All of the writers wish to acknowledge the support of the Oak Ridge National Laboratory under Subcontract 1491. Two of us (J. A. S. A. and J. J. W. R.) have received substantial funds under Grant C-009 from the Robert A. Welch Foundation for studies on the geochemistry of thorium. The Robert A. Welch Foundation under Grant K-054b also provided one of us (J. A. S. A.) with funds for both the development and construction of the laboratory and field radiometric equipment. The writers would like to thank numerous individuals for their many courtesies and stimulating discussions regarding the present work. In particular, the help of the following is acknowledged: A. P. Butler, Jr., D. Gottfried, E. S. Larsen III, and George Phair of the U.S. Geological Survey; K. B. Brown, D. Crouse, and F. Hurst of the Oak Ridge National Laboratory; J. B. Lyons of Dartmouth College; and M. P. Billings of Harvard University.

¹ U.S. Atomic Energy Commission, Technical Information Service Report TID-8201 (1959).

² McKelvey, V. E., *Am. J. Sci.*, **258-A**, 234-241 (1960).

³ Adams, J. A. S., J. K. Osmond, and J. J. W. Rogers, *Physics and Chemistry of the Earth*, **3**, 298-348 (1959).

⁴ Kline, M.-C., J. A. S. Adams, and J. J. W. Rogers, *Geological Society of America Special Paper* **68**, 211 (1961).

⁵ Billings, M. P., and N. B. Keevil, *Geol. Soc. Am. Bull.*, **57**, 797-828 (1946).

⁶ Whitfield, J. M., Ph.D. Thesis, Rice University (1958).

⁷ Butler, A. P., Jr., *U.S. Geological Survey Professional Paper* **424-B**, 67-69 (1961).

⁸ Lyons, J. B., *U.S. Geological Survey Professional Paper* **424-B**, 69-71 (1961).

⁹ Billings, M. P., *The Geology of New Hampshire; Part II—Bedrock Geology*: New Hampshire Planning and Development Commission (1956).

¹⁰ Adams, J. A. S., *Geological Society of America Special Paper* **68**, 125 (1961).

¹¹ Ahrens, L. H., *Geochim. et Cosmochim. Acta*, **11**, 205-212 (1957).

¹² Jizba, Z. V., *Geochim. et Cosmochim. Acta*, **16**, 79-82 (1959).

¹³ Rogers, J. J. W., *International Geological Congress, XXI Session, Part XXI*, 275-280 (1960).

¹⁴ Ragland, P. C., Ph.D. Thesis, Rice University (1962).

¹⁵ Bean, R. J., *Geol. Soc. Am. Bull.*, **64**, 509-537 (1953).

¹⁶ Guilbert, J. M., and J. A. S. Adams, *Nucleonics*, **13**, 43 (1955).

¹⁷ Anderson, A. L., *Econ. Geol.*, **56**, 177-197 (1961).

THE CHRONOLOGY OF CALEDONIAN FOLDING IN THE BRITISH ISLES

By W. S. MCKERROW

DEPARTMENT OF GEOLOGY AND MINERALOGY, OXFORD UNIVERSITY

Communicated by James Gilluly, August 22, 1962

The Dalradian areas of Scotland and Ireland had their large major folding and metamorphism before the Arenig; strong pre-Silurian folding is known in the Welsh Borderland and western Ireland; at the end of the Silurian, we know there is folding in Berwickshire and Pembrokeshire; in Middle Old Red Sandstone time, the main movements are in South Wales, in the Welsh Borderland, and near fault zones in Scotland. The evidence for these events is provided by unconformities, radioactive age dating of granites and metamorphism, and the oncoming of Old Red Sandstone facies. It is suggested that the Caledonian orogeny is the sum total of all these events from the Upper Cambrian to the Middle Devonian which occur in, or adjacent to, the Caledonian geosyncline.

The stratigraphical evidence of the times of folding is reviewed with the following

questions in mind: (1) Was there a single maximum folding episode? (2) If so, when was it? (3) If there were several events, were they local or widespread? Were they spasmodic or prolonged? Over what period of time did they take place?

Spieker,³⁴ in examining similar questions, stated that "the way in which we think about orogeny, the time scale, our stratigraphic sections, strongly controls the actual nature of the practical facts we gather in the field." For this reason, if for no other, it seems necessary to review recent work on the Caledonian orogeny.

Major earth movements can be dated by unconformities, effects on sedimentation, igneous activity, and radioactive dating, which all reflect different aspects of orogeny; the same orogenic activity might result in different effects in different areas and at slightly different times. Of these, it is only the unconformity which is both direct proof of movements and capable of accurate stratigraphical control; some emphasis is therefore to be given to breaks in the stratigraphic column.

The evidence is presented geographically, starting in the south of Scotland, then through England to Wales, westwards to Ireland, and finally back to Scotland.

Southern Uplands and North of England.—The oldest widespread breaks in sedimentation in this area are at the base of the Caradocian; they have been recorded in southwest Scotland, Cross Fell, and the Lake District. Williams³⁸ has demonstrated that there have been subsequent sporadic movements throughout the Upper Ordovician in the Girvan area. He considers that recurrent impersistent conglomerates are associated with repeated movements along faults bounding the basin of deposition. The Bala grades up into the Silurian without any big break in places where the contact has been described (Girvan, Moffat, Lake District). In fact, sedimentation seems to have been continuous over the whole area from late Ordovician times until the end of the Wenlock, or possibly well into the Ludlow. Then there was a big break.

The Lower Old Red Sandstone rests unconformably on the Silurian in Berwickshire (where the age of the overlying beds is proved by *Pterygotus*),¹³ near Jedburgh, in the Girvan district, and possibly also in the Lake District where Capewell⁶ has suggested that the Mell Fell Conglomerate might be Lower Old Red Sandstone. This evidence of considerable folding at the end of the Silurian can be linked with the intrusion of some Galloway and Lake District granites. Lambert and Mills²⁵ state that the post-tectonic Shap and Creetown granites have respective ages of 397 and 399 million years (see also ref. 10). Lambert and Mills also consider that it is "most probable that the Caledonian Shap, Creetown, and Leinster granites were intruded in the early part of the interval Upper Silurian to Upper Devonian." Both the major movements and these granites were thus probably either latest Silurian or Lower Devonian. These granites, and others nearby,^{4, 12} were involved in further movements, possibly of Middle Old Red Sandstone age. Stratigraphical evidence of these later movements is provided by the unconformity between the Lower and the Upper Old Red Sandstone in the Midland Valley and in Berwickshire, but it is not clear how great an area was affected by these Middle Old Red movements. There is no certainty that they were continuous southwards to Wales, or even that they occurred everywhere between the Midland Valley and the Lake District. Later igneous activity is indicated by the ages of the Weardale and Skiddaw granites;¹¹ the 365-million-year date suggests a distinctly later (Upper Devonian) age for these two intrusions.²⁴

Wales.—In both North and South Wales, the Lower Cambrian rests unconformably on pre-Cambrian. The first break recorded in the Lower Paleozoic succession is the disconformity at the base of the Middle Cambrian Solva Series in South Wales.

Evidence for stronger and more widespread movement is present at the base of the Arenig, which rests on folded and eroded Cambrian rocks in North Wales (e.g., St. Tudwal's Peninsula) and in South Wales (Pembroke). These pre-Arenig movements were, however, very slight in the Welsh Borderland. Later Ordovician movements have been recorded by Davies⁸ associated with the Cader Idris granophyre. The absence of Llandeilo rocks over much of North Wales suggests that these movements were quite widespread. But, while they may be related to the pre-Caradocian break in the north of England, there is no sign of these movements occurring in the Shelve inlier, where sedimentation is continuous throughout the Ordovician. Some movement did occur near Caer Caradoc, however, between the Tremadocian and the Caradocian. In South Wales, the main Upper Ordovician movements are later than in the north; local breaks occurring at the top of the Caradocian indicate an emergence of relatively short duration.²⁸ Thus, during the Ordovician, the main movements are mostly confined to the geosynclinal areas of Wales. At the end of the Ordovician, the geographical distribution of the main stratigraphical breaks is quite different; they occur in northeast Wales and the Welsh Border but not in the south, west, or northwest of the country. Strong pre-Silurian folding is evident in the Malverns³¹ and the Shelve inlier, both situated to the southeast of the Lower Paleozoic geosyncline. By contrast, there are few concurrent movements in the Welsh geosynclinal areas. Once Silurian sedimentation commenced (which was at different times throughout the Llandovery in different areas), it appears to have been continuous, except in southeast Wales, where local uplifts interrupted the Llandovery succession,² and in south Pembrokeshire, where there is a big break at the base of the Upper Llandovery (Jones, ref. 40).

The immediately post-Silurian breaks may extend throughout the geosynclinal parts of Wales: in the south, the small breaks seen at the base of the Old Red Sandstone in Brecknockshire become larger to the west of Llandeilo, and there is no Upper Ludlow exposed in North Wales. It is possible that the whole geosyncline from Berwick to Pembroke *might* have suffered considerable movement at this time, but it is not possible to be definite about the age of the movements in North Wales or northern England. The Old Red Sandstone of Anglesey is unfossiliferous and is thus of no help in precise dating. We have just seen that in, and at the end of, the Ordovician, fold movements took place at different times in the Welsh geosynclinal areas and in the Borderland area outside the geosyncline. Long distance correlation of breaks in the stratigraphical succession must therefore be proved and not assumed. While the fold movements affecting the Silurian of North Wales may possibly be pre-Lower Old Red Sandstone as in the north of England and Pembroke, they may have taken place at any time between the Upper Ludlow and the Upper Devonian (perhaps in the Middle Devonian, for example—see below).

In southeast Wales and the Welsh Borderland, the main post-Silurian movements are after the Lower Old Red Sandstone, there being an unbroken sequence up to the end of the Breconian, the highest beds of which may, in fact, be of Middle Old Red Sandstone age,³ though no Middle Old Red fishes or plants have been recorded. The post-Lower Old Red Sandstone break is clearly the dominant feature of the

Caledonian folding in this area (see also ref. 19). Although the change in facies suggests regional uplift at the start of the Old Red Sandstone, O. T. Jones²¹ concludes that "in Wales and the adjoining regions, as well as south of the Highland Border, the great earth movements to which the name Caledonian has usually been applied, occurred after the deposition of all the known Lower Old Red Sandstone of these areas." Widespread movements clearly took place both before and after the Lower Old Red Sandstone, but it is most doubtful whether, at either time, they affected a geographically continuous area extending from Pembroke to the Scottish Highlands. The extension of the Old Red Sandstone facies to England at the start of the Devonian (it is there in the Silurian of Scotland) could quite well be due to local uplift (say, in Pembrokeshire and near the Scottish Border) coupled with a change in the relative rates of subsidence and sedimentation in the areas of deposition.

Ireland.—In southeast Ireland, the Carboniferous and Old Red Sandstone rest on folded Cambrian and Ordovician; there is thus no accurate stratigraphic dating of the Caledonian folding. Lambert and Mills²⁵ give an age of 405 million years for the Leinster granite; this suggests that, as the granite post-dates the main folding,⁵ fold movements took place shortly before Lower Old Red Sandstone times.

In the southwest, fossiliferous Ordovician occurs only in the Slieve Bernagh and Slieve Aughty inliers. Because of faulting and poor outcrops, there are no records of inter-Ordovician breaks. The Ordovician-Silurian junction is also obscure, but Ashgillian and Llandovery are both present. It may be that there are no large breaks in the sedimentary record before the end of the Silurian. A continuous Silurian succession from Upper Llandovery to Ludlow is seen in the Dingle Peninsula; here the Dingle Beds rest on the Ludlow with a distinct stratigraphical junction, but, as these oldest Old Red Sandstone beds are unfossiliferous, it is not possible to assess the length of the break (if any) at the end of the Ludlow; it may not be very large. The Dingle Beds are folded along with the Silurian beds and are succeeded, after a marked unconformity, by later Old Red Sandstone beds.³³

Throughout the whole of the southern half of Ireland, the only known Old Red fossils are in the Kiltorcan Beds at the very top of the Devonian; these pass up conformably into the basal Carboniferous beds. It is possible that, except in Dingle, there is no Lower Old Red Sandstone in this large area;¹⁶ the strong unconformity between the Old Red Sandstone and the Lower Paleozoic beds could thus be due to folding at any time between the Ludlow and the Upper Old Red. It is only in Dingle that there is evidence to show that folding took place after the onset of Old Red Sandstone conditions, but there, because of the lack of fossils, precise dating of this break is not possible; we know only that it is post-Ludlow.

North of Galway Bay, there is better stratigraphical evidence for the dating of fold movements. In Connemara, there are strong pointers (but no exposures) to suggest that the Arenig lies on the Connemara Schists.^{27, 9} Age dates suggest that the metamorphism of the Connemara Schists was about 470 million years ago;¹⁵ it is probable that this date corresponds to late Cambrian or Tremadocian, as the Mayo Arenig is not affected by this event. Ordovician or pre-Ordovician movements are also shown to occur further east, by the unconformable relationship of the Caradocian beds near Pomeroy to the Tyrone Igneous Series.

At the end of the Ordovician, there was considerable movement in the northwest

quarter of Ireland. Northwest Galway, south Mayo, and the Curlew Mountains all have Upper Llandovery rocks resting unconformably on an eroded surface of Ordovician; and in the first two of these areas, there is evidence of the removal of several thousand feet of Ordovician beds in this interval. J. F. Dewey (personal communication) has observed pre-Silurian thrusting affecting the Dalradian and Ordovician beds in south Mayo on the southern shores of Clew Bay. But further east, at Pomeroy in County Tyrone, there is no recognizable break between the Ashgillian and Lower Llandovery. It is also possible that Silurian beds follow conformably on the Ordovician at Balbriggan and Portrane to the north of Dublin. Sedimentation may thus have been continuous between the Ordovician and Silurian on both sides of the Irish Sea (cf. North Wales and the Lake District). Further north, in the Longford-County Down continuation of the Scottish Southern Uplands, the relationship of the Silurian to the Ordovician is not at all clear. Upper Caradocian and Ashgillian have only been recorded at one locality¹⁸; there might well be a break between the two systems in much of this region.

Evidence of mid-Silurian movements is restricted to the west; in Galway and Mayo up to 3,000 feet of Lower Wenlock conglomerates and graywackes rest unconformably on Upper Llandovery beds of C5 age.^{27, 1} The graywackes pass up into shales with graptolites assigned to the zone of *Monograptus riccartonensis*. Uplift, erosion of 500 feet of Llandovery beds, and deposition of 3,000 feet of Lower Wenlock thus took place during a time approximately equal to one-sixth of the Upper Llandovery and one-sixth of the Wenlock. This is certainly proof that stratigraphical breaks do not necessarily represent long periods of time.

South of Killary Harbour, the *M. riccartonensis* beds pass up conformably into nonmarine sandstones and shales similar in many ways to the Old Red Sandstone facies;²⁷ the only fossils known are *Lingula* and nonmarine lamellibranchs. In south Mayo too, there is a similar facies in the Upper Silurian west of Louisburg (Dewey, personal communication). In western Ireland, the major change from geosynclinal graywackes occurs earlier than anywhere else in the British Isles except for the Midland Valley of Scotland. These two areas are also unusual in having no big stratigraphical breaks between the two different facies.

As in the south of Ireland, there is little proof in the north of widespread post-Silurian, pre-Lower Old Red Sandstone movements. The only fossiliferous Lower Old Red occurs at Pomeroy,¹⁶ where beds with *Pteraspis* rest on the Llandovery. That this was not a widespread break is suggested by the evidence from the Curlew Mountains, where there is no great discordance between the Silurian and the Old Red Sandstone.⁷

Middle Old Red Sandstone folding in Antrim has been inferred by Wilson³⁹ on the strength of lithological comparison with the Midland Valley of Scotland. There has also been Middle or Upper Old Red folding in the Curlew Mountains.⁷ But there is no direct evidence to date these movements accurately. In western Ireland, Wenlock beds were chloritized and strongly folded before Viséan times. A single determination from the post-tectonic Galway granite gives an age of 365 million years;¹⁵ this corresponds to the Upper Devonian on Kulp's scale²⁴ and suggests that the folding and chloritization may have taken place close to, or during, Middle Devonian times.

Midland Valley of Scotland.—The oldest beds outcropping in the Midland Valley

are the Arenig lavas and sediments alongside the Highland Boundary Fault; at Aberfoyle, Jehu and Campbell²⁰ have described how these are overlain unconformably by Upper Ordovician sediments. The pre-Caradoc break of the Lake District may thus possibly extend right across the Southern Uplands (where it is known) and the whole breadth of the Midland Valley (where it is not seen), (cf. ref. 14, fig. 4).

No large inter-Silurian breaks occur. The Silurian stratigraphy of several Midland Valley inliers has recently been revised;^{26, 32} it now appears probable that much of it is of Old Red Sandstone facies. The fact that the end of the Silurian does not everywhere coincide with the first appearance of these nonmarine facies shows that this marked change in sedimentation cannot be used as evidence of a major event occurring throughout Scotland at this time. Lamont²⁶ shows "that nonmarine conditions supervened at various times in different areas of the Silurian and that orogenic conditions, erosion phenomena, and red beds are as likely in the Gala as in the Ludlow." Parts of the Midland Valley seem to have had their main orogeny well before the Devonian and possibly before the Silurian. At the base of the Old Red Sandstone, local breaks are known in the Pentlands and near Girvan; but, as there is no contemporary break at Lesmahagow, these are probably only of local significance. Kennedy²³ considers that the Midland Valley acted as a foreland with respect to the late Silurian folding in the Southern Uplands. In the Midland Valley, no direct evidence of pre-Silurian movements is now exposed; the major movements seen are of Middle Old Red Sandstone age. These are especially strong near the boundary faults. No Middle Old Red Sandstone faunas or floras are known in the area.

Scottish Highlands.—From direct stratigraphical and fossil evidence, all that can be proved about the age of the Dalradian folding and metamorphism in the Grampian Highlands is that it is probably post-Middle Cambrian and certainly pre-Lower Old Red Sandstone. The plant and fish remains at Oban show without doubt that, in contrast to the Midland Valley, all the major movements (except along and near to the large faults) occurred before the Devonian. It is likely that the *Pagetia*-bearing Leny Limestone was involved in most of these movements.³⁵ Some of them are thus post-Middle Cambrian and are probably to be correlated with Giletti, Moor bath, and Lambert's¹⁵ event of 475 million years. The evidence from Ireland (see above) suggests that this age is pre-Ordovician. The Upper Cambrian may therefore be the time of the largest Caledonian events over the whole of the Dalradian area which extends from the Grampians to Connemara.

But there were undoubtedly other events, both before and after the Upper Cambrian. From the Aberfoyle evidence cited above, Kennedy^{22, 23} has suggested major folding in mid-Ordovician times. This is possible, but there is little evidence to prove movement at this time north of the Highland Border area. From other work,^{29, 30} it is clear that several tectonic events occurred in the Highlands, and a comparison with known events in Lower Paleozoic areas suggests that while some (like those in the Upper Cambrian) may be widespread, many may have only local significance.

Radioactive age work seems the most profitable field for future work in the Highlands. Already results are indicating several events in the Highlands, but much more work remains to be done.

Northeast Scotland also provides evidence of some minor later movements. Correlations of the Old Red with the European Devonian^{37, 38} suggest that Lower Old Red Sandstone may be present in the Highlands, and that breaks are present both before and after the Middle Old Red Sandstone.

The Dalradian areas of Scotland and Ireland had their large major folding and metamorphism before the Arenig; strong pre-Silurian folding is known in the Welsh Borderland and western Ireland, it is also possible in northeast Ireland and the Midland Valley; at the end of the Silurian, we know there was folding in Berwickshire and Pembrokeshire, and possibly in intermediate areas and in southeast Ireland (it is likely that north of a line from Galway to the Forth there was an already stable block of folded beds by this time); in Middle Old Red Sandstone times, the main movements are in South Wales, in the Welsh Borderland, and near fault zones in Scotland (Fig. 1). The division on this map of Caledonian folding into five episodes is a considerable over-simplification: the rocks in and near the British Caledonian geosyncline show many more than five distinct folding episodes. Some are local, others fairly widespread, but probably at no one time was the whole area suffering from strong simultaneous fold movements. The evidence suggests that many of the breaks occurred during short periods of time separated by much longer intervals when there was gentle subsidence or no movements at all.

Widespread folding took place in Middle Old Red Sandstone times; in fact, this is the only time in the Paleozoic when it is not possible to prove continuous sedimentation at least somewhere in the British Isles. Folding *could* have taken place almost everywhere south of the Scottish and Irish Dalradian areas during the Middle Devonian; but is this the age of the Caledonian orogeny? There were many orogenic events before the Middle Devonian, and there are many folding episodes that cannot be well dated. Is it not more in keeping with the facts to suggest that the Caledonian orogeny is the sum total of all these events from the Upper Cambrian to the Devonian?

Other aspects of the orogeny than folding, such as the Silurian and Devonian granites, the metamorphism of the Dalradian and Moine Series, and the oncoming of Old Red Sandstone sediments, are also seen to occur at different times throughout the British Isles (in the instances where they can be dated; there may be much more diversification in time than is apparent).

It is now suggested that these movements in, and adjacent to, the Caledonian geosyncline which took place between the Upper Cambrian and Middle Devonian are all truly Caledonian: many events occurred in Scotland through this period and

FIG. 1.—The distribution of Caledonian folding in the British Isles. Broken lines indicate uncertainty in either time or in area. Each of the five groups of folding covers movements during an appreciable geological time; all the folding in any one group is not necessarily closely associated.

practically all have been termed "Caledonian" by some past authors. It is meaningless to call those in the Upper Ordovician "Taconic" and those in the mid-Devonian "Acadian." These two names should be restricted to movements in the Taconic and Acadian geosynclines respectively. Different episodes of movement in the Caledonian geosyncline can best be labeled (as is done in this paper) in a stratigraphical sense, e.g., pre-Caradocian, mid-Devonian, etc. If movements in the British Isles can occur at distinct times within a system or part of a system, is it at all probable that Taconic or Acadian movements occurred exactly simultaneously on both sides of the Atlantic in (even allowing for continental drift) distinct geosynclines? Woodward⁴¹ has shown that there are several distinct deformations in the eastern United States; we have the same in Britain.

Gilluly¹⁷ pointed out that "stratigraphers assign unconformities, not at random, but displace them in the direction of stage boundaries." We must be careful not to displace the Caledonian orogeny towards any one period of time just to satisfy preconceived ideas. The truth about this complicated series of events will be found only by objective examination of all the evidence available.

The writing of this paper has been possible only after many discussions with many friends, to all of whom I am much indebted; especial thanks go to my colleagues R. St. J. Lambert and H. G. Reading and to Professor T. Neville George.

¹ Anderson, J. G. C., *Geol. Mag.*, **97**, 265-275 (1960).

² Andrew, G., and O. T. Jones, *Quart. J. Geol. Soc. (London)*, **81**, 407-414 (1925).

³ Ball, H. W., and D. L. Dineley, *Bull. Brit. Mus. (N. H.) Geol.*, **5**, 177-310 (1961).

⁴ Blyth, F. G. H., *Proc. Geol. Assoc.*, **65**, 224-250 (1954).

⁵ Brindley, J. C., in *A View of Ireland* (British Association for the Advancement of Science 1957), pp. 15-22.

⁶ Capewell, J. G., *Quart. J. Geol. Soc. (London)*, **109**, 23-46 (1955).

⁷ Charlesworth, H. A. K., *Proc. R. Irish Acad.*, **61**, B, 3, 51-58 (1960).

⁸ Davies, R. G., *Quart. J. Geol. Soc. (London)*, **115**, 189-216 (1959).

⁹ Dewey, J. F., *Geol. Mag.*, **98**, 399-405 (1961).

¹⁰ Dodson, M. H., J. A. Miller, and D. York, *Nature*, **190**, 800-802 (1961).

¹¹ Dodson, M. H., and S. Moorbath, *Nature*, **190**, 900 (1961).

¹² Firman, R. J., *Quart. J. Geol. Soc. (London)*, **114**, 317-347 (1960).

¹³ Geikie, A., *The Geology of Eastern Berwickshire*, Memoirs of the Geological Survey of Great Britain (1863).

¹⁴ George, T. N., *Trans. Geol. Soc. (Glasgow)*, **24**, 32-107 (1960).

¹⁵ Gilletti, B. J., S. Moorbath, and R. St. J. Lambert, *Quart. J. Geol. Soc. (London)*, **117**, 233-272 (1961).

¹⁶ Gill, W. D., in *A View of Ireland* (British Association for the Advancement of Science, 1957), pp. 22-32.

¹⁷ Gilluly, J., *Bull. Geol. Soc. Am.*, **60**, 561-590 (1949).

¹⁸ Harper, J. C., *Proc. Liverpool Geol. Soc.*, **20**, 48-67 (1948).

¹⁹ Holland, C. H., *Quart. J. Geol. Soc. (London)*, **114**, 449-482 (1959).

²⁰ Jchu, T. J., and R. Campbell, *Trans. Roy. Soc. Edin.*, **52**, 175-212 (1917).

²¹ Jones, O. T., *Quart. J. Geol. Soc. (London)*, **111**, 323-351 (1956).

²² Kennedy, W. Q., *Quart. J. Geol. Soc. (London)*, **110**, 357-390 (1955).

²³ Kennedy, W. Q., *Trans. Geol. Soc. (Glasgow)*, **23**, 106-133 (1958).

²⁴ Kulp, J. L., *Science*, **133**, 1105-1114 (1961).

²⁵ Lambert, R. St. J., and A. A. Mills, *Ann. N. Y. Acad. Sci.*, **91**, 378-388 (1961).

²⁶ Lamont, A., in *Report of the Eighteenth International Geological Congress, London, 1948*, part 10 (1952), pp. 27-32.

²⁷ McKerrow, W. S., and C. J. Campbell, *Sci. Proc. Roy. Dublin Soc., Ser. A*, **1**, 27-51 (1960).

- ²⁸ Pringle, J., and T. N. George, in *South Wales*, British Regional Geology (2d ed.; London: 1948).
- ²⁹ Ramsay, J. G., *Quart. J. Geol. Soc. (London)*, **113**, 271–307 (1958).
- ³⁰ Rast, N., *Quart. J. Geol. Soc. (London)*, **114**, 25–46 (1958).
- ³¹ Reading, H. G., and Poole, A. B., *Geol. Mag.*, **98**, 295–300 (1961).
- ³² Rolfe, W. D. I., *Proc. Geol. Soc. (London)*, 1585, 48–52 (1961).
- ³³ Shackleton, R. M., *Proc. R. Irish Acad.*, **46**, 1–12 (1940).
- ³⁴ Spieker, E. M., *Bull. Am. Assn. Petrol. Geologists*, **40**, 1769–1815 (1956).
- ³⁵ Stone, M., *Geol. Mag.*, **94**, 265–276 (1957).
- ³⁶ Tarlo, L. B., *Quart. J. Geol. Soc. (London)*, **117**, 193–213 (1961).
- ³⁷ Westoll, T. S., in *Report of the Eighteenth International Geological Congress, London, 1948*, part 11, Geological Survey and Museum (1951), pp. 5–21.
- ³⁸ Williams, A., *Proc. Geol. Soc. (London)*, **1580**, 102–107 (1960).
- ³⁹ Wilson, H. E., *Proc. R. Irish Acad.*, **55**, B, 283–320 (1953).
- ⁴⁰ Wood, A., and A. J. Smith, *Quart. J. Geol. Soc. (London)*, **114**, 163–195 (1958).
- ⁴¹ Woodward, H. P., *Bull. Am. Assoc. Petrol. Geologists*, **41**, 10, 2312–2327 (1957).

A NEW FORMALISM IN PERTURBATION THEORY USING CONTINUED FRACTIONS

BY RICHARD BELLMAN AND JOHN M. RICHARDSON

THE RAND CORPORATION, SANTA MONICA, AND HUGHES RESEARCH LABORATORIES,
MALIBU, CALIFORNIA

Communicated by H. S. Vandiver, September 24, 1962

1. *Introduction.*—A problem of continuing interest is that of obtaining approximate solutions of the functional equation

$$L(u) + (a(p) + \lambda b(p))u = 0, \quad (1.1)$$

where L is a linear differential operator, in terms of the solution of the unperturbed equation

$$L(u) + a(p)u = 0. \quad (1.2)$$

Using the Green's function of (1.2), or equivalent techniques, and regarding the term involving λ as a forcing term, we can convert (1.1) into an equation of the form

$$u = f + \lambda T(u), \quad (1.3)$$

where T is a linear transformation.

We shall present a new approach to problems of this nature using the classical technique of continued fractions. In subsequent papers, we shall discuss the rigorous aspects and the value of this approach in obtaining rational approximations outside the circle of convergence of the Liouville-Neumann solution of (1.3).

2. *Continued fraction expansion.*—From (1.3) we have

$$T^n(u) = T^n(f) + \lambda T^{n+1}(u), \quad n = 0, 1, 2, \dots \quad (2.1)$$

Hence, combining the n th and $(n + 1)$ st expressions,