PROBABILITY GENERATING FUNCTIONS AND THEIR ITERATES*

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1. *Introduction.*—The usual definition of probability generating functions applies to nonnegative integer-valued random variables. In this paper the term probability generating function (pgf) of a random variable X will be used to denote the function

$$f(s) = Es^X = \int_0^\infty s^x dF(x) \quad (1.1)$$

where X is nonnegative, but not necessarily integer-valued, and where F denotes the distribution function of X . The paper deals with properties of (1.1) in their relation to probabilistic properties of X .

2. *Types of Probability Generating Functions.*—For $0 \leq s \leq 1$, the probability generating function f is a real-valued nondecreasing function of its argument with $0 \leq f(0) \leq f(1) = 1$. It is well known that, for X integer-valued, the function f is convex. In general, this remains true only as long as $P(0 < X < 1) = 0$; more precisely, for any positive integer k we have

LEMMA 2.1. *If $P(0 < X < k - 1, X \text{ not integer}) = 0$, then*

- (i) $f^{(j)}(0) = j!P(X = j) \geq 0$; $j = 0, 1, 2, \dots, k - 1$
- (ii) $f^{(j)}(s)$ is nonnegative and nondecreasing for

$$j = 0, 1, \dots, k - 1; \quad 0 < s.$$

Furthermore, if $P(k - 1 < X < k) > 0$, then

- (iii) $f^{(k)}(0) = +\infty$, $f^{(k)}(s) \geq 0$, $0 < s$.

Proof: To show (i) and the first part of (iii) we write (1.1) as

$$f(s) = \sum_{i=0}^{k-1} p_i s^i + \int_{(k-1)^+}^\infty s^x dF(x), \text{ where } p_i = P(X = i).$$

Thus, for $j = 0, 1, \dots, k - 1$,

$$\begin{aligned} f^{(j)}(0) &= j!p_j + \lim_{s \rightarrow 0} \frac{1}{s^j} \sum_{i=0}^j (-1)^{(j-i)} \binom{j}{i} \int_{(k-1)^+}^\infty (is)^x dF(x) \\ &= j!p_j + \lim_{s \rightarrow 0} \sum_{i=0}^j (-1)^{(j-i)} \binom{j}{i} \int_{(k-1)^+}^\infty i^x s^{(x-j)} dF(x). \end{aligned}$$

Each of the integrands being bounded and monotone decreasing to zero assertion (i) follows. For $j = k$, however, we obtain

$$f^{(k)}(0) = \lim_{s \rightarrow 0} \sum_{i=0}^k (-1)^{(k-i)} \binom{k}{i} \int_{(k-1)^+}^\infty i^x s^{(x-k)} dF(x).$$

Since $P(k - 1 < X < k - 1 + a) > 0$ for some $a < 1$, one has, as $s \rightarrow 0$,

$$\begin{aligned} \int_{(k-1)^+}^{\infty} i^x s^{(x-k)} dF(x) &\geq \int_{(k-1)^+}^{k-1+a} i^x s^{(x-k)} dF(x) \\ &\geq i^{(k-1)} s^{a-1} P(k-1 < X < k-1+a) \rightarrow \infty \end{aligned}$$

which proves the first part of (iii). Assertion (ii) and the second part of (iii) can be shown by differentiating (1.1) under the integral sign. It is seen that $f^{(j)}(s)$ is even strictly positive unless with probability one X takes only the values $0, 1, \dots, j-1$. In particular, $P(0 < X < 1) = 0$ and $P(X > 1) > 0$ imply that $f(s)$ is strictly convex. Thus, in this case the equation $f(s) = s$ can have at most one solution q in the open unit interval. In the general case, this will be true as a consequence of the following lemma.

LEMMA 2.2. *If $P(X = 0 \text{ or } X = 1) < 1$, then the equation $f''(s) = 0$ has at most one solution $\rho > 0$.*

Proof: Let $0 < s < t$. Then, for $0 < x \neq 1$ we have $x(x-1)s^x < x(x-1)t^x s/t$, and therefore the relation

$$s^2 f''(s) = \int_0^{\infty} x(x-1)s^x dF(x) < \frac{s}{t} \int_0^{\infty} x(x-1)t^x dF(x) = s t f''(t),$$

which shows that not both $f''(s)$ and $f''(t)$ can be zero.

Differentiating (1.1) under the integral sign at $s = 1$ (taking left derivatives if needed), we obtain

LEMMA 2.3. *The kk th factorial moment $EX(X-1)\dots(X-k+1)$ of X is given by*

$$\mu_k = f^{(k)}(1) \tag{2.1}$$

which may be finite or not.

Let $p = P(X = 0)$ and $m = \mu_1 = EX$; then we can summarize the results obtained so far:

THEOREM 2.1. *The probability generating function $f(s)$, $0 \leq s \leq 1$, of a distribution on $[0, \infty[$ is of one of the types listed below and drawn in Figure 1:*

	0-Type ($p = 0$)	p -Type ($0 < p < 1$)	Trivial ($p = 1$)
Linear	$m = 1$	$m = 1 - p < 1$	$m = 0$
Convex	$m > 1$	$\begin{cases} 1 - p < m < 1 \\ m = 1 \\ m > 1 \end{cases}$	
Concave	$m < 1$	$m < 1$	
ρ -Type	$\begin{cases} m < 1 \\ m = 1 \\ m > 1 \end{cases} \begin{cases} \rho < q \\ \rho = q \\ \rho > q \end{cases}$	$\begin{cases} m < 1 \\ m = 1 \\ m > 1 \end{cases} \begin{cases} \rho < q \\ \rho = q \\ \rho > q \end{cases}$	

In fact, any function not of one of the types listed has been excluded by Lemma 2.1 or Lemma 2.2.

THEOREM 2.2. *There are probability generating functions of any of the types listed in Theorem 2.1.*

LINEAR TYPE:

$p=0, m=1$

$0<p<1, m<1$

$p=1, m=0$

CONVEX TYPE:

$p=0, m>1$

$0<p<1, m<1$

$0<p<1, m=1$

$0<p<1, m>1$

CONCAVE TYPE:

$p=0, m<1$

$0<p<1, m<1$

p -O-TYPE:

$p=0, m<1$

$p=0, m=1$

$p=0, m>1, \rho<q$

$p=0, m>1, \rho>q$

p -p-TYPE:

$0<p<1, m<1$

$0<p<1, m=1$

$0<p<1, m>1, \rho<q$

$0<p<1, m>1, \rho>q$

FIG. 1.—Graphs of typical probability generating functions.

Proof: Let α, β, λ be nonnegative constants; let Y be Poisson-distributed with parameter λ , and let $X = \alpha Y + \beta$. It can then be seen, inspecting the probability generating function f of X and the first two derivatives of f , that the parameters α, β, λ can be adjusted in such a way that f is of any desired type except linear. The linear types correspond to $P(X = 0) = 1 - P(X = 1) = \rho$.

3. *Iterates and Compositions of Probability Generating Functions.*—Iterates of pgf's play a basic role in the investigation of branching sequences (refs. 2 and 3). If the "branching conditions" change from generation to generation, then these iterates have to be replaced by compositions of different pgf's.

THEOREM 3.1. *The types of probability generating functions as listed in Theorem 1.1 are invariant under iteration. More precisely, if f and g have one of the following properties in common, then their composition $f \circ g$ has this same property:*

- (a) *linear type, convex type, concave type, ρ -type;*
- (b) *0-type, p -type;*
- (c) *$m < 1, m = 1, m > 1$;*
- (d) *existence of q with $f(q) = q$.*

Proof: All these assertions involve the values of $f \circ g$ and of its first two derivatives and can be shown using

$$(f \circ g)'(s) = f'(g(s))g'(s) \quad (3.1)$$

$$(f \circ g)''(s) = f''(g(s))(g'(s))^2 + f'(g(s))g''(s). \quad (3.2)$$

For (d) we give a direct proof, not involving the derivatives, which shows even a little more: If $s < \min(q_f, q_g)$, then $g(s) > s$, $f(s) > s$ and thus $f(g(s)) \geq s$; conversely, $s > \max(q_f, q_g)$ will imply $f(g(s)) < s$. Thus, $q_{f \circ g}$ exists and is intermediate between q_f and q_g .

The author does not know whether the subtypes $\rho < q$, $\rho = q$, $\rho > q$ of the ρ -type with $m > 1$ are invariant under composition; however, under iteration even a stronger statement can be made.

THEOREM 3.2. *If $f(q) = q$, $0 < q \leq 1$, then for all iterates f_n we have $f_n(q) = q$; moreover, $f_n(s) \rightarrow q$ for $0 < s < 1$. If $f(1) = 1$ is the only solution of $f(s) = s$, then $\lim_{n \rightarrow \infty} f_n(s) = 1$; if $f(0) = 0$ and $f(1) = 1$ are the only solutions of $f(s) = s$, then $f_n(s) \rightarrow 0$ for all $s \in [0, 1[$ in the convex case, or $f_n(s) \rightarrow 1$ for all $s \in]0, 1]$ in the concave and ρ -type cases.*

If f is of the ρ -type with $m > 1$, then if $f(q) = q$ and $f_n''(\rho_n) = 0$, the following is true:

- (i) *if $\rho_1 = q$, then $q = \rho_1 = \rho_2 = \dots$;*
- (ii) *if $\rho_1 < q$, then $q > \rho_1 > \rho_2 > \dots$;*
- (iii) *if $\rho_1 > q$, then $q < \rho_1 < \rho_2 < \dots$.*

Proof: The first part of the theorem is shown exactly as for probability generating functions of integer-valued variables (see, e.g., ref. 2, pp. 6 and 7). To prove the second part, we need two versions of (3.2).

$$f_{n+1}''(s) = f_n''(f(s))(f'(s))^2 + f_n'(f(s))f''(s), \quad (3.3)$$

$$f_{n+1}'(s) = f''(f_n(s))(f_n'(s))^2 + f'(f_n(s))f_n''(s). \quad (3.4)$$

Statement (i) follows readily from either one of these relations. Next, show (ii), putting $s = \rho$, in (3.3) it is seen that $\rho_n < \rho_1$; $n = 2, 3, \dots$; then assuming $\rho_n <$

ρ_{n+1} for some n and using (3.4) first with $s = \rho_{n+1}$ and then with $s = \rho_n$, it follows that $\rho_{n-1} < \rho_n$ which leads to a contradiction. The proof of (iii) is analogous.

Remark 3.1: The probabilistic meaning of p and m is obvious; q is the probability $P(Z_n \rightarrow 0 | Z_0 = 1)$ for a branching sequence with basic probability generating function f (refs. 1 and 2). It is not clear, however, what the probabilistic meaning of ρ could be.

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¹ Bühler, W. J., "Limit theorems for branching processes," Ph.D. thesis, University of California, Berkeley, 1966.

² Harris, T. E., *The Theory of Branching Processes* (Berlin-Göttingen-Heidelberg: Springer-Verlag, 1963).

³ Jiřina, M., "Stochastic branching processes with continuous state space," *Czech. Math. J.*, **8**, 292-313 (1958).