TWO KINDS OF THETA CONSTANTS AND PERIOD RELATIONS ON A RIEMANN SURFACE

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Abstract.—It was recognized in Riemann's work more than one hundred years ago and proved recently by Rauch (cf. *Bull. Am. Math. Soc.*, **71**, 1–39 (1965)) that the $g(g + 1)/2$ unnormalized periods of the normal differentials of first kind on a compact Riemann surface S of genus $g \geq 2$ with respect to a canonical homology basis are holomorphic functions of $3g - 3$ complex variables, "the" moduli, which parametrize the space of Riemann surfaces near S and, hence, that there are $(g - 2)(g - 3)/2$ holomorphic relations among those periods. Eighty years ago, Schottky exhibited the one relation for $g = 4$ as the vanishing of an explicit homogeneous polynomial in the Riemann theta constants. Sixty years ago, Schottky and Jung conjectured a result which implies Schottky's earlier one and some generalizations for higher genera.

Here, we formulate Schottky and Jung's conjecture precisely and, on the basis of a recent result of Farkas (these PROCEEDINGS, **62**, 320 (1969)), prove it. We then derive Schottky's result (we believe for the first time correctly) and exhibit a typical relation of this kind for $g = 5$ (we can do this for any genus). We do not prove that *our* relations imply *all* relations, but there are some indications that they do.

1. *Introduction.*—In this note, which is a sequel to an earlier one,¹ we bring to a provisional conclusion the program outlined in section 1 there. Our main result, Theorem 1 of section 2 below, is the generalization of Theorem 1 of reference 1, anticipated there. As remarked in section 4 of reference 1, Schottky's relation² for $g = 4$ and analogous relations among the periods of the normal abelian integrals for higher genera result immediately from Theorem 1 in the form of certain homogeneous polynomials in the Riemann theta constants equated to zero. In section 4 herein we derive Schottky's result and a typical relation for $g = 5$. Our derivation of Schottky's result seems to us to be the first comprehensible one in the literature. We have *not* yet proved that our method gives *all* the $(g - 2)(g - 3)/2$ relations among the $g(g + 1)/2$ periods. In this respect we call attention to a paper by Andreotti and Mayer,³ who prove by quite a different method the existence of polynomials in the Riemann theta constants and their derivatives whose vanishing implies *all* the period relations and who indicate a process for eliminating the derivatives. It would seem a considerable labor at best to exhibit explicitly their relations of the first kind and much more for the second kind, and, at any rate, they do not obtain our relations (in particular not Schottky's in his form). Ours (actually they were, in effect, conjectured by Schottky and Jung in 1909⁴), on the other hand, can be written down with ease (section 4).

The proof of Theorem 1 follows the lines conjectured in section 3 of reference 1. Farkas has found⁵ the appropriate generalization (Theorem 2) of

Theorem 2 there, and only the vagueness of his earlier process⁶ for generating certain characteristics has stood in the way of completing the argument. The authors have then observed that Farkas' process gives the desired answer on suitable hyperelliptic surfaces and that the answer can be "propagated" into all of the Torelli space T^g by variational methods. All of this is outlined in section 2 herein. In section 3 the hyperelliptic and propagation arguments are sketched. We use freely the notation, definitions, and lemmata of reference 1. A detailed version will appear elsewhere.

2. Main Theorem and Outline.

THEOREM 1. *On a Riemann surface S of genus $g \geq 2$ with canonical homology basis $\gamma_1, \dots, \gamma_g; \delta_1, \dots, \delta_g$ the associated Schottky and Riemann theta constants (ref. 1, Definitions 2 and 4) satisfy the proportionalities*

$$\frac{\eta^2 \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}}{\theta \begin{bmatrix} 0\epsilon \\ 0\epsilon' \end{bmatrix}} = \theta \frac{\eta^2 \begin{bmatrix} \mu \\ \mu' \end{bmatrix}}{\begin{bmatrix} 0\mu \\ 0\mu' \end{bmatrix} \theta \begin{bmatrix} 0\epsilon \\ 1\epsilon' \end{bmatrix}} \quad (1)$$

for all $2^{g-2}(2^{g-1} + 1)$ even $(g - 1)$ -characteristics $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$, $\begin{bmatrix} \mu \\ \mu' \end{bmatrix}$, where $\begin{bmatrix} 0\epsilon \\ 0\epsilon' \end{bmatrix}$, $\begin{bmatrix} 0\epsilon \\ 1\epsilon' \end{bmatrix}$, $\begin{bmatrix} 0\mu \\ 0\mu' \end{bmatrix}$, $\begin{bmatrix} 0\mu \\ 1\mu' \end{bmatrix}$ are the g -characteristics obtained by bordering $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ (resp. $\begin{bmatrix} \mu \\ \mu' \end{bmatrix}$) by zeros on the left and then adding (matrixwise) the g -characteristic $\chi = \begin{bmatrix} 0 & \dots & 0 \\ 1, 0, \dots, 0 \end{bmatrix}$.

Definition 1: $\hat{S}$ is the two-sheeted unramified covering of S associated with the homology basis $\gamma_1, \dots, \gamma_g; \delta_1, \dots, \delta_g$ on S and the characteristic χ . That is, all the cycles of the basis except δ_1 lift to cycles (one on each sheet), while twice δ_1 lifts to a cycle. $\hat{\gamma}_1, \dots, \hat{\gamma}_{2g-1}; \hat{\delta}_1, \dots, \hat{\delta}_{2g-1}$ is the canonical homology basis on $\hat{S}$ (which has genus $2g - 1$ by the Riemann-Hurwitz relation) obtained as follows: $\hat{\delta}_1$ is the (unique) lift of twice δ_1 ; $\hat{\gamma}_1$ is either lift of γ_1 (the two lifts are homologous); $\hat{\gamma}_i, \hat{\gamma}_{i+g-1}$ are the two lifts of γ_i ; and $\hat{\delta}_i, \hat{\delta}_{i+g-1}$ are the two lifts of δ_i , $2 \leq i \leq g$, where the latter are chosen to have the proper intersections with the former. $d\hat{\zeta}_i, i = 1, \dots, 2g - 1$ are the normal abelian differentials on $\hat{S}$ with respect to this basis, and $\hat{\pi}$ is the matrix of their periods over the $\hat{\delta}$'s (see Lemma 2 of ref. 1).

THEOREM 2.⁵ *A sufficient condition for the validity of (1) is*

$$\theta \begin{bmatrix} 0 & \epsilon & \epsilon \\ 1 & \epsilon' & \epsilon' \end{bmatrix} = 0 \quad (2)$$

for all $2^{g-2}(2^{g-1} - 1)$ odd $(g - 1)$ -characteristics $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$, where the left side is the Riemann theta constant on $\hat{S}$ (Definition 1) with the $(2g - 1)$ -characteristic whose first column is as shown, whose second through g th columns are the first through $(g - 1)$ st columns of $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$, and whose $(g + 1)$ st through $(2g - 1)$ st columns are the same columns repeated.

THEOREM 3. Equations (2), in fact, hold.

LEMMA 1. By means of the construction of Definition 1 one can define a holomorphic map $f: T^g \rightarrow T^{2g-1}$, the Torelli spaces⁷ of the indicated genera. Under f the left sides of (2) become functions and, hence, holomorphic functions on T^g . In particular, to prove Theorem 3 it is sufficient to prove (2) in a neighborhood of some point in T^g .

THEOREM 4. There is a hyperelliptic Torelli surface $t_0 \in T^g$ and a neighborhood of t_0 in T^g such that equations (2) hold there.

3. Sketch of Proof of Theorem 4.

Definition 2: Let $\tilde{S}$ be a Riemann surface of genus G with a canonical homology basis $\tilde{\gamma}_1, \dots, \tilde{\gamma}_G$; $\tilde{\delta}_1, \dots, \tilde{\delta}_G$ and normal differentials of first kind $d\tilde{\xi}_1, \dots, d\tilde{\xi}_G$ with respect to that basis. The period matrix of these differentials is $(I_G, \tilde{\pi})$, where I_G is the $G \times G$ unit matrix. A $(G-)$ half-period $\begin{pmatrix} \mu \\ \mu' \end{pmatrix}$ is a G -vector whose i th component, $i = 1, \dots, G$, is $\mu'_i/2 + \sum_j \tilde{\pi}_{ij} \mu_j/2$, where μ and μ' are G -vectors of integers. The G -characteristic $\begin{bmatrix} \mu \\ \mu' \end{bmatrix}$ is the symbol of $\begin{pmatrix} \mu \\ \mu' \end{pmatrix}$. Let $P_0 \in \tilde{S}$ be fixed and denoted as the base point. Given a divisor $a^n = P_1^{n_1} \dots P_m^{n_m}$, $n = n_1 + \dots + n_m$, $P_1, \dots, P_m \in \tilde{S}$, on $\tilde{S}$, define $\tilde{\xi}(a^n)$ to be the G -vector

$$\tilde{\xi}_i(a^n) = \sum_{l=1}^m n_l \int_{P_0}^{P_l} d\tilde{\xi}_i, \quad i = 1, \dots, G,$$

where some definite set of paths of integration from P_0 to $P_1, \dots, P_m$ has been chosen (the same for each i). Two G -vectors which differ by an integral linear combination of the columns of the period matrix are said to be congruent (modd. periods), written $\equiv$. Hence up to congruence the choice of paths in $\tilde{\xi}(a^n)$ is irrelevant.

LEMMA 2.⁸ With the assumptions and notation of Definition 2, a necessary and sufficient condition for

$$\theta \begin{bmatrix} \mu \\ \mu' \end{bmatrix} = 0, \quad (3)$$

where the left side is the indicated Riemann theta constant on $\tilde{S}$ with respect to the basis $\tilde{\gamma}_1$, etc., is that there exist a base point P_0 and a divisor a^{g-1} on $\tilde{S}$ such that

$$\begin{bmatrix} \mu \\ \mu' \end{bmatrix} \equiv \tilde{\xi}(a^{g-1}) + \tilde{K}(P_0), \quad (4)$$

where the left side is the half-period whose symbol is $\begin{bmatrix} \mu \\ \mu' \end{bmatrix}$ and $\tilde{K}(P_0)$ is the G -vector of Riemann constants on $\tilde{S}$ (with respect to $\tilde{\gamma}_1$, etc., and the base point P_0).

The following is a sharpened version of the principal device in reference 5.

THEOREM 5. Let S and $\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g$ be as in section 2 and $\hat{S}, \hat{\gamma}_1, \dots, \hat{\gamma}_{2g-1}; \hat{\delta}_1, \dots, \hat{\delta}_{2g-1}$ as in Definition 1. Given $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$, an odd $(g-1)$ -characteristic, there exists at least one divisor a , $\deg a = g-1$, on S such that for any base point $P_0 \in S$

$$\zeta(a) + K(P_0) \equiv \begin{pmatrix} 0\epsilon \\ 1\epsilon' \end{pmatrix}, \quad (5)$$

where the notation for the half-period is now self-explanatory. If $\hat{a}$ is the lift of a on $\hat{S}$, $\deg \hat{a} = 2g - 2 = G - 1$, $G = 2g - 1$, and if as base point $\hat{P}_0$ on $\hat{S}$ one chooses one of the lifts of P_0 , then

$$\hat{\zeta}(\hat{a}) + \hat{K}(\hat{P}_0) \equiv \begin{pmatrix} \mu \\ \mu' \end{pmatrix}, \quad (6)$$

where $\begin{pmatrix} \mu \\ \mu' \end{pmatrix}$ is some half-period on $\hat{S}$. Furthermore, $(\hat{a})^2$ is the divisor of a differential of first kind of $\hat{S}$. Finally, $\begin{pmatrix} \mu \\ \mu' \end{pmatrix}$ does not depend on $\hat{a}$ (i.e., a) but only on $\hat{P}_0$ (i.e., P_0) and, via the map f of Lemma 1, on the Torelli surface t determined by S and γ_1 , etc., and $\begin{pmatrix} \mu \\ \mu' \end{pmatrix}$ depends holomorphically on t and P_0 ; in particular, in a sufficiently small neighborhood of t in T^g , if a continuous choice of P_0 is made (and this can be done), the symbol $\begin{bmatrix} \mu \\ \mu' \end{bmatrix}$ is constant.

Sketch of proof: All of Theorem 5 except the final statement is, in essence, in reference 5. The final statement is deduced by explicitly computing $\hat{\zeta}(\hat{a})$ by expressing the vector $d\hat{\zeta}$ in terms of the vectors $d\zeta$ and $d\omega$ on S as in Lemma 2 of reference 1, in particular, using the form of $\hat{\pi}$ indicated by formula (5) there. One obtains, using relations among homology on S and S ,

$$\zeta(\hat{a}) \equiv \begin{pmatrix} g-1 & \epsilon & \epsilon' \\ 2 & \epsilon' & \epsilon' \end{pmatrix} + \nu, \quad (7)$$

where ν is a G -vector with components $\nu_1 = -2K_1(P_0)$, $\nu_\alpha = -K_\alpha(P_0) + \tau_{0(\alpha-1)}$, $\nu_{\alpha+g-1} = -K_\alpha(P_0) - \tau_{0(\alpha-1)}$, $\alpha = 2, \dots, g$. Here $\tau_{0\alpha}$ = period of $d\omega_\alpha$ over δ_1 ($d\omega_\alpha$ viewed as an abelian differential on $\hat{S}$). One sees that ν depends only on t (via f) and P_0 . Multiplying (7) by 2 and noting that twice a half-period is congruent to 0 and $2\hat{\zeta}(\hat{a}) \equiv -2\hat{K}(\hat{P}_0)$, since $(\hat{a})^2$ is the divisor of a differential, one has $2\nu \equiv -2\hat{K}(\hat{P}_0)$, i.e.,

$$\nu \equiv -\hat{K}(\hat{P}_0) + \begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}, \quad (8)$$

where $\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}$ is a half-period on S . Substituting (8) in (7), one has (6) with $\begin{pmatrix} \mu \\ \mu' \end{pmatrix}$ equal the sum of the half-periods in (7) and (8). The possibility of a continuous choice of P_0 results from the application of a suitable variational method.^{9, 10}

THEOREM 6. *There is a hyperelliptic Riemann surface S of genus g and on S a canonical homology basis $\gamma_1, \dots, \gamma_g; \delta_1, \dots, \delta_g$ and a base point P_0 such that $\hat{S}$ with homology basis $\hat{\gamma}_1, \dots, \hat{\gamma}_{2g-1}; \hat{\delta}_1, \dots, \hat{\delta}_{2g-1}$ constructed per Definition 1 has the following property: if a $(g-1)$ -divisor a on S satisfies*

$$\zeta(a) + K(P_0) \equiv \begin{pmatrix} 0 & \epsilon \\ 1 & \epsilon' \end{pmatrix} \quad (9)$$

and if $\hat{P}_0 \in \hat{S}$ is a suitable lift of P_0 , then

$$\hat{\zeta}(\hat{a}) + \hat{K}(\hat{P}_0) \equiv \begin{pmatrix} 1 & \epsilon & \epsilon \\ 1 & \epsilon' & \epsilon' \end{pmatrix}. \quad (10)$$

FIG. 1.

Note that the $(g-1)$ -characteristic $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ is not restricted to be odd but, in fact, can be proved to be so.

Sketch of proof: Let S be the hyperelliptic surface which is two-sheeted over the z -plane with branch points at $0, 1, \infty$ and $\lambda_1, \dots, \lambda_{2g-1}$ where $1 < \lambda_1 < \dots < \lambda_{2g-1}$ and with branch slits as drawn in Figure 1. Let $\gamma_1, \dots, \gamma_g$; $\delta_1, \dots, \delta_g$ be the homology basis indicated in Figure 1. Setting $z = \zeta^2$, one gets over the ζ -plane the hyperelliptic surface with branch slits as indicated in Figure 2 (0 and ∞ are not branch points). It is easy to verify that this surface is a model of $\hat{S}$ and that the homology basis indicated will serve as in Definition 1. $\hat{S}$ admits the involution T which is the composition of sheet interchange and reflection through $\zeta = 0$. $\hat{\gamma}_1$, etc., behave under T as in Lemma 2 of reference 1.

We set $P_0 = 1$ on S and $\hat{P}_0 = 1$ on $\hat{S}$. The theory of hyperelliptic surfaces¹¹ shows that we may take the points of a to be branch points of S ; that, further, $\zeta(\lambda_1), \dots, \zeta(\infty), \zeta(0)$ is a fundamental (maximal azygetic) set of half-periods; and, finally, that the sum of these which have odd (even) symbols is $K(1)$. Making the computations yields

$$K(P_0) = K(1) \equiv \left(1 \mid \begin{array}{ccc} g & 1 & \dots 1 \\ (g-1) & (g-2) & \dots 1 \end{array} \right) \equiv \begin{pmatrix} 1 & K \\ 1 & K' \end{pmatrix}, \quad (11)$$

where the vertical bar is simply a convenience to set off blocks of $g-1$ columns.

A similar computation on $\hat{S}$ yields

$$\hat{K}(\hat{P}_0) = \hat{K}(1) \equiv \left(g \mid \begin{array}{ccc} 1 & \dots 1 \\ (g-1) & \dots 1 \end{array} \mid \begin{array}{ccc} 1 & \dots 1 \\ (g-1) & \dots 1 \end{array} \right). \quad (12)$$

We now compute $\hat{\zeta}(\hat{a})$ in (10) as follows. From (9) and (11) one has

$$\hat{\zeta}(a) \equiv \begin{pmatrix} 1 & \epsilon + K \\ 0 & \epsilon' + K' \end{pmatrix}. \quad (13)$$

We make the vital observation that a contains either θ or ∞ but not both. This follows by replacing $\zeta(P)$ for each $P \in a$ by the half-period of the funda-

mental set equal to it and observing that the presence of both 0 and ∞ or of neither would give $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the first column of (13). Suppose first that 0 is in a , so that a consists of a $(g - 2)$ -divisor b and 0. The lift $\hat{a}$ of a then consists of a right-hand lift $\hat{b}_1$ of b and one lift of 0, say, $\hat{0}_1$, the point over 0 on the top sheet of $\hat{S}$ and the images of these under T , and $\hat{b}_2$, the left-hand lift of b , and $\hat{0}_2$, the point over 0 on the lower sheet of $\hat{S}$. Comparing Figures 1 and 2 and inspecting the table of half-periods on S , one sees that the half-period $\hat{\zeta}(\hat{b}_1)$ has the symbol obtained from that of the half-period to which $\zeta(b)$ is congruent by simply adding $g - 1$ zeros to the right of both rows of the latter. But

$$\begin{aligned} \zeta(b) &\equiv \begin{pmatrix} 1 & | & \epsilon + K \\ 0 & | & \epsilon' + K' \end{pmatrix} - \zeta(0) \equiv \begin{pmatrix} 1 & | & \epsilon + K \\ 0 & | & \epsilon' + K' \end{pmatrix} \\ &\quad - \begin{pmatrix} 1 & | & 10 \dots 0 \\ 0 & | & 0 \dots 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & | & \epsilon + K \\ 0 & | & \epsilon' + K' \end{pmatrix} + \begin{pmatrix} 0 & | & 10 \dots 0 \\ 0 & | & 0 \dots 0 \end{pmatrix}, \end{aligned}$$

as one sees from the table of half-periods on S . Hence

$$\hat{\zeta}(\hat{b}_1\hat{0}_1) \equiv \begin{pmatrix} 0 & | & \epsilon + K & | & 0 \dots 0 \\ 0 & | & \epsilon' + K' & | & 0 \dots 0 \end{pmatrix} + \begin{pmatrix} 0 & | & 10 \dots 0 & | & 0 \dots 0 \\ 0 & | & 0 \dots 0 & | & 0 \dots 0 \end{pmatrix} + \int_1^{\hat{0}_1} d\hat{\zeta}.$$

Looking at Figure 2, one sees immediately then that

$$\begin{aligned} \hat{\zeta}(\hat{b}_2\hat{0}_2) &\equiv \begin{pmatrix} 0 & | & 0 \dots 0 & | & \epsilon + K \\ 0 & | & 0 \dots 0 & | & \epsilon' + K' \end{pmatrix} + \begin{pmatrix} 0 & | & 0 \dots 0 & | & 10 \dots 0 \\ 0 & | & 0 \dots 0 & | & 0 \dots 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} (g-2) & | & (g-2) 0 \dots 0 & | & (g-2) 0 \dots 0 \\ 0 & | & 0 \dots 0 & | & 0 \dots 0 \end{pmatrix} + \int_1^{\hat{0}_2} d\hat{\zeta}, \end{aligned}$$

where the third term is the result of integrating $d\hat{\zeta}$ from 1 to the $g - 2$ points of $\hat{b}_2$ on $\hat{S}$. Adding the last two equations, adding $\hat{K}(\hat{P}_0)$ from (12), and using (11), one obtains

$$\hat{\zeta}(\hat{a}) + \hat{K}(\hat{P}_0) \equiv \begin{pmatrix} 0 & | & \epsilon & | & \epsilon \\ 1 & | & \epsilon' & | & \epsilon' \end{pmatrix} + \int_1^{\hat{0}_1} d\hat{\zeta} + \int_1^{\hat{0}_2} d\hat{\zeta}.$$

But the last two terms cancel, since all differentials on a hyperelliptic surface, in particular $\hat{S}$, change sign under sheet interchange. Thus (10) and with it

FIG. 2.

Theorem 6 is proved (the proof is virtually unchanged with ∞ instead of 0).

The first part of Theorem 5 implies that (9) holds for odd $(g-1)$ -characteristics $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$. Hence (10) holds, which, by the last part of Theorem 5, implies Theorem 4, hence Theorem 1.

4. *Schottky's Identity for $g=4$ and Its Generalizations for $g=5$.*—According to Schottky,² we obtain a relation among general theta constants of dimension 3, in particular, among Schottky constants of genus 4, as follows. Take a syzygetic group of half-periods of degree 2, $G = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}, \begin{pmatrix} \mu \\ \mu' \end{pmatrix}, \begin{pmatrix} \lambda + \mu \\ \lambda' + \mu' \end{pmatrix} \right\}$, and find three 3-characteristics $\begin{bmatrix} \alpha \\ \alpha' \end{bmatrix}, \begin{bmatrix} \beta \\ \beta' \end{bmatrix}, \begin{bmatrix} \gamma \\ \gamma' \end{bmatrix}$ such that the twelve 3-characteristics obtained by composing those three with the elements of G are all even. For example, take $\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix} = \begin{pmatrix} 100 \\ 000 \end{pmatrix}, \begin{pmatrix} \mu \\ \mu' \end{pmatrix} = \begin{pmatrix} 010 \\ 000 \end{pmatrix}, \begin{bmatrix} \alpha \\ \alpha' \end{bmatrix} = \begin{bmatrix} 000 \\ 000 \end{bmatrix}, \begin{bmatrix} \beta \\ \beta' \end{bmatrix} = \begin{bmatrix} 001 \\ 000 \end{bmatrix}, \begin{bmatrix} \gamma \\ \gamma' \end{bmatrix} = \begin{bmatrix} 000 \\ 001 \end{bmatrix}$. Then the relation is (there is a definite choice of signs, but they are irrelevant as one sees later)

$$\eta \begin{bmatrix} 000 \\ 000 \end{bmatrix} \eta \begin{bmatrix} 100 \\ 000 \end{bmatrix} \eta \begin{bmatrix} 010 \\ 000 \end{bmatrix} \eta \begin{bmatrix} 110 \\ 000 \end{bmatrix} \pm \eta \begin{bmatrix} 001 \\ 000 \end{bmatrix} \eta \begin{bmatrix} 101 \\ 000 \end{bmatrix} \eta \begin{bmatrix} 011 \\ 000 \end{bmatrix} \eta \begin{bmatrix} 111 \\ 000 \end{bmatrix} \\ \pm \eta \begin{bmatrix} 000 \\ 001 \end{bmatrix} \eta \begin{bmatrix} 100 \\ 001 \end{bmatrix} \eta \begin{bmatrix} 010 \\ 001 \end{bmatrix} \eta \begin{bmatrix} 110 \\ 001 \end{bmatrix} = 0. \quad (R_1)$$

Substituting from (1) (taking square roots), one has one form of Schottky's relation among the Riemann theta constants, and, hence, among the periods of the normal integrals of first kind on S for $g=4$:

$$\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0 \quad (R_2)$$

(here the signs are indeterminate but again irrelevant), or in rational form with sign ambiguities removed,

$$r_1^2 + r_2^2 + r_3^2 - 2(r_1 r_2 + r_1 r_3 + r_2 r_3) = 0, \quad (R_2')$$

$$r_1 = \theta \begin{bmatrix} 0000 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0000 \\ 1000 \end{bmatrix} \theta \begin{bmatrix} 0100 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0100 \\ 1000 \end{bmatrix} \theta \begin{bmatrix} 0010 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0010 \\ 1000 \end{bmatrix} \theta \begin{bmatrix} 0110 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0110 \\ 1000 \end{bmatrix}, \\ r_2 = \theta \begin{bmatrix} 0001 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0001 \\ 1000 \end{bmatrix} \theta \begin{bmatrix} 0101 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0101 \\ 1000 \end{bmatrix} \theta \begin{bmatrix} 0011 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0011 \\ 1000 \end{bmatrix} \theta \begin{bmatrix} 0111 \\ 0000 \end{bmatrix} \theta \begin{bmatrix} 0111 \\ 1000 \end{bmatrix}, \\ r_3 = \theta \begin{bmatrix} 0000 \\ 0001 \end{bmatrix} \theta \begin{bmatrix} 0000 \\ 1001 \end{bmatrix} \theta \begin{bmatrix} 0100 \\ 0001 \end{bmatrix} \theta \begin{bmatrix} 0100 \\ 1001 \end{bmatrix} \theta \begin{bmatrix} 0010 \\ 0001 \end{bmatrix} \theta \begin{bmatrix} 0010 \\ 1001 \end{bmatrix} \theta \begin{bmatrix} 0110 \\ 0001 \end{bmatrix} \theta \begin{bmatrix} 0110 \\ 1001 \end{bmatrix}.$$

To obtain a typical relation in genus $g=5$, we again write a typical relation among Schottky theta constants by choosing a syzygetic group G of half-periods of degree 2 and then an azygetic set of four 4-characteristics, which, when composed with G , give sixteen even 4-characteristics. We choose $\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix} = \begin{pmatrix} 1000 \\ 0000 \end{pmatrix}, \begin{pmatrix} \mu \\ \mu' \end{pmatrix} = \begin{pmatrix} 0100 \\ 0000 \end{pmatrix}$. A suitable azygetic set of characteristics is $\begin{bmatrix} 0000 \\ 0000 \end{bmatrix}, \begin{bmatrix} 0001 \\ 0000 \end{bmatrix},$

$\begin{bmatrix} 0000 \\ 0001 \end{bmatrix}, \begin{bmatrix} 0011 \\ 0011 \end{bmatrix}$. Take any of these and compose it with G and form the product of the four Schottky theta constants whose characteristics are thus obtained. Doing this for each member of the azygetic set, one obtains four products whose sum with suitable signs is zero (analogous to (R_1)). Again substituting square roots from (1) and rationalizing, one obtains

$$\Pi(\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} \pm \sqrt{r_4}) = (r_1 + r_2 - r_3 - r_4)^4 - 8(r_1 r_2 + r_3 r_4) \times (r_1 + r_2 - r_3 - r_4)^2 + 16(r_1 r_2 + r_3 r_4)^2 - 64 r_1 r_2 r_3 r_4 = 0, \quad (R_3)$$

$$\begin{aligned} r_1 &= \theta \begin{bmatrix} 00000 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 00000 \\ 10000 \end{bmatrix} \theta \begin{bmatrix} 01000 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 01000 \\ 10000 \end{bmatrix} \\ &\quad \times \theta \begin{bmatrix} 00100 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 00100 \\ 10000 \end{bmatrix} \theta \begin{bmatrix} 01100 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 01100 \\ 10000 \end{bmatrix}, \\ r_2 &= \theta \begin{bmatrix} 00001 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 00001 \\ 10000 \end{bmatrix} \theta \begin{bmatrix} 01001 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 01001 \\ 10000 \end{bmatrix} \\ &\quad \times \theta \begin{bmatrix} 00101 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 00101 \\ 10000 \end{bmatrix} \theta \begin{bmatrix} 01101 \\ 00000 \end{bmatrix} \theta \begin{bmatrix} 01101 \\ 10000 \end{bmatrix}, \\ r_3 &= \theta \begin{bmatrix} 00000 \\ 00001 \end{bmatrix} \theta \begin{bmatrix} 00000 \\ 10001 \end{bmatrix} \theta \begin{bmatrix} 01000 \\ 00001 \end{bmatrix} \theta \begin{bmatrix} 01000 \\ 10001 \end{bmatrix} \\ &\quad \times \theta \begin{bmatrix} 00100 \\ 00001 \end{bmatrix} \theta \begin{bmatrix} 00100 \\ 10001 \end{bmatrix} \theta \begin{bmatrix} 01100 \\ 00001 \end{bmatrix} \theta \begin{bmatrix} 01100 \\ 10001 \end{bmatrix}, \\ r_4 &= \theta \begin{bmatrix} 00011 \\ 00011 \end{bmatrix} \theta \begin{bmatrix} 00011 \\ 10011 \end{bmatrix} \theta \begin{bmatrix} 01011 \\ 00011 \end{bmatrix} \theta \begin{bmatrix} 01011 \\ 10011 \end{bmatrix} \\ &\quad \times \theta \begin{bmatrix} 00111 \\ 00011 \end{bmatrix} \theta \begin{bmatrix} 00111 \\ 10011 \end{bmatrix} \theta \begin{bmatrix} 01111 \\ 00011 \end{bmatrix} \theta \begin{bmatrix} 01111 \\ 10011 \end{bmatrix}, \end{aligned}$$

where the product in the initial identity is over all eight choices of sign and the rational form is derived by the primitive device of repeated transposition and squaring.

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