ON THE DECOMPOSITION OF STRESS AND STRAIN TENSORS INTO SPHERICAL AND DEVIATORIC PARTS

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Abstract.—It is well known that Hooke's law for a linearly elastic, isotropic solid may be written in the form of two relations that involve only the spherical or only the deviatoric parts of the tensors of stress and strain. The example of the linearly elastic, transversely isotropic solid is used to show that this decomposition is not, in general, feasible for linearly elastic, anisotropic solids. The discussion is extended to a large class of work-hardening rigid, plastic solids, and it is shown that the considered decomposition can only be achieved for the incompressible solids of this class.

The decomposition of stress and strain tensors into spherical and deviatoric parts is widely used in solid mechanics. For infinitesimal strains, it separates changes of volume from changes of shape. Its mechanical importance stems from the fact that, for many solids, the first kind of change depends only on the spherical part, and the second, only on the deviatoric part of the stress tensor. In linear isotropic elasticity, for instance, the spherical parts of stress and strain are related by the bulk modulus, and the deviatoric parts, by the shear modulus. In view of the important role of the decomposition in many branches of solid mechanics, it is not surprising that an essential restriction on its use has occasionally been overlooked: for anisotropic solids, the decomposition is, in general, devoid of physical meaning. The following examples illustrate this fact.

Consider first a linearly elastic, transversely isotropic solid. Taking the plane of isotropy as the x_1x_2 -plane of the rectangular Cartesian coordinates x_i ($i = 1, 2, 3$), and denoting by σ_{ij} and ϵ_{ij} the components of stress and strain, we have the constitutive equation

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{bmatrix} = \begin{bmatrix} A + 2B & A & 2C & 0 & 0 & 0 \\ A & A + 2B & 2C & 0 & 0 & 0 \\ 2C & 2C & 2D & 0 & 0 & 0 \\ 0 & 0 & 0 & 2B & 0 & 0 \\ 0 & 0 & 0 & 0 & 2E & 0 \\ 0 & 0 & 0 & 0 & 0 & 2E \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}, \quad (1)$$

where $A, B, \dots, E$ are elastic constants.

According to (1), the mean normal strain $\epsilon = (\epsilon_{11} + \epsilon_{22} + \epsilon_{33})/3$ is given by

$$\epsilon = \frac{2}{3} \{ [A + B + C](\sigma_{11} + \sigma_{22}) + [2C + D]\sigma_{33} \}. \quad (2)$$

If ϵ is to be proportional to the mean normal stress $\sigma = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$, the bracketed expressions in (2) must have the same value. Thus, the mean normal strain will be independent of the stress deviator if and only if

$$A + B = C + D. \quad (3)$$

Note that this condition is fulfilled for the isotropic, linearly elastic solid for which (1) holds with $2C = A$, $2B = 2D = 2E = A + 2B$.

If (3) is satisfied, a purely deviatoric state of stress does not cause any volume change and thus produces a purely deviatoric strain. Conversely for $\epsilon = 0$ to be a consequence of $\sigma = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3 = 0$, the condition (3) must obviously be satisfied. Thus, (3) is a necessary and sufficient condition for the decomposition of the constitutive law (1) into a relation involving only the spherical parts and a relation involving only the deviatoric parts of stress and strain. Since the general transversely isotropic solid with the constitutive relation (1) does not satisfy (3), it does not admit this decomposition.

To be specific, the preceding discussion was restricted to a particular kind of anisotropy. Quite generally, however, the decomposition of the constitutive law into spherical and deviatoric parts had the following implications: (i) a deviatoric stress cannot cause a change in volume and thus produces a purely deviatoric strain; (ii) a spherical stress cannot cause deviatoric strains and thus produces a purely spherical strain. Conversely, if these statements hold true, it follows from the linearity of the constitutive law that it can be decomposed in the desired fashion.

Note that, in linear elasticity, each of the two statements above follows from the other. This may be deduced from Betti's principle, which is a consequence of the existence of a quadratic strain energy function. Indeed, let ϵ_{ij} with $\epsilon_{ii} = 0$ (summation convention!) be the purely deviatoric response to the deviatoric stress σ_{ij} (with $\sigma_{ii} = 0$), and suppose that the spherical stress $\sigma'_{ij} = \sigma' \delta_{ij}$, where δ_{ij} is the Kronecker delta, produces the strain

$$\epsilon'_{ij} = e'_{ij} + \epsilon' \delta_{ij}, \quad (4)$$

where e'_{ij} is deviatoric ($e'_{ii} = 0$). By Betti's principle, we have

$$\sigma_{ij} \epsilon'_{ij} = \sigma'_{ij} \epsilon_{ij} = \sigma' \epsilon_{ii} = 0. \quad (5)$$

Substituting (4) into (5), we obtain

$$\sigma_{ij} e'_{ij} + \sigma_{ii} \epsilon' = \sigma_{ij} \epsilon'_{ij} = 0. \quad (6)$$

Since this must hold regardless of the choice of the deviatoric stress σ_{ij} , it follows that $e'_{ij} = 0$, that is, the strain caused by the spherical stress is spherical. The statement (ii) above thus is seen to be a consequence of (i). The converse can be established in a similar manner.

It is worth noting that an analog to Betti's principle may be used in a similar fashion to investigate the decomposition of constitutive laws for rigid, work-hardening solids into spherical and deviatoric parts. Let

$$f \equiv f(\sigma_{ij}, \epsilon_{ij}, \alpha_1, \dots, \alpha_n) = 0 \quad (7)$$

be the yield condition in the state specified by the stresses σ_{ij} , the plastic strains ϵ_{ij} , and parameters $\alpha_1, \dots, \alpha_n$ that are supposed to reflect the history of loading. For example, if the loading history includes periods of rigid comportment separated by periods of plastic flow, α_1 may be the mechanical energy dissipated in the transition from the virgin to the present state, α_2 may be the energy dissipated

up to the beginning of the last period of rigid comportment, etc. As was shown by Prager,¹ the constitutive law

$$\dot{\epsilon}_{ij} = \begin{cases} 0 & \text{if } f < 0, \\ -\frac{\partial f}{\partial \sigma_{ij}} \left\langle \frac{\partial f}{\partial \sigma_{kl}} \dot{\sigma}_{kl} \right\rangle / \left(\frac{\partial f}{\partial \sigma_{pq}} \frac{\partial f}{\partial \sigma_{pq}} \right) & \text{if } f = 0, \end{cases} \quad (8)$$

where

$$\langle x \rangle = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x > 0, \end{cases} \quad (9)$$

has an appropriate mathematical structure provided that

$$\left. \begin{aligned} &\sigma_{ij} \partial f / \partial \sigma_{ij} > 0 \\ &\frac{\partial f}{\partial \epsilon_{pq}} \frac{\partial f}{\partial \sigma_{pq}} < 0 \end{aligned} \right\} \text{ for } f = 0. \quad (10)$$

In particular, the constitutive law (8) assures unique strain rate fields for a wide class of boundary value problems.

In discussing the decomposition of the constitutive law (8) into spherical and deviatoric parts, we only need to be concerned with states at the instantaneous yield limit $f = 0$ and stress rates $\dot{\sigma}_{ij}$ that produce plastic flow. The angle brackets in (8) may then be dropped. For a given state of the solid, that is, for given states of stress and strain and given values of the parameters $\alpha_1, \dots, \alpha_n$, consider the stress rates $\dot{\sigma}_{ij}$ and $\dot{\sigma}'_{ij}$, and denote the corresponding strain rates by $\dot{\epsilon}_{ij}$ and $\dot{\epsilon}'_{ij}$. As is readily seen from (8), Betti's principle is valid in the form

$$\dot{\sigma}_{ij} \dot{\epsilon}_{ij} = \dot{\sigma}'_{ij} \dot{\epsilon}_{ij}. \quad (11)$$

(Indeed, when the angle brackets are omitted in (8), the coefficient of $\dot{\sigma}_{kl}$ is symmetric with respect to the index pairs i, j and k, l .) This reciprocity principle can be used to discuss the decomposition of (8) in the same manner in which the classical reciprocity principle has been used above for an elastic solid. It follows that the constitutive law (8) can be decomposed in the desired manner if and only if a deviatoric stress rate produces a purely deviatoric strain rate.

To investigate the consequences of this condition, it will be convenient to set $\sigma_{ij} = s_{ij} + \sigma \delta_{ij}$ with $s_{ii} = 0$ and write the yield function f as a function $\bar{f}$ of ϵ_{ij} , s_{ij} , σ , $\alpha_1, \dots, \alpha_n$. If the denominator in (8) is abbreviated by $-D$, with $D > 0$ by (10), the constitutive law (8) furnishes the mean normal strain rate

$$\dot{\epsilon} = \frac{1}{3D} \frac{\partial \bar{f}}{\partial \sigma} \left(\frac{\partial \bar{f}}{\partial s_{kl}} \dot{s}_{kl} + \frac{\partial \bar{f}}{\partial \sigma} \dot{\sigma} \right). \quad (12)$$

If this is to vanish when $\dot{\sigma}_{kl}$ is deviatoric, that is, when $\dot{\sigma} = 0$, we must have $\partial \bar{f} / \partial \sigma = 0$. In this case, however, $\dot{\epsilon} = 0$ even if $\dot{\sigma}_{kl}$ is not purely deviatoric. In other words, the considered constitutive law can be decomposed in the desired fashion only if it describes an *incompressible* rigid, work-hardening solid.

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¹ Prager, W., *Proc. Roy. Neth. Acad. Sci., Ser. B*, 61, 176-182 (1958).