) - 2c_2(V) = 3\tau(V) \leq 3(b_2(V) - 2) = 3(c_2(V) + 4(h^{1, 0}(V) - 1)) \leq 6c_2(V),$$

or

$$c_1^2(V) - 8c_2(V) \leq 0.$$

This also means that in this case $(c_1^2(V), c_2(V)) \in D$.

Besides the projective plane, there are connected algebraic surfaces V , constructed by F. Hirzebruch,¹⁰ and another type, constructed by K. Kodaira (unpublished), for which $(c_1^2(V), c_2(V)) \in D_2$. On the other hand, it is possible, using Propositions 5, 6, and 7 to find pairs of integers (p, q) , $p + q = 0(12)$, $(p, q) \in D_2$, for which there is no compact, connected complex surface V with $c_1^2(V) = p$, $c_2(V) = q$.

It follows from Theorem 1 and Proposition 3 that for an infinite number of triples (l, m, n) , the compact connected differentiable 4-manifolds $W_{l, m, n}$ have almost complex structures, but no complex structure. It is easy to construct along the same lines other examples of this type, and we mention a few of them.

Let S^i be an oriented i -sphere, and P as before the naturally oriented underlying differentiable manifold of the complex projective plane. We claim that $V = P + S^1 \times S^3 + S^1 \times S^3$ has almost complex structures, but no complex structure. The first claim can be checked easily, using Theorem 2. It turns out that $c_1^2(V) = 1$. Now it follows readily from the Atiyah-Singer Riemann-Roch theorem and some general results of Kodaira that a complex surface V with $c_1^2(V) > 0$ is algebraic. However, V does not admit an algebraic structure, for in that case $b_1(V) = 2$ would imply that V would admit a holomorphic map onto an elliptic curve, namely, its Albanese torus, but it is easy to see that as algebraic surface V with $b_2(V) = 1$ does not admit any holomorphic map onto a curve.¹¹ Another example of the same type is provided by a connected sum $S^1 \times S^3 + S^1 \times S^3 + S^2 \times S^2$. As was observed by A. Howard, the same method can be used to construct *simply connected*, compact, connected, differentiable 4-manifolds with almost complex structures, but no complex structure. Namely, let R be the naturally oriented underlying differentiable manifold of a nonsingular algebraic surface of degree 4 in the 3-dimensional complex projective space. Then a connected sum of $2k + 1$ copies of R has the required property, provided that $k \geq 1$.

¹ Milnor, J., "On the cobordism ring Ω^* and a complex analogue, I," *Am. J. Math.*, **82**, 505-521 (1960); Milnor, J., "On the cobordism ring Ω^* and a complex analogue, II," in preparation.

² Hirzebruch, F., and H. Hopf, "Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten," *Math. Ann.*, **136**, 156-172 (1958).

³ Milnor, J., "Sommes des variétés différentiables et structures différentiables des sphères," *Bull. Soc. Math. France*, **87**, 439-444 (1959).

⁴ Castelnuovo, G., "Sulle superficie aventi il genere aritmetico negativo," in *Memorie Scelte* (Bologna, 1937).

⁵ See, for example, Weil, A., *Variétés Kähleriennes* (Paris, 1958).

⁶ Campedelli, L., "Ancora sul computo dell'invariante di Zeuthen-Segre per una superficie algebrica," *Rend. Lincei*, **19**, 860-866 (1934).

⁷ Šafarevič, I. R., *Algebraic Surfaces* (Moscow, 1965), p. 53.

⁸ Kodaira, K., "On the structure of compact complex analytic surfaces, I," *Am. J. Math.*, **86**, 751-798 (1964).

⁹ See references 2 or 8.

¹⁰ Hirzebruch, F., "Automorphe Formen und der Satz von Riemann-Roch," in *Symp. Int. Topologia Algebraica* (Mexico, 1958), pp. 129-144.

¹¹ Remmert, R., and A. Van de Ven, "Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten," *Topology*, **2**, 137-157 (1963).