PARTLY GENTLE PERTURBATION WITH APPLICATION TO PERTURBATION BY ANNIHILATION-CREATION OPERATORS*

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For a self-adjoint operator H_0 with an absolutely continuous spectrum, Friedrichs¹ used the notion of gentle perturbation to prove that for small values of λ the perturbation $H = H_0 + \lambda V$ has also an absolutely continuous spectrum and is unitarily equivalent with H_0 , provided that V satisfies certain gentleness conditions. Rejto extended this result to the case of "partial gentleness" in which it is only assumed that the perturbation is gentle in some interval.

In this paper a new definition of partly gentle perturbation is given, which in a sense is more general than Rejto's.² In applying it to perturbation of the Laplacian by a multiplication operator, fairly strong results are obtained.

The main application is to perturbation by annihilation-creation operators, suggested by problems of quantum-field theory. We consider only nonlocal interactions with smooth kernels and no vacuum interaction, and in this case Theorems 7, 8, and 9 show that the perturbation by annihilation-creation operators produces partial unitary equivalence.

(1) *A General Theory of Partly Gentle Perturbations.*—Let H_0 be a self-adjoint operator acting in an abstract Hilbert space $\mathcal{H}$. We will assume that $\mathcal{H}$ is separable. The domain of H_0 will be denoted by D_{H_0} . Let V be a symmetric operator with domain containing the domain of H_0 .

We will assume that V is essentially bounded by H_0 , and by this we mean that there exist two positive real numbers a and b such that

$$\|V\psi\| = a\|\psi\| + b\|H_0\psi\|$$

for all $\psi \in \mathcal{H}$.

In this case it is well known that the operator

$$H = H_0 + \lambda V$$

is self-adjoint with domain D_{H_0} , for $|\lambda| < b^{-1}$.

Let $E(\lambda)$ and $E_0(\lambda)$ be the family of spectral projections corresponding to H and H_0 , i.e.,

$$H = \int \lambda dE(\lambda), \quad H_0 = \int \lambda dE_0(\lambda).$$

Let E^I or E_0^I denote the spectral projection onto the interval I of the real line corresponding to H or H_0 , i.e.,

$$E^I = \int_I dE(\lambda), \quad E_0^I = \int_I dE_0(\lambda).$$

We will say that two separable Banach spaces B_1 and B_2 have the regular intersection property iff:

(i) There is a sequence of elements $\{f_n\}$ in $B_1 \cap B_2$ such that $\{f_n\}$ is dense in B_1 and in B_2 .

(ii) Both identity mappings I_1 and I_2 are closed:

$$\begin{aligned} I_1: B_1 &\rightarrow B_2 \text{ with domain } B_1 \cap B_2, \\ I_2: B_2 &\rightarrow B_1 \text{ with domain } B_1 \cap B_2. \end{aligned}$$

Consider now the self-adjoint operator

$$H = H_0 + \lambda V$$

defined on D_{H_0} for $|\lambda| < b$, and let $R_0(z) = (z - H_0)^{-1}$ be the resolvent of H_0 . We shall say that the perturbation $H = H_0 + \lambda V$, where V is essentially bounded by H_0 , is gentle on the interval I iff there is a Banach space B , such that B and $\mathcal{H}$ have the regular intersection property and the following four conditions are satisfied.

I. For all $\lambda \in I$ and all $\epsilon > 0$: $R_0(\lambda \pm i\epsilon)$ maps B into its dual space B^* , and the norms of these mappings are uniformly bounded in λ and ϵ .

II. For all $\lambda \in I$: there exist mappings, $R_0(\lambda +)$ and $R_0(\lambda -)$ of B into B^* , such that $R_0(\lambda \pm i\epsilon)$ converges to $R_0(\lambda \pm)$ in the weak * topology. That is: For all f and g in B , $(f, R_0(\lambda \pm i\epsilon)g)$ converges to $(f, R_0(\lambda \pm)g)$ as ϵ tends to zero.

III. For all $\lambda \in I$ and for all $\epsilon > 0$: $VR_0(\lambda + i\epsilon)$ maps B into B , and the norms of these mappings are uniformly bounded in λ and ϵ .

IV. For all $\lambda \in I$: $VR_0(\lambda \pm i\epsilon)$ converges strongly to $VR_0(\lambda \pm)$. That is: For all $f \in B$, $\|(VR_0(\lambda \pm i\epsilon) - VR_0(\lambda \pm))f\|_B$ tends to zero as ϵ tends to zero. $\| \quad \|$ is the norm in B .

THEOREM 1. Suppose that the perturbation $H = H_0 + \lambda V$ is gentle on the interval I , then H_0 has an absolutely continuous spectrum in the interval I .

This theorem is in fact implied by condition I alone. To see this, we observe that condition I implies that for a dense set of elements in $\mathcal{H}$, namely for all elements f in $B \cap \mathcal{H}$.

$$(f, R_0(\lambda \pm i\epsilon)f)$$

is uniformly bounded in $\lambda \in I$ and $\epsilon > 0$. This implies that H_0 has an absolutely continuous spectrum in the interval I .

THEOREM 2. If the perturbation

$$H = H_0 + \lambda V$$

is gentle on the interval I , the perturbation

$$H_0 = H - \lambda V$$

is also gentle on the interval I . That is: For $|\lambda|$ small enough, conditions I, II, III, and IV hold with the resolvent of H replacing the resolvent of H_0 everywhere.

Theorem 3, our main theorem, states that the parts of the operators H and H_0 corresponding to the spectral interval I are unitarily equivalent if the perturbation is gentle on this interval. Its validity is the reason we call a perturbation satisfying conditions I to IV gentle on the interval I .

THEOREM 3. Suppose that the perturbation $H = H_0 + \lambda V$ is gentle on the interval I ; then for all λ with $|\lambda| < \min(\lambda_0, b^{-1})$ we can find a partial isometry, U_λ , of $\mathcal{H}$ into $\mathcal{H}_0$, such that

$$U_\lambda^* U_\lambda = E^I, \quad U_\lambda U_\lambda^* = E_0^I,$$

and

$$E^I H = U_\lambda {}^* E_0^I U_0 U_\lambda, \quad E_0^I H_0 = U_\lambda E^I H U_\lambda {}^*;$$

moreover, U_λ is analytic in λ for $|\lambda| < \min(\lambda_0, b)$. Here λ_0 is the uniform bound of the norms according to condition III.

(2) *Perturbation of the Negative Laplacian by a Multiplication Operator.*—The results described can be applied to the case in which $H_0 = -\Delta$, where Δ is the Laplacian in three dimensions. The perturbing operator V will be a multiplication operator given by multiplication by a real function $V(x)$. The operator is regarded as a self-adjoint operator in the Hilbert space $\mathcal{H} = L_2(R^3)$ on its natural domain of definition. We then note

THEOREM 4. If $V(x)$ is in $L_{3/2}(R^3)$, the perturbation

$$H = -\Delta + \lambda V$$

is gentle on the whole real line.

By Theorem 3 this statement implies that, for $V(x)$ in $L_{3/2}(R^3)$, the operators

$$H = -\Delta + \lambda V \quad \text{and} \quad H_0 = -\Delta$$

are unitarily equivalent for $|\lambda| < \lambda_0$, where λ_0 is given by a constant times the $L_{3/2}$ -norm of $V(x)$. Accordingly, there is a unitary operator mapping $L_2(R^3)$ onto itself such that, for $|\lambda| < \lambda_0$,

$$H_0 = U_\lambda H U_\lambda {}^*$$

with U_λ is analytic in λ . This result was recently obtained independently by Kato³ using other methods.

We shall prove Theorem 4 by giving a Banach space B that satisfies the conditions I to IV.

We take $B = L_{6/5}(R^3)$, and recall that the resolvent of $-\Delta$ is given by the kernel

$$R_0(z) = \frac{e^{-\sqrt{-z}|x-y|}}{4\pi|x-y|}, \quad \operatorname{Re} \sqrt{-z} \geq 0.$$

Sobolev's inequality gives us that the kernel given by

$$\frac{1}{4\pi} \frac{1}{|x-y|}$$

maps $L_{6/5}(R^3)$ into $L_6(R^3)$, which is the dual space of $L_{6/5}(R^3)$.

Since $\frac{1}{4\pi} \frac{1}{|x-y|}$ is larger than the absolute value of the kernel of $R_0(z)$, this

verifies condition I, and using dominated convergence we also get condition II.

Let $q = 6/5$, $q' = 6$, and let $V \in L_p$. Hölder's inequality implies that V maps $L_{q'}$ into L_q , where $p^{-1} + (q')^{-1} = q^{-1}$, and this gives us $p = 3/2$. Since thus V is in $L_{3/2}(R^3)$, it maps B^* into B with bounded norm, and this gives us condition III. Condition IV follows again by dominated convergence.

Let Δ now be the Laplacian in n dimensions, $n > 3$. Theorem 4 generalizes to this case under somewhat stronger conditions on the function $V(x)$.

THEOREM 5. *If the real function $V(x)$ is bounded and integrable, the perturbation $H = -\Delta + \lambda V$ is gentle on the whole real line.*

Again Theorem 3 implies that for $V(x)$ bounded and integrable, there is a λ_0 , depending on the L_∞ and the L_1 -norm of $V(x)$, such that, for $\lambda \leq \lambda_0$, the operators $H = -\Delta + \lambda V$ and $H_0 = -\Delta$ are unitarily equivalent, i.e., $H_0 = U_\lambda H U_\lambda^*$, where U_λ is analytic in λ for $|\lambda| \leq \lambda_0$ and unitary.

Let H and H_0 be as above, and define

$$U_\lambda^+ = \text{strong} \lim_{t \rightarrow \infty} e^{iHt} e^{-iH_0 t}, \quad U_\lambda^- = \text{strong} \lim_{t \rightarrow -\infty} e^{iHt} e^{-iH_0 t},$$

whenever the strong limits exist. If U_λ^+ and U_λ^- exist, we define the *scattering operator* by

THEOREM 6. *Let $H_0 = -\Delta$ in n dimensions $n \geq 3$, and let $H = -\Delta + \lambda V$, where V stands for multiplication by a function $V(x)$. If $V(x)$ is bounded and integrable, there is a bound $\lambda_0 > 0$, depending only on the L_∞ and the L_1 -norm of $V(x)$. For $|\lambda| < \lambda_0$, the operators U_λ^+ and U_λ^- exist and are unitary, and depend analytically on λ . Hence the scattering operator S_λ exists for $|\lambda| < \lambda_0$ and depends analytically on λ .*

(3) *Partly Gentle Perturbation by Annihilation-Creation Operators.*—In the following the undisturbed operator H_0 will be of the same type as the free field energy operator in quantum-field theory and the disturbance V is built up in terms of annihilation-creation operators.

The disturbing operator V in a relativistic quantum-field theory has the following characteristic properties: it is Lorentz invariant, it is local, and it is built up in terms of annihilation-creation operators. We shall consider perturbation problems in which the disturbing operator V has only the last property, namely, that it is built up in terms of annihilation-creation operators. Moreover, we shall assume that the kernels which express V in terms of annihilation-creation operators have certain smoothness properties. In order to simplify the exposition, we shall consider only the case where there is one fermion field which interacts with itself.

The undisturbed operator H_0 will be given with respect to a specific representation of the Hilbert space $\mathcal{H}$, the so-called particle representation. An element f in $\mathcal{H}$ is represented by a sequence of complex valued functions

$$f = \{f_0, f_1, f_2, \dots\}$$

where f_0 is just a complex constant and f_n for $n \geq 1$ are antisymmetric functions $f_n(v_1 \dots v_n)$ of n real variables $v_1 \dots v_n$, each of which has as domain the interval $[\alpha, \infty)$. The values of f_n are vectors of some "accessory" complex Hilbert space.

By saying that the function $f_n(v_1 \dots v_n)$ is antisymmetric we mean that

$$f_n(v_1 \dots v_n) = \frac{1}{n!} \sum_{\sigma} (-1)^{\sigma} f(v_{\sigma(1)} \dots v_{\sigma(n)}),$$

where the sum on the right-hand side is over all permutation σ of the numbers $1 \dots n$.

We shall assume that α is positive, $\alpha = m_0^2$, where m_0 is the rest mass of the free particle. The inner product in $\mathcal{H}$ is given by $(f, g) = \sum_{n=0}^{\infty} n! (f_n, g_n)$ where

$$(f_n, g_n) = \int_{\alpha}^{\infty} \dots \int_{\alpha}^{\infty} \bar{f}_n(v_1 \dots v_n) g_n(v_1 \dots v_n) dv_1 \dots dv_n.$$

The undisturbed operator H_0 is given by

$$(H_0 f)_n(v_1 \dots v_n) = \sum_{i=1}^n v_i \cdot f_n(v_1 \dots v_n),$$

with $(H_0 f)_0 = 0$. Obviously H_0 is self-adjoint on its natural domain of definition. The annihilation operator $a(\omega)$ is defined by

$$(a(\omega)f)_n(v_1 \dots v_n) = (n+1)f_{n+1}(\omega, v_1 \dots v_n).$$

The creation operator $a^+(\omega)$ is defined as the adjoint of $a(\omega)$.

Actually, $a(\omega)$ and $a^+(\omega)$ are improper operators. Introducing for any test function h in $L_2(\alpha, \infty)$

$$a(h) = \int_{\alpha}^{\infty} a(\omega)h(\omega)d\omega, \quad a^+(h) = \int_{\alpha}^{\infty} a^+(\omega)h(\omega)d\omega,$$

we get well-defined operators with $a(h)^* = a^+(\bar{h})$.

The fact that $f_n(v_1 \dots v_n)$ is antisymmetric implies the identity $a(h)a^+(\bar{h}) + a^+(\bar{h})a(h) = \|h\|_2^2$ and this again implies $\|a(h)\| \leq \|h\|_2$. Hence $a(h)$ and $a^+(\bar{h})$ are bounded operators for h in L_2 .

We shall consider disturbing operators of the form $V = \sum_{1 \leq j, k \leq N} V_{jk}$ where N may be any number.

The operators V_{jk} are expressed in terms of annihilation-creation operators as

$$V_{jk} = \int_{\alpha}^{\infty} \dots \int_{\alpha}^{\infty} dv_1 \dots dv_j dw_1 \dots dw_k V_{jk}(v_1 \dots v_j | w_1 \dots w_k) \cdot a^+(v_1) \dots a^+(v_j) a(w_1) \dots a(w_k),$$

where the kernel $V_{jk}(v_1 \dots v_j | w_1 \dots w_k)$ is a complex-valued function that is antisymmetric in the variables $v_1 \dots v_j$ and in $w_1 \dots w_k$.

Using the definition of $a^+(\omega)$ and of $a(\omega)$ we can give the action of the operator V_{jk} directly in terms of its kernel $V_{jk}(v_1 \dots v_j | w_1 \dots w_k)$. Let $m = j - k + n$, then

$$(V_{jk}f)_m(v_1 \dots v_m) = \binom{n}{k} k! \text{Asym} \int_{\alpha}^{\infty} \dots \int_{\alpha}^{\infty} dw_1 \dots dw_k V_{jk}(v_1 \dots v_j | w_1 \dots w_k) \cdot f_n(w_1 \dots w_k, v_{j+1} \dots v_m).$$

Here we have introduced the notion

$$\text{Asym } g_m(v_1 \dots v_m) = \frac{1}{m!} \sum_{\sigma} (-1)^{\sigma} g_m(v_{\sigma(1)} \dots v_{\sigma(m)})$$

where g_m is a function of m real variables, and the summation on the right-hand side is taken over all permutations of the numbers $1, \dots, m$.

We will say that the kernels $V_{jk}(v_1 \dots v_j | w_1 \dots w_k)$ are smooth if their Fourier transforms are in L_1 and in L_{∞} . What is actually needed is a somewhat weaker notion of smoothness, namely that some "gentleness norms" used by Friedrichs in reference 1 should be finite. The details of this will be given in a later publication.

We point out that due to the fact that V contains no terms of the form V_{0k} or

V_{j0} , our perturbation is actually a perturbation in the subspace $\mathcal{H}_0$ spanned by all elements such that $f_0 = 0$.

We are now in a position to state the theorem about perturbation by annihilation-creation operators.

THEOREM 7. *Suppose V is symmetric and essentially bounded by H_0 , and if V has the form*

$$V = \sum_{1 \leq j, k \leq N} V_{jk}$$

where the kernels $V_{jk}(v_1 \dots v_j | w_1 \dots w_k)$ are all smooth, then the perturbation $H = H_0 + \lambda V$ in $\mathcal{H}_0$ is gentle on any bounded interval I on the real axis.

The proof of this theorem is rather involved and will be given in a later publication.

Together with Theorem 3 this theorem shows that for any bounded interval I the projections $E^I H$ and $E_0^I H_0$ are unitarily equivalent for $|\lambda| \leq \lambda_0$ where $\lambda_0 > 0$ and depend on the interval I .

In terms of the scattering operator S defined as in section (2), by means of the two operators

$$U^\pm = \text{strong} \lim_{t \rightarrow \pm \infty} e^{iHt} e^{-iH_0 t}$$

we state

THEOREM 8. *Under the conditions of Theorem 7, the strong limits above exist for all values of λ and define operators $U^\pm$ that will be isometries in $\mathcal{H}$.*

This theorem was proved under slightly different hypotheses by Chestjakov,⁴ and his proof carries over to our case with only small adjustments.

From Theorem 8 it follows that the scattering operator S_λ is defined for all values of λ , and from the definition of S_λ it is easy to see that it must commute with H_0 . This shows that S_λ has the form

$$S_\lambda = \int S_\lambda(\omega) dE_0(\omega),$$

where $E_0(\omega)$ is the spectral family of projections corresponding to H_0 . Here $S_\lambda(\omega)$ is an operator acting in the space obtained from the space $\mathcal{H}$ by the restriction

$$\sum_{i=1}^n v_i = \omega. \quad \text{We now state}$$

THEOREM 9. *There is a positive function $\lambda_0(\omega)$, $\lambda_0(\omega) > 0$ for all ω , such that, for any fixed energy ω , $S_\lambda(\omega)$ is a unitary operator for $|\lambda| \leq \lambda_0(\omega)$.*

$$S_\lambda(\omega) S_\lambda^*(\omega) = S_\lambda^*(\omega) S_\lambda(\omega) = 1.$$

Moreover, $S_\lambda(\omega)$ is analytic in λ for $|\lambda| \leq \lambda_0(\omega)$.

The proof of this theorem follows closely the proof of Theorem 7.

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³ Kato, T., "Wave operators and similarity for some non-self-adjoint operators," *Math. Ann.*, **162**, 275 (1966).

⁴ Chestjakov, A. L., "Ob operatore ressejanija v prostranstve vtorichnovo kvantovanija," *Dokl. Akad. Nauk SSSR*, **158** (1964).